Iranian Journal of Mathematical Sciences and Informatics
Vol. 14, No. 2 (2019), pp 1-8
DOI: 10.7508/ijmsi.2019.02.001

Graded r-Ideals

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Abstract. Let G be a group with identity e and R be a commutative G-graded ring with nonzero unity 1. In this article, we introduce the concept of graded r-ideals. A proper graded ideal P of a graded ring R is said to be a graded r-ideal if whenever $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$, then $b \in P$. We study and investigate the behavior of graded r-ideals to introduce several results. We introduced several characterizations for graded r-ideals; we proved that P is a graded r-ideal of R if and only if $aP = aR \bigcap P$ for all $a \in h(R)$ with $Ann(a) = \{0\}$. Also, P is a graded r-ideal of R if and only if $P = (P : a)$ for all $a \in h(R)$ with $Ann(a) = \{0\}$. Moreover, P is a graded r-ideal of R if and only if whenever A, B are graded ideals of R such that $AB \subseteq P$ and $A \bigcap r(h(R)) \neq \phi$, then $B \subseteq P$. In this article, we introduce the concept of a huz-rings. A graded ring R is said to be a huz-ring if every homogeneous element of R is either a zero divisor or a unit. In fact, we proved that R is a huz-ring if and only if every graded ideal of R is a graded r-ideal. Moreover, assuming that R is a graded domain, we proved that $\{0\}$ is the only graded r-ideal of R.

Keywords: Graded ideals, Graded prime ideals, Graded r-ideals.

2000 Mathematics Subject Classification: 13A02

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Received 24 November 2016; Accepted 23 July 2018

1. Introduction

Let G be a group with identity e. A ring R is said to be a G-graded ring if there exist additive subgroups R_g of R such that $R = \bigoplus\limits_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous of degree g and R_e (the identity component of R) is a subring of R and $1 \in R_e$. For $x \in R$, x can be written uniquely as $\sum\limits_{g\in G} x_g$ where x_g is the component of x in R_g. Also we write $h(R) = \bigcup\limits_{g\in G} R_g$ and $supp(R, G) = \{g \in G : R_g \neq 0\}$. For more details, see [3].

Let R be a G-graded ring and P be an ideal of R. Then P is called a G-graded ideal if $P = \bigoplus_{g\in G}(P \cap R_g)$, i.e., if $x \in P$ and $x = \sum_{g\in G} x_g$, then $x_g \in P$ for all $g \in G$. An ideal of a graded ring need not be graded; see the following example.

Example 1.1. Consider $R = \mathbf{Z}[i]$ and $G = \mathbf{Z}_2$. Then R is G-graded by $R_0 = \mathbf{Z}$ and $R_1 = i\mathbf{Z}$. Now, $P = \langle 1+i\rangle$ is an ideal of R with $1+i \in P$. If P is a graded ideal, then $1 \in P$, so $1 = a(1 + i)$ for some $a \in R$, i.e., $1 = (x + iy)(1 + i)$ for some $x, y \in \mathbf{Z}$. Thus $1 = x - y$ and $0 = x + y$, i.e., $2x = 1$ and hence $x = \frac{1}{2}$ a contradiction. So, P is not graded ideal.

Throughout this article, R will be a commutative ring with nonzero unity 1. For $a \in R$, we define $Ann(a) = \{r \in R : ra = 0\}$. An element $a \in R$ is said to be a regular element if $Ann(a) = \{0\}$, the set of all regular elements of R is denoted by $r(R)$. If A is a subset of R and P is an ideal of R, then we define $(P : A) = \{r \in R : rA \subseteq P\}$.

The notion of r-ideals was introduced and studied by Rostam Mohamadian in [2]. A proper ideal P of R is said to be an r-ideal (resp. pr-ideal) if whenever $a, b \in R$ such that $ab \in P$ and $Ann(a) = \{0\}$, then $b \in P$ (resp. $b^n \in P$ for some $n \in \mathbf{N}$).

In this article, we introduce the concept of graded r-ideals. A proper graded ideal P of a graded ring R is said to be a graded r-ideal (resp. graded pr-ideal) if whenever $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$, then $b \in P$ (resp. $b^n \in P$ for some $n \in \mathbf{N}$). We study and investigate the behavior of graded r-ideals to introduce several results.

We introduce several characterizations for graded r-ideals; we prove that P is a graded r-ideal of R if and only if $aP = aR \bigcap P$ for all $a \in h(R)$ with $Ann(a) = \{0\}$. Also, P is a graded r-ideal of R if and only if $P = (P : a)$ for all $a \in h(R)$ with $Ann(a) = \{0\}$. Moreover, P is a graded r-ideal of R if and only if whenever A, B are graded ideals of R such that $AB \subseteq P$ and $A \bigcap r(h(R)) \neq \phi$, then $B \subseteq P$.

A proper graded ideal of a graded ring R is said to be graded prime if whenever $a, b \in h(R)$ such that $ab \in P$, then either $a \in P$ or $b \in P$ ([1]). We prove that the intersection of two graded r-ideals is a graded r-ideal. On the other hand, if the intersection of two non-comparable graded prime ideals is a graded r-ideal, then both ideals are graded r-ideals. Moreover, we prove that every graded maximal r-ideal is graded prime.

If P is a graded r-ideal of R, we prove that P_e is an r-ideal of R_e and $(P : a)$ is a graded r-ideal of R for all $a \in h(R) - P$. Also, we prove that if R is $\mathbb{Z}$-graded, then P is a graded pr-ideal of R if and only if $\sqrt{P}$ is a graded r-ideal of R.

In this article, we introduce the concept of huz-rings. A graded ring R is said to be a huz-ring if every homogeneous element of R is either a zero divisor or a unit. In fact, we prove that R is a huz-ring if and only if every graded ideal of R is a graded r-ideal. Moreover, assuming that R is a graded domain, we prove that $\{0\}$ is the only graded r-ideal of R.

2. Graded r-Ideals

In this section, we introduce and study the concept of graded r-ideals.

Definition 2.1. Let R be a G-graded ring. A proper graded ideal P of R is said to be a graded r-ideal (resp. graded pr-ideal) if whenever $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$, then $b \in P$ (resp. $b^n \in P$ for some $n \in \mathbf{N}$).

Note that for a graded ideal P of a G-graded ring R, $P_g = P \bigcap R_g$ for $g \in G$.

Theorem 2.2. *Let R be a G-graded ring and P be a graded ideal of R. Then P is a graded r-ideal if and only if $aP = aR \bigcap P$ for every $a \in h(R)$ with $Ann(a) = \{0\}$.*

Proof. ($\Rightarrow$) Let $a \in h(R)$ such that $Ann(a) = \{0\}$. Then $aP \subseteq P$ and $aP \subseteq aR$, i.e., $aP \subseteq aR \bigcap P$. Let $x \in aR \bigcap P$. Then $x = az \in P$ for some $z \in R$. Since R is G-graded, $z = \sum\limits_{g \in G} z_g$ and then $x = \sum\limits_{g \in G} az_g \in P$ and since P is a graded ideal, $az_g \in P$ for all $g \in G$. Since P is a graded r-ideal, $z_g \in P$ for all $g \in G$ and then $z = \sum\limits_{g \in G} z_g \in P$ which implies that $x = az \in aP$. Hence, $aP = aR \bigcap P$. ($\Leftarrow$) Let $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Then $ab \in aR \bigcap P = aP$ and then $ab = ax$ for some $x \in P$ which implies that $a(b - x) = 0$. Since $Ann(a) = \{0\}$, $b - x = 0$, i.e., $b = x \in P$. Hence, P is a graded r-ideal. □

Theorem 2.3. *Let R be a G-graded ring and P be a graded ideal of R. If $aP_g = aR_h \bigcap P_g$ for all $g, h \in G$ and for all $a \in h(R)$ with $Ann(a) = \{0\}$, then P is a graded r-ideal of R.*

Proof. Let $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Then there exist $g, h \in G$ such that $a \in R_g$ and $b \in R_h$ and then $ab \in R_gR_h \bigcap P \subseteq R_{gh} \bigcap P = P_{gh}$. Now, $ab \in aR_h \bigcap P_{gh} = aP_{gh}$, i.e., $ab = ay$ for some $y \in P_{gh}$ and then $a(b - y) = 0$. Since $Ann(a) = \{0\}$, $b = y \in P_{gh} \subseteq P$. Hence, P is a graded r-ideal of R. □

Theorem 2.4. *Let R be a G-graded ring and P be a graded ideal of R. Then P is a graded r-ideal if and only if $P = (P : a)$ for all $a \in h(R)$ with $Ann(a) = \{0\}$.*

Proof. Suppose that P is a graded r-ideal of R. Let $a \in h(R)$ with $Ann(a) = \{0\}$. Clearly, $P \subseteq (P : a)$. Let $y \in (P : a)$. Then $ya \in P$. Since R is G-graded, $y = \sum\limits_{g \in G} y_g$ and then $ya = \sum\limits_{g \in G} y_g a \in P$ and since P is graded, $y_g a \in P$ for all $g \in G$. Since P is a graded r-ideal, $y_g \in P$ for all $g \in G$ and then $y = \sum\limits_{g \in G} y_g \in P$. Hence, $P = (P : a)$. Conversely, let $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Then $b \in (P : a) = P$. Hence, P is a graded r-ideal of R. □

Theorem 2.5. *Let R be a G-graded ring and P be a graded ideal of R. If $P_g = (P_g :_{R_h} a)$ for all $g, h \in G$ and for all $a \in h(R)$ with $Ann(a) = \{0\}$, then P is a graded r-ideal of R.*

Proof. Let $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Then $a \in R_g$ and $b \in R_h$ for some $g, h \in G$ and then $ab \in R_gR_h \bigcap P \subseteq R_{gh} \bigcap P = P_{gh}$, i.e., $b \in (P_{gh} :_{R_h} a) = P_{gh} \subseteq P$. Hence, P is a graded r-ideal of R. □

Theorem 2.6. *Let R be a G-graded ring and P be a graded ideal of R. Then P is a graded r-ideal if and only if whenever A, B are graded ideals of R such that $AB \subseteq P$ and $A \bigcap r(h(R)) \neq \phi$, then $B \subseteq P$.*

Proof. Suppose that P is a graded r-ideal of R. Let A, B be two graded ideals of R such that $AB \subseteq P$ and $A \bigcap r(h(R)) \neq \phi$. Since $A \bigcap r(h(R)) \neq \phi$, there exists $a \in A \bigcap r(h(R))$. Let $g \in G$ and $b \in B_g$. Then $ab \in AB_g \subseteq AB \subseteq P$. Since P is a graded r-ideal, $b \in P$. So, $B_g \subseteq P$ for all $g \in G$ which implies that $B \subseteq P$. Conversely, let $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Then $A = \langle a \rangle$ and $B = \langle b \rangle$ are graded ideals of R such that $AB \subseteq P$ and $a \in A \bigcap r(h(R))$. By assumption, $B \subseteq P$ and then $b \in P$. Hence, P is a graded r-ideal of R. □

Theorem 2.7. *If R is a G-graded domain, then $\{0\}$ is a unique graded r-ideal of R.*

Proof. Let P be a nonzero proper graded ideal of R. Then there exists $0 \neq a = \sum\limits_{g \in G} a_g \in P$ and then $a_g \in P$ for all $g \in G$ since P is graded. Since R is a domain, $Ann(a_g) = \{0\}$ with $1.a_g \in P$. If P is a graded r-ideal, then $1 \in P$ which is a contradiction. Hence, $\{0\}$ is the only graded r-ideal of R. $\square$

Lemma 2.8. *If R is a G-graded ring, then R_e contains all homogeneous idempotent elements of R.*

Proof. Let $0 \neq x \in h(R)$ be an idempotent. Then $x \in R_g$ for some $g \in G$ and then $x = x^2 \in R_g \bigcap R_{g^2}$. Since $0 \neq x \in R_g \bigcap R_{g^2}$, $g^2 = g(\in G)$ which implies that $g = e$. Hence, $x \in R_e$. $\square$

Theorem 2.9. *Let R be a G-graded ring. Suppose that $\{x_i : i \in \Gamma\}$ is a set of homogeneous idempotent elements in R_e. Then $P = \sum_{i\in\Gamma} R_e x_i$ is an r-ideal of R_e.*

Proof. Let $a, b \in R_e$ such that $ab \in P$ and $Ann(a) = \{0\}$. Let $z = \prod_{k=1}^{n}(1 - x_{i_k})$ where $ab = \sum_{j=1}^{n} r_j x_{i_j}$ for some $r_1.....r_n \in R_e$. Then $abz = 0$. Since $Ann(a) = \{0\}$, $bz = 0$. On the other hand, there exists $r \in P$ such that $z = 1 - r$ and then $b(1 - r) = 0$ which implies that $b = br \in P$. Hence, P is an r-ideal of R_e. $\square$

The next lemma is well known and clear; so we omit the proof.

Lemma 2.10. *If P_1 and P_2 are graded ideals of a graded ring R, then $P_1 \bigcap P_2$ is a graded ideal of R.*

Theorem 2.11. *Let R be a G-graded ring. If P_1 and P_2 are graded r-ideals of R, then $P_1 \bigcap P_2$ is a graded r-ideal of R.*

Proof. By Lemma 2.10, $P_1 \bigcap P_2$ is a graded ideal of R. Let $a, b \in h(R)$ such that $ab \in P_1 \bigcap P_2$ and $Ann(a) = \{0\}$. Then $ab \in P_1$. Since P_1 is a graded r-ideal, $b \in P_1$. Similarly, $b \in P_2$ and hence $b \in P_1 \bigcap P_2$. Therefore, $P_1 \bigcap P_2$ is a graded r-ideal of R. $\square$

Theorem 2.12. *Let R be a G-graded ring and P_1, P_2 be graded prime ideals of R which are not comparable. If $P_1 \bigcap P_2$ is a graded r-ideal of R, then P_1 and P_2 are graded r-ideals of R.*

Proof. Let $a, b \in h(R)$ such that $ab \in P_1$ and $Ann(a) = \{0\}$. Suppose that $y \in P_2 - P_1$. Then $aby \in P_1 \bigcap P_2$. Since $P_1 \bigcap P_2$ is graded r-ideal, $by \in P_1 \bigcap P_2$ and then $by \in P_1$. Since P_1 is graded prime and $y \notin P_1$, $b \in P_1$. Hence, P_1 is a graded r-ideal of R. Similarly, P_2 is a graded r-ideal of R. $\square$

If P is a graded ideal of a G-graded ring R, then $\sqrt{P}$ need not to be a graded ideal of R; see ([4], Exercises 17 and 13 on pp. 127-128). We introduce the following.

Lemma 2.13. *If P is a graded ideal of a $\mathbb{Z}$-graded ring R, then $\sqrt{P}$ is a graded ideal of R.*

Proof. Clearly, $\sqrt{P}$ is an ideal of R. Let $x \in \sqrt{P}$ and write $x = \sum_{i=1}^{t} x_i$ where $x_i \in R_{n_i}$ and $n_1 < n_2 < ...... < n_t$. Then $x^k \in P$ for some positive integer k. Of course, $x^k = x_1^k +$ (higher terms) and as P is graded, we should have that $x_1^k \in P$. Thus, $x_1 \in \sqrt{P}$ which implies that $x - x_1 \in \sqrt{P}$. Now, induct on the number of homogeneous components to conclude that $x_i \in \sqrt{P}$ for all $1 \leq i \leq t$. Hence, $\sqrt{P}$ is a graded ideal of R. □

Theorem 2.14. *Let R be a $\mathbb{Z}$-graded ring and P be a graded ideal of R. Then P is a graded pr-ideal of R if and only if $\sqrt{P}$ is a graded r-ideal of R.*

Proof. Suppose that P is a graded pr-ideal of R. By Lemma 2.13, $\sqrt{P}$ is a graded ideal of R. Let $a, b \in h(R)$ such that $ab \in \sqrt{P}$ and $Ann(a) = \{0\}$. Then $a^n b^n = (ab)^n \in P$ for some $n \in \mathbf{N}$. Since $a, b \in h(R)$, there exist $g, h \in G$ such that $a \in R_g$ and $b \in R_h$ and then $a^n \in R_{g^n}$ and $b^n \in R_{h^n}$ which implies that $a^n, b^n \in h(R)$ such that $a^n b^n \in P$. Clearly, $Ann(a^n) = \{0\}$ and since P is a graded pr-ideal, $b^{nm} = (b^n)^m \in P$ for some $m \in \mathbf{N}$ which implies that $b \in \sqrt{P}$. Hence, $\sqrt{P}$ is a graded r-ideal of R. Conversely, let $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Then $ab \in \sqrt{P}$ and since $\sqrt{P}$ is a graded r-ideal, $b \in \sqrt{P}$ which implies that $b^n \in P$ for some $n \in \mathbf{N}$. Hence, P is a graded pr-ideal of R. □

Using Theorem 2.14 and Theorem 2.2, we have the next corollary.

Corollary 2.15. *Let R be a $\mathbb{Z}$-graded ring and P be a graded ideal of R. Then P is a graded pr-ideal if and only if $a\sqrt{P} = aR \bigcap \sqrt{P}$ for every $a \in h(R)$ with $Ann(a) = \{0\}$.*

Using Theorem 2.14 and Theorem 2.4, we have the next corollary.

Corollary 2.16. *Let R be a $\mathbb{Z}$-graded ring and P be a graded ideal of R. Then P is graded pr-ideal if and only if $\sqrt{P} = (\sqrt{P} : a)$ for all $a \in h(R)$ with $Ann(a) = \{0\}$.*

Theorem 2.17. *If P is a graded r-ideal of a G-graded ring R, then $(P : a)$ is a graded r-ideal of R for all $a \in h(R) - P$.*

Proof. Let $a \in h(R) - P$. Clearly, $(P : a)$ is an ideal of R. Let $x \in (P : a)$. Then $x \in R$ such that $xa \in P$. Since R is graded, $x = \sum_{g \in G} x_g$ where $x_g \in R_g$. Since $a \in h(R)$, $a \in R_h$ for some $h \in G$ and then $x_g a \in R_g R_h \subseteq R_{gh}$, i.e., $x_g a \in h(R)$ for all $g \in G$. Now, $xa = \sum_{g \in G} x_g a \in P$. Since P is a graded, $x_g a \in P$ for all $g \in G$, i.e., $x_g \in (P : a)$ for all $g \in G$. Hence, $(P : a)$ is a graded ideal of R.

Let $b, c \in h(R)$ such that $bc \in (P : a)$ and $Ann(b) = \{0\}$. Then $bca \in P$. Since P is a graded r-ideal, $ca \in P$ which implies that $c \in (P : a)$. Therefore, $(P : a)$ is a graded r-ideal of R. □

Theorem 2.18. *Every graded maximal r-ideal of a graded ring R is graded prime.*

Proof. Let P be a graded maximal r-ideal of R. Suppose that $a, b \in h(R)$ such that $ab \in P$ and $a \notin P$. Then by Theorem 2.17, $(P : a)$ is a graded r-ideal of R. Clearly, $P \subseteq (P : a)$ and $b \in (P : a)$. By maximality of P, $P = (P : a)$ and then $b \in P$. Hence, P is a graded prime ideal of R. □

Definition 2.19. A graded ring R is said to be an huz-ring if every homogeneous element of R is either a zero divisor or a unit.

The next theorem gives an example on huz-rings.

Theorem 2.20. *Every graded finite ring is an huz-ring.*

Proof. Let R be a G-graded finite ring. Assume that $a \in h(R)$. Then $a \in R_g$ for some $g \in G$. Define $\phi : R_{g^{-1}} \to R_e$ by $\phi(x) = ax$. If ϕ is injective, then since R is finite, ϕ is surjective and as $1 \in R_e$, $1 = ax$ for some $x \in R_{g^{-1}}$ and then a is a unit. Suppose that ϕ is not injective. Then there exist $x, y \in R_{g^{-1}}$ with $x \neq y$ such that $ax = ay$. But then $a(x - y) = 0$ and $x - y \neq 0$, so a is a zero divisor. □

If we drop the finite condition in Theorem 2.20, then the result is not true in general. See the following example.

EXAMPLE 2.21. Let $G = \mathbb{Z}$. Then clearly, the semigroup ring $R[X; \mathbb{Z}]$ is a $\mathbb{Z}$-graded ring. If R is a field, then $R[X; \mathbb{Z}]$ is a huz-ring; and if $R = \mathbb{Z}$, then $R[X; \mathbb{Z}]$ is not a huz-ring.

Finally, we prove that a graded ring R is an huz-ring if and only if every proper graded ideal of R is a graded r-ideal.

Theorem 2.22. *A graded ring R is a huz-ring if and only if every proper graded ideal of R is a graded r-ideal.*

Proof. Suppose that R is an huz-ring. Let P be a proper graded ideal of R. Assume that $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Since $Ann(a) = \{0\}$, a is not zero divisor and since R is huz, a is a unit and then $b = a^{-1}(ab) \in P$. Hence, P is a graded r-ideal of R. Conversely, let $a \in h(R)$ such that a is not a zero divisor. Then $Ann(a) = \{0\}$. Suppose that $P = \langle a \rangle$. If P is proper, then P is a graded r-ideal of R by assumption. Let $b \in h(R)$. Then $ab \in P$ and then $b \in P$ since P is a graded r-ideal. So, $h(R) \subseteq P$. Since $1 \in R_e \subseteq h(R)$, $1 \in P$ which is a contradiction. So, $P = R$, then $1 \in P$ and then $1 = xa$ for some $x \in R$ which implies that a is a unit and hence R is an huz-ring. □

Acknowledgments

The authors extend their thanks and gratitude to the referees for their efforts in reviewing the article.

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