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On the 2-Adjointable Operators and Superstability of them between 2-Pre Hilbert C^* -Module Spaces

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ABSTRACT. In this paper, first, we introduce the new concept of 2-inner product on Banach modules over a C^* -algebra. Next, we present the concept of 2-linear operators over a C^* -algebra. Our result improve the main result of the paper [Z. Lewandowska, On 2-normed sets, Glasnik Mat., 38(58) (2003), 99-110]. In the end of this paper, we define the notions 2-adjointable mappings between 2-pre Hilbert C*-modules and prove superstability of them in the spirit of Hyers-Ulam-Rassias.

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1. Introduction

The concept of 2-inner product has been intensively studied by many authors in the last three decades. The basic definitions and elementary properties of 2-inner product spaces can be found in [1] and [2].

Recently, M.Frank and e.t. defined the notion ϕ -perturbation of an adjointable mapping and proved the superstability of an adjointable mapping on Hilbert C^* -modules(see [3]).

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In this paper, first, we introduce the definition 2-pre Hibert C^* -module spaces and give several important properties. Next, we present the concept of 2-linear operators over a C^* -algebra which coincides with Lewandowska's definition (see [4, 5]). Also, we define 2-adjoinable mappings between 2-pre Hilbert C^* -modules and prove an analogue of ϕ -perturbation of adjoitable mappings in paper([3]).

We refer the interested reader to monographs [6, 7, 8, 9] and references therein for more information.

2. 2-Pre Hilbert Modules

Let X be a left module over a C^* -algebra A. An action of $a \in A$ on X is denoted by $a.x \in X$, $x \in X$.

Definition 2.1. A 2-pre Hilbert A-module is a left A-module X equipped with A-valued function defined on $X \times X \times X$ satisfing the following conditions:

- I_1) (x, x|z) is a positive element in A for any $x, z \in X$ and (x, x|z) = 0 if and only if x and z are linearly dependent;
 - I_2) (x, x|z) = (z, z|x) for any $x, z \in X$;
 - I_3) $(y, x|z) = (x, y|z)^*$ for any $x, y, z \in X$;
 - I_4) $(\alpha x + x', y|z) = \alpha(x, y|z) + (x', y|z)$ for any $\alpha \in \mathbb{C}$ and $x, x', y, z \in X$;
 - I_5) (ax, y|z) = a(x, y|z) for any $x, y, z \in X$ and any $a \in A$.

The map (.,.|.) is called A-valued 2-inner product and (X,(.,.|.)) is called 2-pre Hilbert C^* -module space.

EXAMPLE 2.2. Every 2-inner product space is a 2-pre Hilbert C-module.

EXAMPLE 2.3. Let A be a C^* -algebra and $J \subseteq A$ be a left ideal. Then J can be equipped with the structure of 2-pre Hilbert A-module with A-valued inner product $(x,y|z) := xy^*zz^* - xz^*zy^*$ for any $x,y,z \in A$.

Definition 2.4. Let X be a 2-pre Hilbert A-module. we can define a function $||.|.||_X$ on $X \times X$ by $||x|z||_X = ||(x,x|z)||^{\frac{1}{2}}$ for all $x,z \in X$.

Lemma 2.5. $||.|.||_X$ satisfies the following conditions:

- *N*1) $||ax|z||_X \le ||a|| \ ||x|z||_X$ for any $x, z \in X$ and $a \in A$;
- $N2) \ (x,y|z) \ (y,x|z) \leq \|y|z\|_X^2 \ (x,x|z) \ for \ any \ x,y,z \in X;$
- $N3) ||(x,y|z)||^2 \le ||(x,x|z)|| ||(y,y|z)||$

Proof. N1 is obvious; N3 follows from N2, so let us prove N2.

Let ϕ be a positive linear functional on A. Then $\phi((.,.|.))$ is usual 2-inner product on X. Applying the Schwartz inequality for 2-inner product (see [2],

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page 3) we obtain for all $x, y, z \in X$,

$$\begin{split} \phi((x,y|z)\ (y,x|z)) &= \phi((x,y|z)y,x\ |z)) \\ &\leq \phi((x,x|z))^{\frac{1}{2}}\ \phi(((x,y|z)y\ ,(x,y|z)y\ |z))^{\frac{1}{2}} \\ &\leq \phi((x,x|z))^{\frac{1}{2}}\ \phi((x,y|z)\ (y,y|z)\ (x,y|z)^*)^{\frac{1}{2}} \\ &\leq \phi((x,x|z))^{\frac{1}{2}}\ ||(y,y|z)||^{\frac{1}{2}}\ \phi((x,y|z)\ (y,x|z))^{\frac{1}{2}}. \end{split}$$

Thus, for any positive linear functional ϕ , we have

$$\phi((x,y|z) \ (y,x|z)) \le ||y|z||_X^2 \ \phi((x,x|z))$$

hence

$$(x,y|z) (y,x|z) \le ||y|z||_X^2 (x,x|z).$$

Theorem 2.6. The function $||.|.||_X$ is a 2-norm on X.

Proof. Now, we verify that $||.|.||_X$ satisfies the following properties of 2-norms:

- 1) I_3 and I_4 show that $||\alpha x|y||_X = ||(\alpha x, \alpha x|y)||^{\frac{1}{2}} = |\alpha| ||x|y||_X$ for all $x, y \in X$ and $\alpha \in \mathbb{C}$.
- 2) I_1 follows that $||x|y||_X = 0$ if and only if x and y are linearly dependent for all $x, y \in X$.
 - 3) it follows from I_2 that $||x|y||_X = ||(x, x|y)||^{\frac{1}{2}} = ||y|x||_X$ for all $x, y \in X$.
 - 4) By proposition 2.5 (N3), we have

$$\begin{aligned} ||x+x'|y||_X^2 &= ||(x+x',x+x'|y)|| = ||(x,x|y) + (x',x|y + (x,x'|y) + (x',x'|y))|| \\ &\leq ||(x,x|y)|| + 2||(x,x'|y)|| + ||(x',x'|y)|| \\ &\leq (||(x,x|y)||^{\frac{1}{2}} + ||(x',x'|y)||^{\frac{1}{2}})^2 = (||x|y||_X + ||x'|y||_X)^2 \end{aligned}$$

for all $x, x', y \in X$. This show that $(X, ||.||_X)$ is a 2-normed space.

3. 2-adjointable mappings

In continue, we let A be a C^* -algebra. Now, we start with following definition. **Definition 3.1.** Let X and Y be two 2-pre Hilbert A-modules. An operator $f: X \times X \to Y$ is said to be A-2 linear if it satisfies the following conditions:

- 1) f(x+y,z+w) = f(x,z) + f(x,w) + f(y,z) + f(y,w) for all $x,y,z,w \in X$;
- 2) $f(\alpha x, \beta y) = \alpha \overline{\beta} f(x, y)$ for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in X$;
- 3) f(ax, by) = a. $b^* f(x, y)$ for all $x, y \in X$ and $a, b \in A$.

EXAMPLE 3.2. Let X be a 2-pre Hilbert A-module and $z \in X$. Define $f: X \times X \to A$ by f(x,y) = (x,y|z). Then f is a A- 2 linear operator.

Definition 3.3. Let X and Y be two 2-pre Hilbert A-modules. A mapping $f: X \times X \to Y$ is called 2-adjointable if there exists a mapping $g: Y \times Y \to X$ such that

$$(f(x,y), s \mid t) = (x, y \mid g(s,t))$$
 (3.1)

for all $x, y \in X$ and $s, t \in Y$. The mapping g is denoted by f^* and is called the 2-adjointable of f.

Lemma 3.4. Let X be a 2-pre Hilbert A-module and $\dim(X) > 1$. If (x, y|z) = 0 for all $y, z \in X$, then x = 0.

Proof. Suppose $x \neq 0$. Let x and y be linearly independent. Then by hypothesis (x, x, |y|) = 0 and this is contradiction.

Lemma 3.5. Every 2-adjonable mapping is A- 2 linear.

Proof. Let $f: X \times X \to Y$ be a 2-adjoinable mapping. Then there exists a mapping $g: Y \times Y \to X$ such that (3.1) holds. For every $x, y, z, w \in X$, every $s, t \in Y$, every $\alpha, \beta \in \mathbb{C}$, every $a, b \in A$, we have

$$(f(\alpha ax + y, \beta bz + w), s \mid t) = (\alpha ax + y, \beta bz + w \mid g(s, t))$$

$$= \alpha \overline{\beta} ab^* (x, z \mid g(s, t)) + \alpha a (x, w \mid g(s, t)) + \overline{\beta} b^* (y, z \mid g(s, t)) + (y, w \mid g(s, t))$$

$$= \alpha \overline{\beta} ab^* (f(x, z), s \mid t) + \alpha a (f(x, w), s \mid t) + \overline{\beta} b^* (f(y, z), s \mid t) + (f(y, w), s \mid t)$$

$$= (\alpha \overline{\beta} ab^* f(x, z) + \alpha a f(x, w) + \overline{\beta} b^* f(y, z) + f(y, w), s \mid t).$$

It follows from lemma 3.4 that f is A-2 linear.

4. Superstability of 2-adjoinable mappings

In this section, X and Y denote 2-pre Hilbert A-modules and $\dim(X) > 1$, $\dim(Y) > 1$ and $\phi: X^2 \times Y^2 \to [0, \infty)$ is a function. We start our work with following definition.

Definition 4.1. A (not necessarily A-2 linear) mapping $f: X \times X \to Y$ is called

 ϕ -perturbation of an 2-adjointable mapping if there exists a mapping (not necessarily A-2- linear) $g:Y\times Y\to X$ such that

$$||(f(x,y),s \mid t) - (x,y \mid g(s,t))|| \le \phi(x,y,s,t)$$
(4.1)

for all $x, y \in X$ and $s, t \in Y$.

Theorem 4.2. Let $f: X \times X \to Y$ be a ϕ -perturbation of a 2-adjointable mapping with corresponding mapping $g: Y \times Y \to X$. Suppose for some sequence c_n of non-zero

complex numbers the following conditions hold:

$$\lim_{n \to \infty} |c_n|^{-1} \phi(c_n x, y, s, t) = 0 \quad (x, y \in X, s, t \in Y)$$
 (4.2)

$$\lim_{n \to \infty} |c_n|^{-1} \phi(x, y, c_n s, t) = 0 \quad (x, y \in X, s, t \in Y)$$
(4.3)

Then f is 2-adjointable and hence f is A-2 linear.

Proof. Let $\lambda \in \mathbb{C}$ be an arbitrary number. Putting λx instead x in (4.1), we get

$$||(f(\lambda x, y), s \mid t) - (\lambda x, y \mid g(s, t))|| \le \phi(\lambda x, y, s, t)$$

multiplication of (4.1) by $|\lambda|$, we have

$$\|(\lambda f(x,y), s \mid t) - \lambda(x,y \mid g(s,t))\| \le |\lambda|\phi(x,y,s,t)$$

Thus,

$$||(f(\lambda x, y), s \mid t) - (\lambda f(x, y), s \mid t)|| \le \phi(\lambda x, y, s, t) + |\lambda|\phi(x, y, s, t)$$
(4.4)

Replacing $c_n s$ by s in (4.4), we get

$$||f(\lambda x, y), s | t) - (\lambda f(x, y), s | t)|| \le |c_n|^{-1} \phi(\lambda x, y, c_n s, t) + |\lambda| |c_n|^{-1} \phi(x, y, c_n s, t)$$

hence, as $n \to \infty$, applying (4.3) we obtain

$$(f(\lambda x, y), s \mid t) - (\lambda f(x, y), s \mid t) = 0 \quad (\lambda \in \mathbb{C}, x, y \in X, s, t \in Y).$$

It follows from proposition 3.4 that

$$f(\lambda x, y) = \lambda f(x, y) \quad (\lambda \in \mathbb{C}, x, y \in X)$$
(4.5)

Now, we take $c_n x$ instead x in (4.1) to get

$$||(f(c_n x, y), s \mid t) - (c_n x, y \mid g(t, s))|| \le \phi(c_n x, y, s, t).$$

It follows from (4.5) that

$$||(f(x,y),s \mid t) - (x,y \mid g(s,t))|| \le |c_n|^{-1}\phi(c_nx,y,s,t)$$

hence, as $n \to \infty$, applying (4.2) we get

$$(f(x,y), s \mid t) = (x, y \mid g(s,t) \quad (x, y \in X, s, t \in Y).$$

Therefore f is 2-adjointable and by Lemma 3.5, f is A-2 linear.

In the following, we let $c_n = a^n$ that a > 1. we get the following results.

Corollary 4.3. If $f: X \times X \to Y$ is a ϕ -perturbation of a 2-adjointable mapping, where

 $\phi(x,y,s,t) = \epsilon \|x|y\|_X^p \|s|t\|_Y^q$ ($\epsilon \geq 0$, 0 , <math>0 < q < 1), then f is 2-adjointable and hence f is A-2-linear.

Corollary 4.4. If $f: X \times X \to Y$ is a ϕ -perturbation of a 2-adjointable mapping, where

 $\phi(x, y, s, t) = \epsilon_1 \|x\|y\|_X^p + \epsilon_2 \|s\|t\|_Y^q \ (\epsilon_1 \ge 0, \ \epsilon_2 \ge 0, \ 0 Then <math>f$ is 2-adjointable and hence f is A-2 linear.

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References

- Y.J. Cho, P.C.S. Lin, S.S. Kim, A. Misiak, Theory of 2-Inner Product Spaces, Nova Science Publishers, Inc., New York, 2001.
- 2. S.S. Dragomir, Y.J. Cho, S.S. Kim, A. Sofo, Some Boas-Bellman Type Inequalitys in 2-Inner Product Space, *JIPAM*, **6**(2),(2005), article 55.
- 3. M. Frank, P. Găvruta, M.S. Moslehian, Superstability of Adjointable Mappings on Hilbert C*- Modules, Appl. Anal. Discrete Math., No3, (2009), 39-45.
- 4. Z. Lewandowska, On 2-Normed Sets, Glasnik Mat., 38(58), (2003), 99-110.
- Z. Lewandowska, Bounded 2-Linear Operators on 2-Normed Sets, Glas. Mat. Ser. III, 39(59)(2), (2004), 301-312.
- A, Ashyani, H, Mohammadinejad, O, RabieiMotlagh, Stability Analysis of Mathematical Model of Virus Therapy for Cancer, *Iranian Journal of Mathematical Sciences and Informatics*, 11(2), (2016), 97-110.
- H, Sadeghi, Generalized Approximate Amenability of Direct Sum of Banach Algebras, Iranian Journal of Mathematical Sciences and Informatics, 13(1), (2018), 75-87.
- M.E. Gordji, M. Ramezani, Approximate inner products on Hilbert C*-modules; A Fixed Point Approach, Operators and Matrices, 6(4), (2012), 757-766.
- M.E. Gordji, M. Ramezani, Y.J. Cho, H. Baghani, Approximate Lie brackets: A Fixed Point Approach, Journal of Inequalities and Applications, (2012), doi:10.1186/1029-242X-2012-125.