Iranian Journal of Mathematical Sciences and Informatics
Vol. 14, No. 1 (2019), pp 121-126
DOI: 10.7508/ijmsi.2019.01.010

On the 2-Adjointable Operators and Superstability of them between 2-Pre Hilbert C^*-Module Spaces

Maryam Ramezani[*,a], Hamid Baghani[b]

[a]Department of Mathematics, University of Bojnord, Bojnord, Iran.
[b]Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran.

E-mail: m.ramezani@ub.ac.ir
E-mail: h.baghani@gmail.com

Abstract. In this paper, first, we introduce the new concept of 2-inner product on Banach modules over a C^*-algebra. Next, we present the concept of 2-linear operators over a C^*-algebra. Our result improve the main result of the paper [Z. Lewandowska, *On 2-normed sets*, Glasnik Mat., **38(58)** (2003), 99-110]. In the end of this paper, we define the notions 2-adjointable mappings between 2-pre Hilbert C*-modules and prove superstability of them in the spirit of Hyers-Ulam-Rassias.

Keywords: C^*-Algebra , 2-Adjointable mapping, Superstability.

2000 Mathematics subject classification: 46L08, 46L09.

1. Introduction

The concept of 2-inner product has been intensively studied by many authors in the last three decades. The basic definitions and elementary properties of 2-inner product spaces can be found in [1] and [2].

Recently, M.Frank and e.t. defined the notion ϕ-perturbation of an adjointable mapping and proved the superstability of an adjointable mapping on Hilbert C^*-modules(see [3]).

[*]Corresponding Author

Received 13 August 2016; Accepted 24 September 2017

In this paper, first, we introduce the definition 2-pre Hibert C^*-module spaces and give several important properties. Next, we present the concept of 2-linear operators over a C^*-algebra which coincides with Lewandowska's definition (see [4, 5]). Also, we define 2-adjoinable mappings between 2-pre Hilbert C^*-modules and prove an analogue of ϕ-perturbation of adjoitable mappings in paper([3]).

We refer the interested reader to monographs [6, 7, 8, 9] and references therein for more information.

2. 2-Pre Hilbert Modules

Let X be a left module over a C^*-algebra A. An action of $a \in A$ on X is denoted by $a.x \in X$, $x \in X$.

Definition 2.1. A 2-pre Hilbert A-module is a left A-module X equipped with A-valued function defined on $X \times X \times X$ satisfing the following conditions:
I_1) $(x, x|z)$ is a positive element in A for any $x, z \in X$ and $(x, x|z) = 0$ if and only if x and z are linearly dependent;
I_2) $(x, x|z) = (z, z|x)$ for any $x, z \in X$;
I_3) $(y, x|z) = (x, y|z)^*$ for any $x, y, z \in X$;
I_4) $(\alpha x + x', y|z) = \alpha(x, y|z) + (x', y|z)$ for any $\alpha \in \mathbb{C}$ and $x, x', y, z \in X$;
I_5) $(ax, y|z) = a(x, y|z)$ for any $x, y, z \in X$ and any $a \in A$.
The map $(.,.|.)$ is called A-valued 2-inner product and $(X,(.,.|.))$ is called 2-pre Hilbert C^*-module space.

Example 2.2. Every 2-inner product space is a 2-pre Hilbert $\mathbb{C}$-module.

Example 2.3. Let A be a C^*-algebra and $J \subseteq A$ be a left ideal. Then J can be equipped with the structure of 2-pre Hilbert A-module with A-valued inner product $(x, y|z) := xy^*zz^* - xz^*zy^*$ for any $x, y, z \in A$.

Definition 2.4. Let X be a 2-pre Hilbert A-module. we can define a function $||.|.||_X$ on $X \times X$ by $||x|z||_X = ||(x, x|z)||^{\frac{1}{2}}$ for all $x, z \in X$.

Lemma 2.5. $||.|.||_X$ *satisfies the following conditions:*
$N1$) $||ax|z||_X \leq ||a||\ ||x|z||_X$ *for any* $x, z \in X$ *and* $a \in A$*;*
$N2$) $(x, y|z)\ (y, x|z) \leq \|y|z\|_X^2\ (x, x|z)$ *for any* $x, y, z \in X$*;*
$N3$) $||(x, y|z)||^2 \leq ||(x, x|z)||\ ||(y, y|z)||$

Proof. $N1$ is obvious; $N3$ follows from $N2$, so let us prove $N2$.
Let ϕ be a positive linear functional on A. Then $\phi((.,.|.))$ is usual 2-inner product on X. Applying the Schwartz inequality for 2-inner product (see [2],

page 3) we obtain for all $x, y, z \in X$,

$$\begin{aligned}\phi((x,y|z)\ (y,x|z)) &= \phi((x,y|z)y, x\ |z)) \\ &\le \phi((x,x|z))^{\frac{1}{2}}\ \phi(((x,y|z)y\ , (x,y|z)y\ |z))^{\frac{1}{2}} \\ &\le \phi((x,x|z))^{\frac{1}{2}}\ \phi((x,y|z)\ (y,y|z)\ (x,y|z)^*)^{\frac{1}{2}} \\ &\le \phi((x,x|z))^{\frac{1}{2}}\ ||(y,y|z)||^{\frac{1}{2}}\ \phi((x,y|z)\ (y,x|z))^{\frac{1}{2}}.\end{aligned}$$

Thus, for any positive linear functional ϕ, we have

$$\phi((x,y|z)\ (y,x|z)) \le ||y|z||_X^2\ \phi((x,x|z))$$

hence

$$(x,y|z)\ (y,x|z) \le ||y|z||_X^2\ (x,x|z).$$

□

Theorem 2.6. *The function* $||.|.||_X$ *is a 2-norm on* X.

Proof. Now, we verify that $||.|.||_X$ satisfies the following properties of 2-norms:

1) I_3 and I_4 show that $||\alpha x|y||_X = ||(\alpha x, \alpha x|y)||^{\frac{1}{2}} = |\alpha|\ ||x|y||_X$ for all $x, y \in X$ and $\alpha \in \mathbb{C}$.

2) I_1 follows that $||x|y||_X = 0$ if and only if x and y are linearly dependent for all $x, y \in X$.

3) it follows from I_2 that $||x|y||_X = ||(x,x|y)||^{\frac{1}{2}} = ||y|x||_X$ for all $x, y \in X$.

4) By proposition 2.5 ($N3$), we have

$$\begin{aligned}||x+x'|y||_X^2 &= ||(x+x', x+x'|y)|| = ||(x,x|y) + (x',x|y + (x,x'|y) + (x',x'|y))|| \\ &\le ||(x,x|y)|| + 2||(x,x'|y)|| + ||(x',x'|y)|| \\ &\le (||(x,x|y)||^{\frac{1}{2}} + ||(x',x'|y)||^{\frac{1}{2}})^2 = (||x|y||_X + ||x'|y||_X)^2\end{aligned}$$

for all $x, x', y \in X$. This show that $(X, ||.|.||_X)$ is a 2-normed space. □

3. 2-adjointable mappings

In continue, we let A be a C^*-algebra. Now, we start with following definition.

Definition 3.1. Let X and Y be two 2-pre Hilbert A-modules. An operator $f : X \times X \to Y$ is said to be A-2 linear if it satisfies the following conditions:

1) $f(x+y, z+w) = f(x,z) + f(x,w) + f(y,z) + f(y,w)$ for all $x, y, z, w \in X$;

2) $f(\alpha x, \beta y) = \alpha\ \overline{\beta}\ f(x,y)$ for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in X$;

3) $f(ax, by) = a.\ b^*.f(x,y)$ for all $x, y \in X$ and $a, b \in A$.

Example 3.2. Let X be a 2-pre Hilbert A-module and $z \in X$. Define $f : X \times X \to A$ by $f(x,y) = (x,y|z)$. Then f is a A- 2 linear operator.

Definition 3.3. Let X and Y be two 2-pre Hilbert A-modules. A mapping $f : X \times X \to Y$ is called 2-adjointable if there exists a mapping $g : Y \times Y \to X$ such that

$$(f(x, y), s \mid t) = (x, y \mid g(s, t)) \tag{3.1}$$

for all $x, y \in X$ and $s, t \in Y$. The mapping g is denoted by f^* and is called the 2-adjointable of f.

Lemma 3.4. *Let X be a 2-pre Hilbert A-module and* $\dim(X) > 1$. *If* $(x, y|z) = 0$ *for all $y, z \in X$, then $x = 0$.*

Proof. Suppose $x \neq 0$. Let x and y be linearly independent. Then by hypothesis $(x, x, |y) = 0$ and this is contradiction. □

Lemma 3.5. *Every 2-adjonable mapping is A- 2 linear.*

Proof. Let $f : X \times X \to Y$ be a 2-adjoinable mapping. Then there exists a mapping $g : Y \times Y \to X$ such that (3.1) holds. For every $x, y, z, w \in X$, every $s, t \in Y$, every $\alpha, \beta \in \mathbb{C}$, every $a, b \in A$, we have

$$\begin{aligned}
&(f(\alpha ax + y, \beta bz + w), s \mid t) = (\alpha ax + y, \beta bz + w \mid g(s, t)) \\
&\quad = \alpha\overline{\beta}ab^* \ (x, z \mid g(s, t)) + \alpha a \ (x, w \mid g(s, t)) + \overline{\beta}b^* \ (y, z \mid g(s, t)) + (y, w \mid g(s, t)) \\
&\quad = \alpha\overline{\beta}ab^* \ (f(x, z), s \mid t) + \alpha a \ (f(x, w), s \mid t) + \overline{\beta}b^* \ (f(y, z), s \mid t) + (f(y, w), s \mid t) \\
&\quad = (\alpha\overline{\beta}ab^* \ f(x, z) + \alpha a \ f(x, w) + \overline{\beta}b^* \ f(y, z) + f(y, w), s \mid t).
\end{aligned}$$

It follows from lemma 3.4 that f is A-2 linear. □

4. Superstability of 2-adjoinable mappings

In this section, X and Y denote 2-pre Hilbert A-modules and $\dim(X) > 1$, $\dim(Y) > 1$ and $\phi : X^2 \times Y^2 \to [0, \infty)$ is a function. We start our work with following definition.

Definition 4.1. A (not necessarily A-2 linear) mapping $f : X \times X \to Y$ is called
ϕ-perturbation of an 2-adjointable mapping if there exists a mapping (not necessarily $A - 2$- linear) $g : Y \times Y \to X$ such that

$$\|(f(x, y), s \mid t) - (x, y \mid g(s, t))\| \leq \phi(x, y, s, t) \tag{4.1}$$

for all $x, y \in X$ and $s, t \in Y$.

Theorem 4.2. *Let $f : X \times X \to Y$ be a ϕ-perturbation of a 2-adjointable mapping with corresponding mapping $g : Y \times Y \to X$. Suppose for some sequence c_n of non-zero*
complex numbers the following conditions hold:

$$\lim_{n\to\infty} |c_n|^{-1}\phi(c_n x, y, s, t) = 0 \quad (x, y \in X, s, t \in Y) \tag{4.2}$$

$$\lim_{n\to\infty} |c_n|^{-1}\phi(x, y, c_n s, t) = 0 \quad (x, y \in X, s, t \in Y) \tag{4.3}$$

Then f is 2-adjointable and hence f is A-2 linear.

Proof. Let $\lambda \in \mathbb{C}$ be an arbitrary number. Putting λx instead x in (4.1), we get

$$\|(f(\lambda x, y), s \mid t) - (\lambda x, y \mid g(s, t))\| \leq \phi(\lambda x, y, s, t)$$

multiplication of (4.1) by $|\lambda|$,we have

$$\|(\lambda f(x, y), s \mid t) - \lambda(x, y \mid g(s, t))\| \leq |\lambda|\phi(x, y, s, t)$$

Thus,

$$\|(f(\lambda x, y), s \mid t) - (\lambda f(x, y), s \mid t)\| \leq \phi(\lambda x, y, s, t) + |\lambda|\phi(x, y, s, t) \tag{4.4}$$

Replacing $c_n s$ by s in (4.4), we get

$$\|f(\lambda x, y), s \mid t) - (\lambda f(x, y), s \mid t)\| \leq |c_n|^{-1}\phi(\lambda x, y, c_n s, t) + |\lambda|\ |c_n|^{-1}\phi(x, y, c_n s, t)$$

hence, as $n \to \infty$, applying (4.3) we obtain

$$(f(\lambda x, y), s \mid t) - (\lambda f(x, y), s \mid t) = 0 \quad (\lambda \in \mathbb{C}, x, y \in X, s, t \in Y).$$

It follows from proposition 3.4 that

$$f(\lambda x, y) = \lambda f(x, y) \quad (\lambda \in \mathbb{C}, x, y \in X) \tag{4.5}$$

Now, we take $c_n x$ instead x in (4.1) to get

$$\|(f(c_n x, y), s \mid t) - (c_n x, y \mid g(t, s))\| \leq \phi(c_n x, y, s, t).$$

It follows from (4.5) that

$$\|(f(x, y), s \mid t) - (x, y \mid g(s, t))\| \leq |c_n|^{-1}\phi(c_n x, y, s, t)$$

hence, as $n \to \infty$, applying (4.2) we get

$$(f(x, y), s \mid t) = (x, y \mid g(s, t) \quad (x, y \in X, s, t \in Y).$$

Therefore f is 2-adjointable and by Lemma 3.5, f is A-2 linear. □

In the following, we let $c_n = a^n$ that $a > 1$. we get the following results.

Corollary 4.3. *If $f : X \times X \to Y$ is a ϕ-perturbation of a 2-adjointable mapping, where*
$\phi(x, y, s, t) = \epsilon\ \|x|y\|_X^p\ \|s|t\|_Y^q$ $(\epsilon \geq 0,\ 0 < p < 1,\ 0 < q < 1)$*, then f is 2-adjointable and hence f is A-2-linear.*

Corollary 4.4. *If $f : X \times X \to Y$ is a ϕ-perturbation of a 2-adjointable mapping, where*
$\phi(x, y, s, t) = \epsilon_1\ \|x|y\|_X^p\ +\ \epsilon_2\ \|s|t\|_Y^q$ $(\epsilon_1 \geq 0,\ \epsilon_2 \geq 0,\ 0 < p < 1,\ 0 < q < 1)$.
Then f is 2-adjointable and hence f is A-2 linear.

Acknowledgments

We would like to thank the referee for his/her careful reading of the paper.

References

1. Y.J. Cho, P.C.S. Lin, S.S. Kim, A. Misiak, *Theory of* 2*-Inner Product Spaces*, Nova Science Publishers, Inc., New York, 2001.
2. S.S. Dragomir, Y.J. Cho, S.S. Kim, A. Sofo, Some Boas-Bellman Type Inequalitys in 2-Inner Product Space, *JIPAM*, **6**(2),(2005), article 55.
3. M. Frank, P. Găvruta, M.S. Moslehian, Superstability of Adjointable Mappings on Hilbert C^*- Modules, *Appl. Anal. Discrete Math.*, **No3**, (2009), 39-45.
4. Z. Lewandowska, On 2-Normed Sets, *Glasnik Mat.*, **38**(58), (2003), 99-110.
5. Z. Lewandowska, Bounded 2-Linear Operators on 2-Normed Sets, *Glas. Mat. Ser. III*, **39(59)**(2), (2004), 301-312.
6. A, Ashyani, H, Mohammadinejad, O, RabieiMotlagh, Stability Analysis of Mathematical Model of Virus Therapy for Cancer, *Iranian Journal of Mathematical Sciences and Informatics*, **11**(2), (2016), 97-110.
7. H, Sadeghi, Generalized Approximate Amenability of Direct Sum of Banach Algebras, *Iranian Journal of Mathematical Sciences and Informatics*, **13**(1), (2018), 75-87.
8. M.E. Gordji, M. Ramezani, Approximate inner products on Hilbert C*-modules; A Fixed Point Approach, *Operators and Matrices*, **6**(4), (2012), 757-766.
9. M.E. Gordji, M. Ramezani, Y.J. Cho, H. Baghani, Approximate Lie brackets: A Fixed Point Approach, *Journal of Inequalities and Applications*, (2012), doi:10.1186/1029-242X-2012-125.