

Renormalized Solutions of Strongly Nonlinear Elliptic Problems with Lower Order Terms and Measure Data in Orlicz-Sobolev Spaces

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ABSTRACT. The purpose of this paper is to prove the existence of a renormalized solution of perturbed elliptic problems

$$-\operatorname{div} \left(a(x, u, \nabla u) + \Phi(u) \right) + g(x, u, \nabla u) = f - \operatorname{div} F,$$

in a bounded open set Ω and $u = 0$ on $\partial\Omega$, in the framework of Orlicz-Sobolev spaces without any restriction on the M - N -function of the Orlicz spaces, where $-\operatorname{div} \left(a(x, u, \nabla u) \right)$ is a Leray-Lions operator defined from $W_0^1 L_M(\Omega)$ into its dual, $\Phi \in C^0(\mathbb{R}, \mathbb{R}^N)$. The function $g(x, u, \nabla u)$ is a non linear lower order term with natural growth with respect to $|\nabla u|$, satisfying the sign condition and the datum μ is assumed to belong to $L^1(\Omega) + W^{-1} E_{\overline{M}}(\Omega)$.

Keywords: Elliptic equation, Orlicz-Sobolev spaces, Renormalized solution.

2000 Mathematics subject classification: 35J15, 35J20.

1. INTRODUCTION

Let Ω be a bounded open set of \mathbb{R}^N , $N \geq 2$, and let M be an N -function. In the present paper we prove an existence result of a renormalized solution of the following strongly nonlinear elliptic problem

$$\begin{cases} A(u) - \operatorname{div} \Phi(u) + g(x, u, \nabla u) = f - \operatorname{div} F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

Received 30 July 2016; Accepted 12 August 2017

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Here, $\Phi \in C^0(\mathbb{R}, \mathbb{R}^N)$, while the function $g(x, u, \nabla u)$ is a non linear lower order term with natural growth with respect to $|\nabla u|$ and satisfying the sign condition. The non everywhere defined nonlinear operator $A(u) = -\operatorname{div}(a(x, u, \nabla u))$ acts from its domain $D(A) \subset W_0^1 L_M(\Omega)$ into $W^{-1} L_{\overline{M}}(\Omega)$. The function $a(x, u, \nabla u)$ is assumed to satisfy, among others, $a(x, u, \nabla u)$ nonstandard growth condition governed by the N -function M , and the source term $f \in L^1(\Omega)$ and $|F| \in E_{\overline{M}}(\Omega)$, \overline{M} stands for the conjugate of M .

We use here the notion of renormalized solutions, which was introduced by R.J. DiPerna and P.-L. Lions in their papers [16, 15] where the authors investigate the existence of solutions of the Boltzmann equation, by introducing the idea of renormalized solution. This concept of solution was then adapted to study (1.1) with $\Phi \equiv 0$, $g \equiv 0$ and $L^1(\Omega)$ -data by F. Murat in [29, 28], by G. Dal Maso et al. in [13] with general measure data and then when f is a bounded Radon measure datum and g grows at most like $|\nabla u|^{p-1}$ by Beta et al. in [9, 10, 11] with $\Phi \equiv 0$ and by Guibé and Mercaldo in [23, 24] when $\Phi(u)$ behaves at most like $|u|^{p-1}$. Renormalization idea was then used in [12] for variational equations and in [30] when the source term is in $L^1(\Omega)$. Recall that to get both existence and uniqueness of a solution to problems with L^1 -data, two notions of solution equivalent to the notion of renormalized solution were introduced, the first is the entropy solution by Bénilan et al. [4] and then the second is the SOLA by Dall'Aglia [14].

The authors in [5] have dealt with the equation (1.1) with $g = g(x, u)$ and $\mu \in W^{-1} E_{\overline{M}}(\Omega)$, under the restriction that the N -function M satisfies the Δ_2 -condition. This work was then extended in [2] for N -functions not satisfying necessarily the Δ_2 -condition. Our goal here is to extend the result in [2] solving the problem (1.1) without any restriction on the N -function M . Recently, a large number of papers was devoted to the existence of solutions of (1.1). In the variational framework, that is $\mu \in W^{-1} E_{\overline{M}}(\Omega)$, an existence result has been proved in [3], Specific examples to which our results apply include the following:

$$\begin{aligned} & -\operatorname{div}\left(|\nabla u|^{p-2} \nabla u + |u|^s u\right) + u|\nabla u|^p = \mu \text{ in } \Omega, \\ & -\operatorname{div}\left(|\nabla u|^{p-2} \nabla u \log^\beta(1 + |\nabla u|) + |u|^s u\right) = \mu \text{ in } \Omega, \\ & -\operatorname{div}\left(\frac{M(|\nabla u|) \nabla u}{|\nabla u|^2} + |u|^s u\right) + M(|\nabla u|) = \mu \text{ in } \Omega, \end{aligned}$$

where $p > 1$, $s > 0$, $\beta > 0$ and μ is a given Radon measure on Ω .

It is our purpose in this paper, to prove the existence of a renormalized solution for the problem (1.1) when the source term has the form $f - \operatorname{div} F$ with $f \in L^1(\Omega)$ and $|F| \in E_{\overline{M}}(\Omega)$, in the setting of Orlicz spaces without any restriction on the N -functions M . The approximate equations provide a $W_0^1 L_M(\Omega)$ bound for the corresponding solution u_n . This allows us to obtain

a function u as a limit of the sequence u_n . Hence, appear two difficulties. The first one is how to give a sense to $\Phi(u)$, the second difficulty lies in the need of the convergence almost everywhere of the gradients of u_n in Ω . This is done by using suitable test functions built upon u_n which make licit the use of the divergence theorem for Orlicz functions. We note that the techniques we used in the proof are different from those used in [2, 5, 12, 17, 25].

Let us briefly summarize the contents of the paper. The Section 2 is devoted to developing the necessary preliminaries, we introduce some technical lemmas. Section 3 contains the basic assumptions, the definition of renormalized solution and the main result, while the Section 4 is devoted to the proof of the main result.

2. PRELIMINARIES

Let $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an N -function, i. e., M is continuous, increasing, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \rightarrow 0$ as $t \rightarrow 0$, and $\frac{M(t)}{t} \rightarrow +\infty$ as $t \rightarrow +\infty$. Equivalently, M admits the representation:

$$M(t) = \int_0^t a(s) ds,$$

where $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is increasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t)$ tends to $+\infty$ as $t \rightarrow +\infty$.

The conjugate of M is also an N -function and it is defined by $\overline{M} = \int_0^t \bar{a}(s) ds$, where $\bar{a} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is the function $\bar{a}(t) = \sup\{s : a(s) \leq t\}$ (see [1]).

An N -function M is said to satisfy the Δ_2 -condition if, for some k ,

$$M(2t) \leq kM(t) \quad \forall t \geq 0, \quad (2.1)$$

when (2.1) holds only for $t \geq t_0 > 0$ then M is said to satisfy the Δ_2 -condition near infinity. Moreover, we have the following Young's inequality

$$st \leq M(t) + \overline{M}(s), \quad \forall s, t \geq 0.$$

Given two N -functions, we write $P \ll Q$ to indicate P grows essentially less rapidly than Q ; i. e. for each $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \rightarrow 0$ as $t \rightarrow +\infty$. This is the case if and only if

$$\lim_{t \rightarrow \infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0.$$

Let Ω be an open subset of \mathbb{R}^N . The Orlicz class $k_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that

$$\int_{\Omega} M(|u(x)|) dx < +\infty \quad (\text{resp. } \int_{\Omega} M\left(\frac{|u(x)|}{\lambda}\right) dx < +\infty \text{ for some } \lambda > 0).$$

The set $L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} M \left(\frac{|u(x)|}{\lambda} \right) dx \leq 1 \right\},$$

and $k_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_M(\Omega)$. The dual of $E_M(\Omega)$ can be identified with $L_{\bar{M}}(\Omega)$ by means of the pairing $\int_{\Omega} uv dx$, and the dual norm of $L_{\bar{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\bar{M},\Omega}$. We now turn to the Orlicz-Sobolev space, $W^1 L_M(\Omega)$ [resp. $W^1 E_M(\Omega)$] is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm

$$\|u\|_{1,M,\Omega} = \sum_{|\alpha| \leq 1} \|D^{\alpha} u\|_{M,\Omega}.$$

Thus, $W^1 L_M(\Omega)$ and $W^1 E_M(\Omega)$ can be identified with subspaces of product of $N + 1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\bar{M}})$ and $\sigma(\prod L_M, \prod L_{\bar{M}})$.

The space $W_0^1 E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1 E_M(\Omega)$ and the space $W_0^1 L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\bar{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1 L_M(\Omega)$. We say that u_n converges to u for the modular convergence in $W^1 L_M(\Omega)$ if for some $\lambda > 0$, $\int_{\Omega} M \left(\frac{D^{\alpha} u_n - D^{\alpha} u}{\lambda} \right) dx \rightarrow 0$ for all $|\alpha| \leq 1$. This implies convergence for $\sigma(\prod L_M, \prod L_{\bar{M}})$. If M satisfies the Δ_2 condition on \mathbb{R}^+ (near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence.

Let $W^{-1} L_{\bar{M}}(\Omega)$ [resp. $W^{-1} E_{\bar{M}}(\Omega)$] denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\bar{M}}(\Omega)$ [resp. $E_{\bar{M}}(\Omega)$]. It is a Banach space under the usual quotient norm (for more details see [1]).

A domain Ω has the segment property if for every $x \in \partial\Omega$ there exists an open set G_x and a nonzero vector y_x such that $x \in G_x$ and if $z \in \bar{\Omega} \cap G_x$, then $z + ty_x \in \Omega$ for all $0 < t < 1$. The following lemmas can be found in [6].

Lemma 2.1. *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let M be an N -function and let $u \in W^1 L_M(\Omega)$ (resp. $W^1 E_M(\Omega)$). Then $F(u) \in W^1 L_M(\Omega)$ (resp. $W^1 E_M(\Omega)$). Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial}{\partial x_i} u & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 2.2. *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. We suppose that the set of discontinuity points of F' is finite. Let M be an N -function, then the mapping $F : W^1 L_M(\Omega) \rightarrow W^1 L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\prod L_M, \prod E_{\overline{M}})$.*

Lemma 2.3. ([21]) *Let Ω have the segment property. Then for each $\nu \in W_0^1 L_M(\Omega)$, there exists a sequence $\nu_n \in \mathcal{D}(\Omega)$ such that ν_n converges to ν for the modular convergence in $W_0^1 L_M(\Omega)$. Furthermore, if $\nu \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$, then*

$$\|\nu_n\|_{L^\infty(\Omega)} \leq (N+1)\|\nu\|_{L^\infty(\Omega)}.$$

We give now the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [8]).

Lemma 2.4. *Let Ω be an open subset of \mathbb{R}^N with finite measure. Let M, P, Q be N -functions such that $Q \ll P$, and let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:*

$$|f(x, s)| \leq c(x) + k_1 P^{-1} M(k_2 |s|),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$.

Then the Nemytskii operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is strongly continuous from $\mathcal{P}(E_M(\Omega), \frac{1}{k_2}) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2}\}$ into $E_Q(\Omega)$.

We will also use the following technical lemma.

Lemma 2.5. ([26]) *If $\{f_n\} \subset L^1(\Omega)$ with $f_n \rightarrow f \in L^1(\Omega)$ a.e. in Ω , $f_n, f \geq 0$ a.e. in Ω and $\int_\Omega f_n(x) dx \rightarrow \int_\Omega f(x) dx$, then*

$$f_n \rightarrow f \text{ in } L^1(\Omega).$$

3. STRUCTURAL ASSUMPTIONS AND MAIN RESULT

Throughout the paper Ω will be a bounded subset of \mathbb{R}^N , $N \geq 2$, satisfying the segment property. Let M and P be two N -functions such that $P \ll M$. Let A be the non everywhere defined operator defined from its domain $\mathcal{D}(\Omega) \subset W_0^1 L_M(\Omega)$ into $W^{-1} L_{\overline{M}}(\Omega)$ given by

$$A(u) := -\operatorname{div} a(\cdot, u, \nabla u),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function. We assume that there exist a nonnegative function $c(x)$ in $E_{\overline{M}}(\Omega)$, $\alpha > 0$ and positive real constants k_1, k_2, k_3 and k_4 , such that for every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, $\xi' \in \mathbb{R}^N$ ($\xi \neq \xi'$) and for almost every $x \in \Omega$

$$|a(x, s, \xi)| \leq c(x) + k_1 \overline{P}^{-1} M(k_2 |s|) + k_3 \overline{M}^{-1} M(k_4 |\xi|), \quad (3.1)$$

$$(a(x, s, \xi) - a(x, s, \xi'))(\xi - \xi') > 0, \quad (3.2)$$

$$a(x, s, \xi)\xi \geq \alpha M(|\xi|). \quad (3.3)$$

Here, $g(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a Carathéodory function satisfying for almost every $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$,

$$|g(x, s, \xi)| \leq b(|s|)(d(x) + M(|\xi|)), \quad (3.4)$$

$$g(x, s, \xi)s \geq 0, \quad (3.5)$$

where $b : \mathbb{R} \rightarrow \mathbb{R}^+$ is a continuous and increasing function while d is a given nonnegative function in $L^1(\Omega)$.

The right-hand side of (1.1) and $\Phi : \mathbb{R} \rightarrow \mathbb{R}^N$, are assumed to satisfy

$$f \in L^1(\Omega) \text{ and } |F| \in E_{\overline{M}}(\Omega), \quad (3.6)$$

$$\Phi \in C^0(\mathbb{R}, \mathbb{R}^N). \quad (3.7)$$

Our aim in this paper is to give a meaning to a possible solution of (1.1). In view of assumptions (3.1), (3.2), (3.3) and (3.6), the natural space in which one can seek for a solution u of problem (1.1) is the Orlicz-Sobolev space $W_0^1 L_M(\Omega)$. But when u is only in $W_0^1 L_M(\Omega)$ there is no reason for $\Phi(u)$ to be in $(L^1(\Omega))^N$ since no growth hypothesis is assumed on the function Φ . Thus, the term $\operatorname{div}(\Phi(u))$ may be ill-defined even as a distribution. This hindrance is bypassed by solving some weaker problem obtained formally through a pointwise multiplication of equation (1.1) by $h(u)$ where h belongs to $C_c^1(\mathbb{R})$, the class of $C^1(\mathbb{R})$ functions with compact support.

Definition 3.1. A measurable function $u : \Omega \rightarrow \mathbb{R}$ is called a renormalized solution of (1.1) if $u \in W_0^1 L_M(\Omega)$, $a(x, u, \nabla u) \in (L_{\overline{M}}(\Omega))^N$, $g(x, u, \nabla u) \in L^1(\Omega)$, $g(x, u, \nabla u)u \in L^1(\Omega)$,

$$\lim_{m \rightarrow +\infty} \int_{\{x \in \Omega : m \leq |u(x)| \leq m+1\}} a(x, u, \nabla u) \nabla u \, dx = 0,$$

and

$$\begin{cases} -\operatorname{div} a(x, u, \nabla u)h(u) - \operatorname{div}(\Phi(u)h(u)) + h'(u)\Phi(u)\nabla u \\ + g(x, u, \nabla u)h(u) = fh(u) - \operatorname{div}(Fh(u)) + h'(u)F\nabla u \text{ in } \mathcal{D}'(\Omega), \end{cases} \quad (3.8)$$

for every $h \in C_c^1(\mathbb{R})$.

Remark 3.2. Every term in the problem (3.8) is meaningful in the distributional sense. Indeed, for h in $C_c^1(\mathbb{R})$ and u in $W_0^1 L_M(\Omega)$, $h(u)$ belongs to $W^1 L_M(\Omega)$ and for φ in $\mathcal{D}(\Omega)$ the function $\varphi h(u)$ belongs to $W_0^1 L_M(\Omega)$. Since $(-\operatorname{div} a(x, u, \nabla u)) \in W^{-1} L_{\overline{M}}(\Omega)$, we also have

$$\begin{aligned} \langle -\operatorname{div} a(x, u, \nabla u)h(u), \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} \\ = \langle -\operatorname{div} a(x, u, \nabla u), \varphi h(u) \rangle_{W^{-1} L_{\overline{M}}(\Omega), W_0^1 L_M(\Omega)} \\ \forall \varphi \in \mathcal{D}(\Omega). \end{aligned}$$

Finally, since Φh and $\Phi h' \in (C_c^0(\mathbb{R}))^N$, for any measurable function u we have $\Phi(u)h(u)$ and $\Phi(u)h'(u) \in (L^\infty(\Omega))^N$ and then $\operatorname{div}(\Phi(u)h(u)) \in W^{-1,\infty}(\Omega)$ and $\Phi(u)h'(u) \in L_M(\Omega)$.

Our main result is the following

Theorem 3.3. *Suppose that assumptions (3.1)–(3.7) are fulfilled. Then, problem (1.1) has at least one renormalized solution.*

Remark 3.4. The condition (3.4) can be replaced by the weaker one

$$|g(x, s, \xi)| \leq d(x) + b(|s|)M(|\xi|),$$

with $b: \mathbb{R} \rightarrow \mathbb{R}^+$ a continuous function belonging to $L^1(\mathbb{R})$ and $d(x) \in L^1(\Omega)$.

Actually the original equation (1.1) will be recovered whenever $h(u) \equiv 1$, but unfortunately this cannot happen in general strong additional requirements on u . Therefore, (3.8) is to be viewed as a weaker form of (1.1).

4. PROOF OF THE MAIN RESULT

From now on, we will use the standard truncation function T_k , $k > 0$, defined for all $s \in \mathbb{R}$ by $T_k(s) = \max\{-k, \min\{k, s\}\}$.

Step 1: Approximate problems. Let f_n be a sequence of $L^\infty(\Omega)$ functions that converge strongly to f in $L^1(\Omega)$. For $n \in \mathbb{N}$, $n \geq 1$, let us consider the following sequence of approximate equations

$$-\operatorname{div} a(x, u_n, \nabla u_n) + \operatorname{div} \Phi_n(u_n) + g_n(x, u_n, \nabla u_n) = f_n - \operatorname{div} F \text{ in } \mathcal{D}'(\Omega), \quad (4.1)$$

where we have set $\Phi_n(s) = \Phi(T_n(s))$ and $g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \frac{1}{n}|g(x, s, \xi)|}$. For fixed $n > 0$, it's obvious to observe that

$$g_n(x, s, \xi)s \geq 0, \quad |g_n(x, s, \xi)| \leq |g(x, s, \xi)| \text{ and } |g_n(x, s, \xi)| \leq n.$$

Moreover, since Φ is continuous one has $|\Phi_n(t)| \leq \max_{|t| \leq n} |\Phi(t)|$. Therefore, applying both Proposition 1, Proposition 5 and Remark 2 of [22] one can deduces that there exists at least one solution u_n of the approximate Dirichlet problem (4.1) in the sense

$$\left\{ \begin{array}{l} u_n \in W_0^1 L_M(\Omega), a(x, u_n, \nabla u_n) \in (L_{\overline{M}}(\Omega))^N \text{ and} \\ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla v dx + \int_{\Omega} \Phi_n(u_n) \nabla v dx \\ + \int_{\Omega} g_n(x, u_n, \nabla u_n) v dx = \langle f_n, v \rangle + \int_{\Omega} F \nabla v dx, \text{ for every } v \in W_0^1 L_M(\Omega). \end{array} \right. \quad (4.2)$$

Step 2: Estimation in $W_0^1 L_M(\Omega)$. Taking u_n as function test in problem (4.2), we obtain

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n dx + \int_{\Omega} \Phi_n(u_n) \nabla u_n dx \\ & + \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n dx = \langle f_n, u_n \rangle + \int_{\Omega} F \nabla u_n dx. \end{aligned} \quad (4.3)$$

Define $\tilde{\Phi}_n \in (C^1(\mathbb{R}))^N$ as $\tilde{\Phi}_n(t) = \int_0^t \Phi_n(\tau) d\tau$. Then formally

$\operatorname{div}(\tilde{\Phi}_n(u_n)) = \Phi_n(u_n) \nabla u_n$, $u_n = 0$ on $\partial\Omega$, $\tilde{\Phi}_n(0) = 0$ and by the Divergence theorem

$$\int_{\Omega} \Phi_n(u_n) \nabla u_n dx = \int_{\Omega} \operatorname{div}(\tilde{\Phi}_n(u_n)) dx = \int_{\partial\Omega} \tilde{\Phi}_n(u_n) \vec{n} ds = 0,$$

where \vec{n} is the outward pointing unit normal field of the boundary $\partial\Omega$ (ds may be used as a shorthand for $\vec{n} ds$). Thus, by virtue of (3.5) and Young's inequality, we get

$$\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n dx \leq C_1 + \frac{\alpha}{2} \int_{\Omega} M(|\nabla u_n|) dx, \quad (4.4)$$

which, together with (3.3) give

$$\int_{\Omega} M(|\nabla u_n|) dx \leq C_2. \quad (4.5)$$

Moreover, we also have

$$\int_{\Omega} g_n(x, u_n, \nabla u_n) u_n dx \leq C_3. \quad (4.6)$$

As a consequence of (4.5) there exist a subsequence of $\{u_n\}_n$, still indexed by n , and a function $u \in W_0^1 L_M(\Omega)$ such that

$$\begin{aligned} u_n & \rightharpoonup u \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega)), \\ u_n & \rightarrow u \text{ strongly in } E_M(\Omega) \text{ and a. e. in } \Omega. \end{aligned} \quad (4.7)$$

Step 3: Boundedness of $(a(x, u_n, \nabla u_n))_n$ in $(L_{\overline{M}}(\Omega))^N$. Let $w \in (E_M(\Omega))^N$ with $\|w\|_M \leq 1$. Thanks to (3.2), we can write

$$\left(a(x, u_n, \nabla u_n) - \left(a(x, u_n, \frac{w}{k_4}) \right) \right) \left(\nabla u_n - \frac{w}{k_4} \right) \geq 0,$$

which implies

$$\begin{aligned} \frac{1}{k_4} \int_{\Omega} a(x, u_n, \nabla u_n) w dx & \leq \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n dx \\ & + \int_{\Omega} a\left(x, u_n, \frac{w}{k_4}\right) \left(\frac{w}{k_4} - \nabla u_n \right) dx. \end{aligned}$$

Thanks to (4.4) and (4.5), one has

$$\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n dx \leq C_5.$$

Define $\lambda = 1 + k_1 + k_3$. By the growth condition (3.1) and Young's inequality, one can write

$$\begin{aligned} & \left| \int_{\Omega} a\left(x, u_n, \frac{w}{k_4}\right) \left(\frac{w}{k_4} - \nabla u_n\right) dx \right| \\ & \leq \left(1 + \frac{1}{k_4}\right) \left(\int_{\Omega} \overline{M}(c(x)) dx + k_1 \int_{\Omega} \overline{M} \overline{P}^{-1} M(k_2 |u_n|) dx \right. \\ & \quad \left. + k_3 \int_{\Omega} M(|w|) dx \right) + \frac{\lambda}{k_4} \int_{\Omega} M(|w|) dx + \lambda \int_{\Omega} M(|\nabla u_n|) dx. \end{aligned}$$

By virtue of [18] and Lemma 4.14 of [20], there exists an N -function Q such that $M \ll Q$ and the space $W_0^1 L_M(\Omega)$ is continuously embedded into $L_Q(\Omega)$. Thus, by (4.5) there exists a constant $c_0 > 0$, not depending on n , satisfying $\|u_n\|_Q \leq c_0$. Since $M \ll Q$, we can write $M(k_2 t) \leq Q(\frac{t}{c_0})$, for $t > 0$ large enough. As $P \ll M$, we can find a constant c_1 , not depending on n , such that $\int_{\Omega} \overline{M} \overline{P}^{-1} M(k_2 |u_n|) dx \leq \int_{\Omega} Q(\frac{|u_n|}{c_0}) + c_1$. Hence, we conclude that the quantity $\left| \int_{\Omega} a(x, u_n, \nabla u_n w) dx \right|$ is bounded from above for all $w \in (E_M(\Omega))^N$ with $\|w\|_M \leq 1$. Using the Orlicz norm we deduce that

$$\left(a(x, u_n, \nabla u_n) \right)_n \text{ is bounded in } (L_{\overline{M}}(\Omega))^N. \quad (4.8)$$

Step 4: Renormalization identity for the approximate solutions. For any $m \geq 1$, define $\theta_m(r) = T_{m+1}(r) - T_m(r)$. Observe that by [19, Lemma2] one has $\theta_m(u_n) \in W_0^1 L_M(\Omega)$. The use of $\theta_m(u_n)$ as test function in (4.2) yields

$$\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx \leq \langle f_n, \theta_m(u_n) \rangle + \int_{\{m \leq |u_n| \leq m+1\}} F \nabla u_n dx,$$

By Hölder's inequality and 4.5 we have

$$\begin{aligned} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx & \leq \langle f_n, \theta_m(u_n) \rangle \\ & + C_6 \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx. \end{aligned}$$

It's not hard to see that

$$\|\nabla \theta_m(u_n)\|_M \leq \|\nabla u_n\|_M.$$

So that by (4.5) and (4.7) one can deduce that

$$\theta_m(u_n) \rightharpoonup \theta_m(u) \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega)).$$

Note that as m goes to ∞ , $\theta_m(u) \rightharpoonup 0$ weakly in $W_0^1 L_M(\Omega)$ for $\sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega))$, and since f_n converges strongly in $L^1(\Omega)$, by Lebesgue's theorem we have

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \langle f_n, \theta_m(u_n) \rangle = 0.$$

By (3.3) we finally have

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx = 0. \quad (4.9)$$

Step 5: Almost everywhere convergence of the gradients. Define $\phi(s) = se^{\lambda s^2}$ with $\lambda = \left(\frac{b(k)}{2\alpha}\right)^2$. One can easily verify that for all $s \in \mathbb{R}$

$$\phi'(s) - \frac{b(k)}{\alpha} |\phi(s)| \geq \frac{1}{2}. \quad (4.10)$$

For $m \geq k$, we define the function ψ_m by

$$\begin{cases} \psi_m(s) = 1 & \text{if } |s| \leq m, \\ \psi_m(s) = m + 1 - |s| & \text{if } m \leq |s| \leq m + 1, \\ \psi_m(s) = 0 & \text{if } |s| \geq m + 1. \end{cases}$$

By virtue of [21, Theorem 4] there exists a sequence $\{v_j\}_j \subset D(\Omega)$ such that $v_j \rightarrow u$ in $W_0^1 L_M(\Omega)$ for the modular convergence and a.e. in Ω . Let us define the following functions $\theta_n^j = T_k(u_n) - T_k(v_j)$, $\theta^j = T_k(u) - T_k(v_j)$ and $z_{n,m}^j = \phi(\theta_n^j) \psi_m(u_n)$. Using $z_{n,m}^j \in W_0^1 L_M(\Omega)$ as test function in (4.2) we get

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx + \int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) dx \\ & + \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n \psi_m'(u_n) \phi(T_k(u_n) - T_k(v_j)) dx \\ & + \int_{\Omega} g_n(x, u_n, \nabla u_n) z_{n,m}^j dx = \int_{\Omega} f_n z_{n,m}^j dx + \int_{\Omega} F \nabla z_{n,m}^j dx. \end{aligned} \quad (4.11)$$

From now on we denote by $\epsilon_i(n, j)$, $i = 0, 1, 2, \dots$, various sequences of real numbers which tend to zero, when n and $j \rightarrow +\infty$, i. e.

$$\lim_{j \rightarrow +\infty} \lim_{n \rightarrow +\infty} \epsilon_i(n, j) = 0.$$

In view of (4.7), we have $z_{n,m}^j \rightharpoonup \phi(\theta^j) \psi_m(u)$ weakly in $L^\infty(\Omega)$ for $\sigma^*(L^\infty, L^1)$ as $n \rightarrow +\infty$, which yields

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f_n z_{n,m}^j dx = \int_{\Omega} f \phi(\theta^j) \psi_m(u) dx,$$

and since $\phi(\theta^j) \rightharpoonup 0$ weakly in $L^\infty(\Omega)$ for $\sigma(L^\infty, L^1)$ as $j \rightarrow +\infty$, we have

$$\lim_{j \rightarrow +\infty} \int_{\Omega} f \phi(\theta^j) \psi_m(u) dx = 0.$$

Thus, we write

$$\int_{\Omega} f_n z_{n,m}^j dx = \epsilon_0(n, j).$$

Thanks to (4.5) and (4.7), we have as $n \rightarrow +\infty$,

$$z_{n,m}^j \rightharpoonup \phi(\theta^j) \psi_m(u) \text{ in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega)),$$

which implies that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} F \nabla z_{n,m}^j dx = \int_{\Omega} F \nabla \theta^j \phi'(\theta^j) \psi_m(u) dx + \int_{\Omega} F \nabla u \phi(\theta^j) \psi'_m(u) dx$$

On the one hand, by Lebesgue's theorem we get

$$\lim_{j \rightarrow +\infty} \int_{\Omega} F \nabla u \phi(\theta^j) \psi'_m(u) dx = 0,$$

on the other hand, we write

$$\begin{aligned} \int_{\Omega} F \nabla \theta^j \phi'(\theta^j) \psi_m(u) dx &= \int_{\Omega} F \nabla T_k(u) \phi'(\theta^j) \psi_m(u) dx \\ &\quad - \int_{\Omega} F \nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) dx, \end{aligned}$$

so that, by Lebesgue's theorem one has

$$\lim_{j \rightarrow +\infty} \int_{\Omega} F \nabla T_k(u) \phi'(\theta^j) \psi_m(u) dx = \int_{\Omega} F \nabla T_k(u) \psi_m(u) dx.$$

Let $\lambda > 0$ such that $M\left(\frac{|\nabla v_j - \nabla u|}{\lambda}\right) \rightarrow 0$ strongly in $L^1(\Omega)$ as $j \rightarrow +\infty$ and $M\left(\frac{|\nabla u|}{\lambda}\right) \in L^1(\Omega)$, the convexity of the N -function M allows us to have

$$\begin{aligned} M\left(\frac{|\nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) - \nabla T_k(u) \psi_m(u)|}{4\lambda \phi'(2k)}\right) \\ = \frac{1}{4} M\left(\frac{|\nabla v_j - \nabla u|}{\lambda}\right) + \frac{1}{4} \left(1 + \frac{1}{\phi'(2k)}\right) M\left(\frac{|\nabla u|}{\lambda}\right). \end{aligned}$$

Then, by using the modular convergence of $\{\nabla v_j\}$ in $(L_M(\Omega))^N$ and Vitali's theorem, we obtain

$$\nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) \rightarrow \nabla T_k(u) \psi_m(u) \text{ in } (L_M(\Omega))^N, \text{ as } j \text{ tends to } +\infty,$$

for the modular convergence, and then

$$\lim_{j \rightarrow +\infty} \int_{\Omega} F \nabla T_k(u) \phi'(\theta^j) \psi_m(u) dx = \int_{\Omega} F \nabla T_k(u) \psi_m(u) dx.$$

We have proved that

$$\int_{\Omega} F \nabla z_{n,m}^j dx = \epsilon_1(n, j).$$

It's easy to see that by the modular convergence of the sequence $\{v_j\}_j$, one has

$$\lim_{j \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n \psi'_m(u_n) \phi(T_k(u_n) - T_k(v_j)) dx = 0,$$

while for the third term in the left-hand side of (4.11) we can write

$$\begin{aligned} &\int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) dx \\ &= \int_{\Omega} \Phi_n(u_n) \nabla T_k(u_n) \phi'(\theta_n^j) \psi_m(u_n) dx - \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx. \end{aligned}$$

Firstly, we have

$$\lim_{j \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(u_n) \phi'(\theta_n^j) \psi_m(u_n) dx = 0.$$

In view of (4.7), one has

$$\Phi_n(u_n) \phi'(\theta_n^j) \psi_m(u_n) \rightarrow \Phi(u) \phi'(\theta^j) \psi_m(u),$$

almost everywhere in Ω as n tends to $+\infty$. Furthermore, we can check that

$$\|\Phi_n(u_n) \phi'(\theta_n^j) \psi_m(u_n)\|_{\overline{M}} \leq \overline{M}(c_m \phi'(2k))|\Omega| + 1,$$

where $c_m = \max_{|t| \leq m+1} \Phi(t)$. Applying [27, Theorem 14.6] we get

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = \int_{\Omega} \Phi(u) \nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) dx.$$

Using the modular convergence of the sequence $\{v_j\}_j$, we obtain

$$\lim_{j \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = \int_{\Omega} \Phi(u) \nabla T_k(u) \psi_m(u) dx.$$

Then, using again the Divergence theorem we get

$$\int_{\Omega} \Phi(u) \nabla T_k(u) \psi_m(u) dx = 0.$$

Therefore, we write

$$\int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) dx = \epsilon_2(n, j).$$

Since $g_n(x, u_n, \nabla u_n) z_{n,m}^j \geq 0$ on the set $\{|u_n| > k\}$ and $\psi_m(u_n) = 1$ on the set $\{|u_n| \leq k\}$, from (4.11) we obtain

$$\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx + \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n^j) dx \leq \epsilon_3(n, j). \quad (4.12)$$

We now evaluate the first term of the left-hand side of (4.12) by writing

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx \\ &= \int_{\Omega} a(x, u_n, \nabla u_n) (\nabla T_k(u_n) - \nabla T_k(v_j)) \phi'(\theta_n^j) \psi_m(u_n) dx \\ & \quad + \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi'_m(u_n) dx \\ &= \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla T_k(v_j)) \phi'(\theta_n^j) dx \\ & \quad - \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx \\ & \quad + \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi'_m(u_n) dx, \end{aligned}$$

and then

$$\begin{aligned}
& \int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx \\
&= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s)) \\
&\quad (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^j) dx \\
&\quad + \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^j) dx \\
&\quad - \int_{\Omega \setminus \Omega_j^s} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) dx \\
&\quad - \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx \\
&\quad + \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi'_m(u_n) dx,
\end{aligned} \tag{4.13}$$

where by χ_j^s , $s > 0$, we denote the characteristic function of the subset

$$\Omega_j^s = \{x \in \Omega : |\nabla T_k(v_j)| \leq s\}.$$

For fixed m and s , we will pass to the limit in n and then in j in the second, third, fourth and fifth terms in the right side of (4.13). Starting with the second term, we have

$$\begin{aligned}
& \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^j) dx \\
& \rightarrow \int_{\Omega} a(x, T_k(u), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta^j) dx,
\end{aligned}$$

as $n \rightarrow +\infty$. Since by lemma (2.4) one has

$$a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^j) \rightarrow a(x, T_k(u), \nabla T_k(v_j) \chi_j^s) \phi'(\theta^j),$$

strongly in $(E_{\overline{M}}(\Omega))^N$ as $n \rightarrow \infty$, while by (4.5)

$$\nabla T_k(u_n) \rightharpoonup \nabla T_k(u),$$

weakly in $(L_M(\Omega))^N$. Let χ^s denote the characteristic function of the subset

$$\Omega^s = \{x \in \Omega : |\nabla T_k(u)| \leq s\}.$$

As $\nabla T_k(v_j) \chi_j^s \rightarrow \nabla T_k(u) \chi^s$ strongly in $(E_M(\Omega))^N$ as $j \rightarrow +\infty$, one has

$$\int_{\Omega} a(x, T_k(u), \nabla T_k(v_j) \chi_j^s) \cdot (\nabla T_k(u) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta^j) dx \rightarrow 0,$$

as $j \rightarrow \infty$. Then

$$\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^j) dx = \epsilon_4(n, j). \tag{4.14}$$

We now estimate the third term of (4.13). It's easy to see that by (3.3), $a(x, s, 0) = 0$ for almost everywhere $x \in \Omega$ and for all $s \in \mathbb{R}$. Thus, from (4.8) we have that $(a(x, T_k(u_n), \nabla T_k(u_n)))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$ for all $k \geq 0$.

Therefore, there exist a subsequence still indexed by n and a function l_k in $(L_{\overline{M}}(\Omega))^N$ such that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup l_k \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M). \quad (4.15)$$

Then, since $\nabla T_k(v_j)\chi_{\Omega \setminus \Omega_j^s} \in (E_{\overline{M}}(\Omega))^N$, we obtain

$$\int_{\Omega \setminus \Omega_j^s} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) dx \rightarrow \int_{\Omega \setminus \Omega_j^s} l_k \nabla T_k(v_j) \phi'(\theta^j) dx,$$

as $n \rightarrow +\infty$. The modular convergence of $\{v_j\}$ allows us to get

$$- \int_{\Omega \setminus \Omega_j^s} l_k \nabla T_k(v_j) \phi'(\theta^j) dx \rightarrow - \int_{\Omega \setminus \Omega^s} l_k \nabla T_k(u) dx,$$

as $j \rightarrow +\infty$. This, proves

$$- \int_{\Omega \setminus \Omega_j^s} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) dx = - \int_{\Omega \setminus \Omega^s} l_k \nabla T_k(u) dx + \epsilon_5(n, j). \quad (4.16)$$

As regards the fourth term, observe that $\psi_m(u_n) = 0$ on the subset $\{|u_n| \geq m+1\}$, so we have

$$\begin{aligned} & - \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = \\ & \quad - \int_{\{|u_n| > k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx. \end{aligned}$$

Since

$$\begin{aligned} & - \int_{\{|u_n| > k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = \\ & \quad - \int_{\{|u| > k\}} l_{m+1} \nabla T_k(u) \psi_m(u) dx + \epsilon_5(n, j), \end{aligned}$$

observing that $\nabla T_k(u) = 0$ on the subset $\{|u| > k\}$, one has

$$- \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = \epsilon_6(n, j). \quad (4.17)$$

For the last term of (4.13), we have

$$\begin{aligned} & \left| \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi'_m(u_n) dx \right| \\ & = \left| \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi'_m(u_n) dx \right| \\ & \leq \phi(2k) \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx. \end{aligned}$$

To estimate the last term of the previous inequality, we use $(T_1(u_n - T_m(u_n))) \in W_0^1 L_M(\Omega)$ as test function in (4.2), to get

$$\begin{aligned} & \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx + \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n dx \\ & + \int_{\{|u_n| \geq m\}} g_n(x, u_n, \nabla u_n) T_1(u_n - T_m(u_n)) dx = \langle f_n, T_1(u_n - T_m(u_n)) \rangle \\ & + \int_{\{m \leq |u_n| \leq m+1\}} F \nabla u_n dx. \end{aligned}$$

By Divergence theorem, we have

$$\int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n dx = 0.$$

Using the fact that $g_n(x, u_n, \nabla u_n) T_1(u_n - T_m(u_n)) \geq 0$ on the subset $\{|u_n| \geq m\}$ and Young's inequality, we get

$$\begin{aligned} & \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx \\ & \leq \langle f_n, T_1(u_n - T_m(u_n)) \rangle + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx. \end{aligned}$$

It follows that

$$\begin{aligned} & \left| \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi'_m(u_n) dx \right| \\ & \leq 2\phi(2k) \left(\int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right). \end{aligned} \quad (4.18)$$

From (4.14), (4.16), (4.17) and (4.18) we obtain

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx \\ & \geq \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s)) \\ & \quad (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^j) dx \\ & \quad - \alpha \phi(2k) \left(\int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right) \\ & \quad - \int_{\Omega \setminus \Omega^s} l_k \cdot \nabla T_k(u) dx + \epsilon_7(n, j). \end{aligned} \quad (4.19)$$

Now, we turn to second term in the left-hand side of (4.12). We have

$$\begin{aligned} & \left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n^j) dx \right| \\ & = \left| \int_{\{|u_n| \leq k\}} g_n(x, T_k(u_n), \nabla T_k(u_n)) \phi(\theta_n^j) dx \right| \\ & \leq b(k) \int_{\Omega} M(|\nabla T_k(u_n)|) |\phi(\theta_n^j)| dx + b(k) \int_{\Omega} d(x) |\phi(\theta_n^j)| dx \\ & \leq \frac{b(k)}{\alpha} \int_{\Omega} a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) |\phi(\theta_n^j)| dx + \epsilon_8(n, j). \end{aligned}$$

Then

$$\begin{aligned}
& \left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n^j) dx \right| \\
& \leq \frac{b(k)}{\alpha} \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s)) \\
& \quad (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) |\phi(\theta_n^j)| dx \\
& \quad + \frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) |\phi(\theta_n^j)| dx \\
& \quad + \frac{b(k)}{\alpha} \int_{\Omega} a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s |\phi(\theta_n^j)| dx + \epsilon_9(n, j).
\end{aligned} \tag{4.20}$$

We proceed as above to get

$$\frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) |\phi(\theta_n^j)| dx = \epsilon_9(n, j)$$

and

$$\frac{b(k)}{\alpha} \int_{\Omega} a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s |\phi(\theta_n^j)| dx = \epsilon_{10}(n, j).$$

Hence, we have

$$\begin{aligned}
& \left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n^j) dx \right| \\
& \leq \frac{b(k)}{\alpha} \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s)) \\
& \quad (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) |\phi(\theta_n^j)| dx + \epsilon_{11}(n, j).
\end{aligned} \tag{4.21}$$

Combining (4.12), (4.19) and (4.21), we get

$$\begin{aligned}
& \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s)) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) \\
& \quad \left(\phi'(\theta_n^j) - \frac{b(k)}{\alpha} |\phi(\theta_n^j)| \right) dx \\
& \leq \int_{\Omega \setminus \Omega^s} l_k \nabla T_k(u) dx + \alpha \phi(2k) \left(\int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right) \\
& \quad + \epsilon_{12}(n, j).
\end{aligned}$$

By (4.10), we have

$$\begin{aligned}
& \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s)) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) dx \\
& \leq 2 \int_{\Omega \setminus \Omega^s} l_k \nabla T_k(u) dx + 4\alpha \phi(2k) \left(\int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right) \\
& \quad + \epsilon_{12}(n, j).
\end{aligned} \tag{4.22}$$

On the other hand we can write

$$\begin{aligned}
& \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) dx \\
&= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s) dx \\
&\quad + \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s) dx \\
&\quad - \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)\chi^s) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) dx \\
&\quad + \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s) dx
\end{aligned}$$

We shall pass to the limit in n and then in j in the last three terms of the right hand side of the above equality. In a similar way as done in (4.13) and (4.20), we obtain

$$\begin{aligned}
& \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s) dx = \epsilon_{13}(n, j), \\
& \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)\chi^s) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) dx = \epsilon_{14}(n, j), \\
& \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s) dx \\
&= \epsilon_{15}(n, j).
\end{aligned} \tag{4.23}$$

So that

$$\begin{aligned}
& \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) dx \\
&= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s) dx \\
&\quad + \epsilon_{16}(n, j).
\end{aligned} \tag{4.24}$$

Let $r \leq s$. Using (3.2), (4.22) and (4.24) we can write

$$\begin{aligned}
0 &\leq \int_{\Omega^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) dx \\
&\leq \int_{\Omega^s} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) dx \\
&= \int_{\Omega^s} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) dx \\
&\leq \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) dx \\
&= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s) dx \\
&\quad + \epsilon_{15}(n, j) \\
&\leq 2 \int_{\Omega \setminus \Omega^s} l_k \nabla T_k(u) dx + 2\alpha\phi(2k) \left(\int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right) \\
&\quad + \epsilon_{17}(n, j).
\end{aligned}$$

By passing to the superior limit over n and then over j

$$\begin{aligned} 0 &\leq \limsup_{n \rightarrow +\infty} \int_{\Omega^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) dx \\ &\leq 2 \int_{\Omega \setminus \Omega^s} l_k \nabla T_k(u) dx + 4\alpha\phi(2k) \left(\int_{\{m \leq |u_n|\}} |f| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right). \end{aligned}$$

Letting $s \rightarrow +\infty$ and then $m \rightarrow +\infty$, taking into account that $l_k \nabla T_k(u) \in L^1(\Omega)$, $f \in L^1(\Omega)$, $|F| \in (E_{\overline{M}}(\Omega))^N$, $|\Omega \setminus \Omega^s| \rightarrow 0$, and $|\{m \leq |u| \leq m+1\}| \rightarrow 0$, one has

$$\int_{\Omega^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) dx, \quad (4.25)$$

tends to 0 as $n \rightarrow +\infty$. As in [20], we deduce that there exists a subsequence of $\{u_n\}$ still indexed by n such that

$$\nabla u_n \rightarrow \nabla u \text{ a. e. in } \Omega. \quad (4.26)$$

Therefore, having in mind (4.8) and (4.7), we can apply [27, Theorem 14.6] to get

$$a(x, u, \nabla u) \in (L_{\overline{M}}(\Omega))^N$$

and

$$a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M). \quad (4.27)$$

Step 6: Modular convergence of the truncations. Going back to equation (4.22), we can write

$$\begin{aligned} &\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\ &\leq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s dx \\ &\quad + \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) dx \\ &\quad + 2\alpha\phi(2k) \left(\int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right) \\ &\quad + 2 \int_{\Omega \setminus \Omega^s} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx + \epsilon_{12}(n, j). \end{aligned}$$

By (4.23) we get

$$\begin{aligned} &\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\ &\leq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s dx \\ &\quad + 2\alpha\phi(2k) \left(\int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right) \\ &\quad + 2 \int_{\Omega \setminus \Omega^s} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx + \epsilon_{18}(n, j). \end{aligned}$$

We now pass to the superior limit over n in both sides of this inequality using (4.27), to obtain

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\ & \leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \chi_j^s dx \\ & \quad + 2\alpha\phi(2k) \left(\int_{\{m \leq |u|\}} |f| dx + \int_{\{m \leq |u| \leq m+1\}} \overline{M}(|F|) dx \right) \\ & \quad + 2 \int_{\Omega \setminus \Omega^s} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx. \end{aligned}$$

We then pass to the limit in j to get

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\ & \leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \chi^s dx \\ & \quad + 2\alpha\phi(2k) \left(\int_{\{m \leq |u|\}} |f| dx + \int_{\{m \leq |u| \leq m+1\}} \overline{M}(|F|) dx \right) \\ & \quad + 2 \int_{\Omega \setminus \Omega^s} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx. \end{aligned}$$

Letting s and then $m \rightarrow +\infty$, one has

$$\limsup_{n \rightarrow +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx.$$

On the other hand, by (3.3), (4.5), (4.26) and Fatou's lemma, we have

$$\int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx.$$

It follows that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx = \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx.$$

By Lemma 2.5 we conclude that for every $k > 0$

$$a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \rightarrow a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u), \quad (4.28)$$

strongly in $L^1(\Omega)$. The convexity of the N -function M and (3.3) allow us to have

$$\begin{aligned} & M\left(\frac{|\nabla T_k(u_n) - \nabla T_k(u)|}{2}\right) \\ & \leq \frac{1}{2\alpha} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) + \frac{1}{2\alpha} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u). \end{aligned}$$

From Vitali's theorem we deduce

$$\lim_{|E| \rightarrow 0} \sup_n \int_E M\left(\frac{|\nabla T_k(u_n) - \nabla T_k(u)|}{2}\right) dx = 0.$$

Thus, for every $k > 0$

$$T_k(u_n) \rightarrow T_k(u) \text{ in } W_0^1 L_M(\Omega),$$

for the modular convergence.

Step 7: Compactness of the nonlinearities. We need to prove that

$$g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \quad (4.29)$$

By virtue of (4.7) and (4.26) one has

$$g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \quad \text{a. e. in } \Omega. \quad (4.30)$$

Let E be measurable subset of Ω and let $m > 0$. Using (3.3) and (3.4) we can write

$$\begin{aligned} & \int_E |g_n(x, u_n, \nabla u_n)| dx \\ &= \int_{E \cap \{|u_n| \leq m\}} |g_n(x, u_n, \nabla u_n)| dx + \int_{E \cap \{|u_n| > m\}} |g_n(x, u_n, \nabla u_n)| dx \\ &\leq b(m) \int_E d(x) dx + b(m) \int_E a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) dx \\ &\quad + \frac{1}{m} \int_\Omega g_n(x, u_n, \nabla u_n) u_n dx. \end{aligned}$$

From (3.5) and (4.6), we deduce that

$$0 \leq \int_\Omega g_n(x, u_n, \nabla u_n) u_n dx \leq C_3.$$

So

$$0 \leq \frac{1}{m} \int_\Omega g_n(x, u_n, \nabla u_n) u_n dx \leq \frac{C_3}{m}.$$

Then

$$\lim_{m \rightarrow +\infty} \frac{1}{m} \int_\Omega g_n(x, u_n, \nabla u_n) u_n dx = 0.$$

Thanks to (4.28) the sequence $\{a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n)\}_n$ is equi-integrable. This fact allows us to get

$$\lim_{|E| \rightarrow 0} \sup_n \int_E a(x, T_m(u_n), \nabla T_m(u_n)) \cdot \nabla T_m(u_n) dx = 0.$$

This shows that $g_n(x, u_n, \nabla u_n)$ is equi-integrable. Thus, Vitali's theorem implies that $g(x, u, \nabla u) \in L^1(\Omega)$ and

$$g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \text{ strongly in } L^1(\Omega).$$

Step 8: Renormalization identity for the solutions. In this step we prove that

$$\lim_{m \rightarrow +\infty} \int_{\{m \leq |u| \leq m+1\}} a(x, u, \nabla u) \nabla u dx = 0. \quad (4.31)$$

Indeed, for any $m \geq 0$ we can write

$$\begin{aligned} & \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx \\ &= \int_{\Omega} a(x, u_n, \nabla u_n) (\nabla T_{m+1}(u_n) - \nabla T_m(u_n)) dx \\ &= \int_{\Omega} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_{m+1}(u_n) dx \\ &\quad - \int_{\Omega} a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) dx. \end{aligned}$$

In view of (4.28), we can pass to the limit as n tends to $+\infty$ for fixed $m \geq 0$

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx \\ &= \int_{\Omega} a(x, T_{m+1}(u), \nabla T_{m+1}(u)) \nabla T_{m+1}(u) dx \\ &\quad - \int_{\Omega} a(x, T_m(u), \nabla T_m(u)) \nabla T_m(u) dx \\ &= \int_{\Omega} a(x, u, \nabla u) (\nabla T_{m+1}(u) - \nabla T_m(u)) dx \\ &= \int_{\{m \leq |u| \leq m+1\}} a(x, u, \nabla u) \nabla u dx. \end{aligned}$$

Having in mind (4.9), we can pass to the limit as m tends to $+\infty$ to obtain (4.31).

Step 9: Passing to the limit. Thanks to (4.28) and Lemma (2.5), we obtain

$$a(x, u_n, \nabla u_n) \nabla u_n \rightarrow a(x, u, \nabla u) \nabla u \text{ strongly in } L^1(\Omega). \quad (4.32)$$

Let $h \in \mathcal{C}_c^1(\mathbb{R})$ and $\varphi \in \mathcal{D}(\Omega)$. Inserting $h(u_n)\varphi$ as test function in (4.2), we get

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n h'(u_n) \varphi dx + \int_{\Omega} a(x, u_n, \nabla u_n) \nabla \varphi h(u_n) dx \\ &+ \int_{\Omega} \Phi_n(u_n) \nabla(h(u_n)\varphi) dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) h(u_n) \varphi dx \\ &= \langle f_n, h(u_n)\varphi \rangle + \int_{\Omega} F \nabla(h(u_n)\varphi) dx. \end{aligned} \quad (4.33)$$

We shall pass to the limit as $n \rightarrow +\infty$ in each term of the equality (4.33). Since h and h' have compact support on \mathbb{R} , there exists a real number $\nu > 0$, such that $\text{supp } h \subset [-\nu, \nu]$ and $\text{supp } h' \subset [-\nu, \nu]$. For $n > \nu$, we can write

$$\Phi_n(t)h(t) = \Phi(T_\nu(t))h(t) \text{ and } \Phi_n(t)h'(t) = \Phi(T_\nu(t))h'(t).$$

Moreover, the functions Φh and $\Phi h'$ belong to $(\mathcal{C}^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$. Observe first that the sequence $\{h(u_n)\varphi\}_n$ is bounded in $W_0^1 L_M(\Omega)$. Indeed, let $\rho > 0$

be a positive constant such that $\|h(u_n)\nabla\varphi\|_\infty \leq \rho$ and $\|h'(u_n)\varphi\|_\infty \leq \rho$. Using the convexity of the N -function M and taking into account (4.5) we have

$$\begin{aligned} \int_{\Omega} M\left(\frac{|\nabla(h(u_n)\varphi)|}{2\rho}\right) dx &\leq \int_{\Omega} M\left(\frac{|h(u_n)\nabla\varphi| + |h'(u_n)\varphi||\nabla u_n|}{2\rho}\right) dx \\ &\leq \frac{1}{2}M(1)|\Omega| + \frac{1}{2} \int_{\Omega} M(|\nabla u_n|) dx \\ &\leq \frac{1}{2}M(1)|\Omega| + \frac{1}{2}C_2. \end{aligned}$$

This, together with (4.7), imply that

$$h(u_n)\varphi \rightharpoonup h(u)\varphi \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}). \quad (4.34)$$

This enables us to get

$$\langle f_n, h(u_n)\varphi \rangle \rightarrow \langle f, h(u)\varphi \rangle.$$

Let E be a measurable subset of Ω . Define $c_\nu = \max_{|t| \leq \nu} \Phi(t)$. Let us denote by $\|v\|_{(M)}$ the Orlicz norm of a function $v \in L_M(\Omega)$. Using strengthened Hölder inequality with both Orlicz and Luxemburg norms, we get

$$\begin{aligned} \|\Phi(T_\nu(u_n))\chi_E\|_{(\overline{M})} &= \sup_{\|v\|_M \leq 1} \left| \int_E \Phi(T_\nu(u_n))v dx \right| \\ &\leq c_\nu \sup_{\|v\|_M \leq 1} \|\chi_E\|_{(\overline{M})} \|v\|_M \\ &\leq c_\nu |E| M^{-1}\left(\frac{1}{|E|}\right). \end{aligned}$$

Thus, we get

$$\lim_{|E| \rightarrow 0} \sup_n \|\Phi(T_\nu(u_n))\chi_E\|_{(\overline{M})} = 0.$$

Therefore, thanks to (4.7) by applying [27, Lemma 11.2] we obtain

$$\Phi(T_\nu(u_n)) \rightarrow \Phi(T_\nu(u)) \text{ strongly in } (E_{\overline{M}})^N,$$

which jointly with (4.34) allow us to pass to the limit in the third term of (4.33) to have

$$\int_{\Omega} \Phi(T_\nu(u_n)) \nabla(h(u_n)\varphi) dx \rightarrow \int_{\Omega} \Phi(T_\nu(u)) \nabla(h(u)\varphi) dx.$$

We remark that

$$|a(x, u_n, \nabla u_n) \nabla u_n h'(u_n)\varphi| \leq \rho a(x, u_n, \nabla u_n) \nabla u_n.$$

Consequently, using (4.32) and Vitali's theorem, we obtain

$$\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n h'(u_n)\varphi dx \rightarrow \int_{\Omega} a(x, u, \nabla u) \nabla u h'(u)\varphi dx.$$

and

$$\int_{\Omega} F \nabla u_n h'(u_n)\varphi dx \rightarrow \int_{\Omega} F \nabla u h'(u)\varphi dx.$$

For the second term of (4.33), as above we have

$$h(u_n)\nabla\varphi \rightarrow h(u)\nabla\varphi \text{ strongly in } (E_M(\Omega))^N,$$

which together with (4.27) give

$$\int_{\Omega} a(x, u_n, \nabla u_n) \nabla \varphi h(u_n) dx \rightarrow \int_{\Omega} a(x, u, \nabla u) \nabla \varphi h(u) dx$$

and

$$\int_{\Omega} F \nabla \varphi h(u_n) dx \rightarrow \int_{\Omega} F \nabla \varphi h(u) dx.$$

The fact that $h(u_n)\varphi \rightharpoonup h(u)\varphi$ weakly in $L^\infty(\Omega)$ for $\sigma^*(L^\infty, L^1)$ and (4.29) enable us to pass to the limit in the fourth term of (4.33) to get

$$\int_{\Omega} g_n(x, u_n, \nabla u_n) h(u_n) \varphi dx \rightarrow \int_{\Omega} g(x, u, \nabla u) h(u) \varphi dx.$$

At this point we can pass to the limit in each term of (4.33) to get

$$\begin{aligned} & \int_{\Omega} a(x, u, \nabla u) (\nabla \varphi h(u) + h'(u) \varphi \nabla u) dx + \int_{\Omega} \Phi(u) h'(u) \varphi \nabla u dx \\ & + \int_{\Omega} \Phi(u) h(u) \nabla \varphi dx + \int_{\Omega} g(x, u, \nabla u) h(u) \varphi dx \\ & = \langle f, h(u) \varphi \rangle + \int_{\Omega} F (\nabla \varphi h(u) + h'(u) \varphi \nabla u) dx, \end{aligned}$$

for all $h \in C_c^1(\mathbb{R})$ and for all $\varphi \in \mathcal{D}(\Omega)$. Moreover, as we have (3.5), (4.6) and (4.30) we can use Fatou's lemma to get $g(x, u, \nabla u)u \in L^1(\Omega)$. By virtue of (4.7), (4.27), (4.29), (4.31), the function u is a renormalized solution of problem (1.1).

ACKNOWLEDGMENTS

We would like to thank the referees for careful reading of the first version of this manuscript and providing many helpful suggestions and comments.

REFERENCES

1. R. Adams, *Sobolev Spaces*, Ac. Press, New York, 1975.
2. L. Aharouch, J. Bennouna, A. Touzani, Existence of Renormalized Solution of Some Elliptic Problems in Orlicz Spaces, *Rev. Mat. Complut.*, **22**(1), (2009), 91–110.
3. A. Aissaoui Fqayeh, A. Benkirane, M. El Moumni, A. Youssfi, Existence of Renormalized Solutions for Some Strongly Nonlinear Elliptic Equations in Orlicz Spaces, *Georgian Math. J.*, **22**(3), (2015), 305–321.
4. P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J. L. Vazquez, An L^1 -theory of Existence and Uniqueness of Solutions of Nonlinear Elliptic Equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **22**(2), (1995), 240–273.
5. A. Benkirane, J. Bennouna, Existence of Renormalized Solutions for Some Elliptic Problems Involving Derivatives of Nonlinear Terms in Orlicz Spaces, Partial differential equations, *Lecture Notes in Pure and Appl. Math.*, **229**, (2002), 125–138.
6. A. Benkirane, A. Elmahi, A Strongly Nonlinear Elliptic Equation Having Natural Growth Terms and L^1 Data, *Nonlinear Analysis*, **39**, (2000), 403–411.
7. A. Benkirane, A. Elmahi, Almost Everywhere Convergence of the Gradients of Solutions to Elliptic Equations in Orlicz Spaces and Application, *Nonlinear Anal. T. M. A.*, **11**(28), (1997), 1769–1784.

8. A. Benkirane, A. Elmahi, An Existence Theorem for a Strongly Nonlinear Elliptic Problem in Orlicz Spaces, *Nonlinear Anal.*, **36** (1999), 11–24.
9. M. F. Betta, A. Mercaldo, F. Murat, M. M. Porzio, Existence and Uniqueness Results for Nonlinear Elliptic Problems with a Lower Order Term and Measure Datum, *C. R. Math. Acad. Sci., Paris* **334**(9), (2002), 757–762.
10. M. F. Betta, A. Mercaldo, F. Murat, M. M. Porzio, Existence of Renormalized Solutions to Nonlinear Elliptic Equations with a Lower-order Term and Right-hand Side a Measure, *J. Math. Pures Appl.*, **82**(9), (2003), 90–124.
11. M. F. Betta, O. Guibé, A. Mercaldo, Neumann Problems for Nonlinear Elliptic Equations with L^1 Data, *J. Differential Equations* **259**, (2015), 898–924.
12. L. Boccardo, D. Giachetti, J.I. Diaz, F. Murat, Existence and Regularity of Renormalized Solutions for Some Elliptic Problems Involving Derivatives of Nonlinear Terms, *J. Differential Equations*, **106**(2), (1993), 215–237.
13. G. Dal Maso, F. Murat, L. Orsina, A. Prignet, Renormalized Solutions of Elliptic Equations with General Measure Data, *Ann. Scuola Norm. Pisa Cl. Sci.*, **28**(4), (1999), 741–808.
14. A. Dall’Aglio, Approximated Solutions of Equations with L^1 Data. Application to the H-convergence of Quasi-Linear Parabolic Equations, *Ann. Mat. Pura Appl.*, **170**(4), (1996), 207–240.
15. R. J. DiPerna, P.-L. Lions, On the Cauchy Problem for Boltzmann Equations: Global Existence and Weak Stability, *Ann. Mat.*, **130**, (1989), 321–366.
16. R.J. DiPerna, P.-L. Lions, Global Existence for the Fokker-Planck-Boltzman Equations, *Comm. Pure Appl. Math.*, **11**(2), (1989), 729–758.
17. S. Djebali, O. Kaviani, T. Moussaoui, Qualitative Properties and Existence of Solutions for a Generalized Fisher-like Equation, *Iranian Journal of Mathematical Sciences and Informatics*, **4**(2), (2009), 65–81.
18. T. K. Donaldson, N. S. Trudinger, Orlicz-Sobolev Spaces and Imbedding Theorems, *J. Funct. Anal.*, **8**, (1971), 52–75.
19. J. Gossez, A Strongly Nonlinear Elliptic Problem in Orlicz-Sobolev Spaces, *Proc. A.M.S. Sympos. Pure Math.*, **45**, (1986), 455–462.
20. J. Gossez, Nonlinear Elliptic Boundary Value Problems for Equations with Rapidly (or slowly) Increasing Coefficients, *Trans. Amer. Math. Soc.*, **190**, (1974), 163–205.
21. J. Gossez, Some Approximation Properties in Orlicz-Sobolev Spaces, *Studia Math.*, **74**, (1982), 17–24.
22. J.-P. Gossez, V. Mustonen, Variational Inequality in Orlicz-Sobolev Spaces, *Nonlinear Anal. Theory Appl.*, **11**, (1987), 379–392.
23. O. Guibé, A. Mercaldo, Existence of Renormalized Solutions to Nonlinear Elliptic Equations with two Lower Order Terms and Measure Data, *Trans. Amer. Math. Soc.*, **360**(2), (2008), 643–669.
24. O. Guibé, A. Mercaldo, Existence and Stability Results for Renormalized Solutions to Noncoercive Nonlinear Elliptic Equations with Measure Data, *Potential Anal.*, **25**(3), (2006), 223–258.
25. B. Hazarika, Strongly Almost Ideal Convergent Sequences in a Locally Convex Space Defined by Musielak-Orlicz Function, *Iranian Journal of Mathematical Sciences and Informatics*, **9**(2), (2014), 15–35.
26. E. Hewitt, K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, Berlin Heidelberg, New York, 1965.
27. M.A. Krasnosel’skiĭ, Y.B. Rutickiĭ, *Convex Functions and Orlicz Space*, Noordhoff, Groningen, 1961.

28. F. Murat, *Soluciones Renormalizadas de EDP Elípticas no Lineales*, Cours à l'Université de Séville, Mars 1992. Publication 93023 du Laboratoire d'Analyse Numérique de l'Université Paris VI, 1993.
29. F. Murat, Équations Elliptiques non linéaires avec Second Membre L^1 ou Mesure, 26^{ème} Congrès National d'Analyse Numérique. Les Karellis, Juin, (1994), 12–24.
30. J.M. Rakotoson, Uniqueness of Renormalized Solutions in a T -set for the L^1 -Data and the Link between Various Formulations, *Indiana Univ. Math. J.*, **43**(2), (1994), 685–702.
31. A. Youssfi, A. Benkirane, M. El Moumni, Bounded Solutions of Unilateral Problems for Strongly Nonlinear Equations in Orlicz Spaces, *E. J. Qualitative Theory of Diff. Equ.*, **21**, (2013), 1–25.