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Renormalized Solutions of Strongly Nonlinear Elliptic Problems with Lower Order Terms and Measure Data in Orlicz-Sobolev Spaces

Mostafa EL MOUMNI

Department of Mathematics, Faculty of Sciences El Jadida, University Chouaib Doukkali, P. O. Box 20, 24000 El Jadida, Morocco.

E-mail: mostafaelmoumni@gmail.com

Abstract. The purpose of this paper is to prove the existence of a renormalized solution of perturbed elliptic problems

$$-\operatorname{div}\Big(a(x,u,\nabla u)+\Phi(u)\Big)+g(x,u,\nabla u)=f-\operatorname{div}F,$$

in a bounded open set Ω and $u=0$ on $\partial\Omega$, in the framework of Orlicz-Sobolev spaces without any restriction on the M N-function of the Orlicz spaces, where $-\operatorname{div}\Big(a(x,u,\nabla u)\Big)$ is a Leray-Lions operator defined from $W_0^1L_M(\Omega)$ into its dual, $\Phi\in C^0(\mathbb{R},\mathbb{R}^N)$. The function $g(x,u,\nabla u)$ is a non linear lower order term with natural growth with respect to $|\nabla u|$, satisfying the sign condition and the datum μ is assumed to belong to $L^1(\Omega)+W^{-1}E_{\overline{M}}(\Omega)$.

Keywords: Elliptic equation, Orlicz-Sobolev spaces, Renormalized solution.

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1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, $N\geq 2$, and let M be an N-function. In the present paper we prove an existence result of a renormalized solution of the following strongly nonlinear elliptic problem

$$\begin{cases} A(u)-\operatorname{div}\Phi(u)+g(x,u,\nabla u)=f-\operatorname{div}F & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases} \tag{1.1}$$

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Here, $\Phi \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^N)$, while the function $g(x, u, \nabla u)$ is a non linear lower order term with natural growth with respect to $|\nabla u|$ and satisfying the sign condition. The non everywhere defined nonlinear operator $A(u) = -\text{div}\ (a(x, u, \nabla u))$ acts from its domain $D(A) \subset W_0^1 L_M(\Omega)$ into $W^{-1} L_{\overline{M}}(\Omega)$. The function $a(x, u, \nabla u)$ is assumed to satisfy, among others, $a(x, u, \nabla u)$ nonstandard growth condition governed by the N-function M, and the source term $f \in L^1(\Omega)$ and $|F| \in E_{\overline{M}}(\Omega)$, $\overline{M}$ stands for the conjugate of M.

We use here the notion of renormalized solutions, which was introduced by R.J. DiPerna and P.-L. Lions in their papers [16, 15] where the authors investigate the existence of solutions of the Boltzmann equation, by introducing the idea of renormalized solution. This concept of solution was then adapted to study (1.1) with $\Phi \equiv 0$, $g \equiv 0$ and $L^1(\Omega)$-data by F. Murat in [29, 28], by G. Dal Maso et al. in [13] with general measure data and then when f is a bounded Radon measure datum and g grows at most like $|\nabla u|^{p-1}$ by Beta et al. in [9, 10, 11] with $\Phi \equiv 0$ and by Guibé and Mercaldo in [23, 24] when $\Phi(u)$ behaves at most like $|u|^{p-1}$. Renormalization idea was then used in [12] for variational equations and in [30] when the source term is in $L^1(\Omega)$. Recall that to get both existence and uniqueness of a solution to problems with L^1-data, two notions of solution equivalent to the notion of renormalized solution were introduced, the first is the entropy solution by Bénilan et al. [4] and then the second is the SOLA by Dall'Aglio [14].

The authors in [5] have dealt with the equation (1.1) with $g = g(x, u)$ and $\mu \in W^{-1} E_{\overline{M}}(\Omega)$, under the restriction that the N-function M satisfies the Δ_2-condition. This work was then extended in [2] for N-functions not satisfying necessarily the Δ_2-condition. Our goal here is to extend the result in [2] solving the problem (1.1) without any restriction on the N-function M. Recently, a large number of papers was devoted to the existence of solutions of (1.1). In the variational framework, that is $\mu \in W^{-1} E_{\overline{M}}(\Omega)$, an existence result has been proved in [3], Specific examples to which our results apply include the following:

$$
\begin{gathered}
-\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u + |u|^s u\Big) + u|\nabla u|^p = \mu \text{ in } \Omega, \\
-\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u \log^\beta(1+|\nabla u|) + |u|^s u\Big) = \mu \text{ in } \Omega, \\
-\operatorname{div}\Big(\frac{M(|\nabla u|)\nabla u}{|\nabla u|^2} + |u|^s u\Big) + M(|\nabla u|) = \mu \text{ in } \Omega,
\end{gathered}
$$

where $p > 1$, $s > 0$, $\beta > 0$ and μ is a given Radon measure on Ω.

It is our purpose in this paper, to prove the existence of a renormalized solution for the problem (1.1) when the source term has the form $f - \text{div}\ F$ with $f \in L^1(\Omega)$ and $|F| \in E_{\overline{M}}(\Omega)$, in the setting of Orlicz spaces without any restriction on the N-functions M. The approximate equations provide a $W_0^1 L_M(\Omega)$ bound for the corresponding solution u_n. This allows us to obtain a function u as a limit of the sequence u_n. Hence, appear two difficulties. The first one is how to give a sense to $\Phi(u)$, the second difficulty lies in the need of the convergence almost everywhere of the gradients of u_n in Ω. This is done by using suitable test functions built upon u_n which make licit the use of the divergence theorem for Orlicz functions. We note that the techniques we used in the proof are different from those used in [2, 5, 12, 17, 25].

Let us briefly summarize the contents of the paper. The Section 2 is devoted to developing the necessary preliminaries, we introduce some technical lemmas. Section 3 contains the basic assumptions, the definition of renormalized solution and the main result, while the Section 4 is devoted to the proof of the main result.

2. Preliminaries

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i. e., M is continuous, increasing, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$, and $\frac{M(t)}{t} \to +\infty$ as $t \to +\infty$. Equivalently, M admits the representation:

$$M(t) = \int_0^t a(s)\,ds,$$

where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t)$ tends to $+\infty$ as $t \to +\infty$.

The conjugate of M is also an N-function and it is defined by $\overline{M} = \displaystyle\int_0^t \bar{a}(s)\,ds$, where $\bar{a} : \mathbb{R}^+ \to \mathbb{R}^+$ is the function $\bar{a}(t) = \sup\{s : a(s) \le t\}$ (see [1]).

An N-function M is said to satisfy the Δ_2-condition if, for some k,

$$M(2t) \le kM(t) \quad \forall t \ge 0, \tag{2.1}$$

when (2.1) holds only for $t \ge t_0 > 0$ then M is said to satisfy the Δ_2-condition near infinity. Moreover, we have the following Young's inequality

$$st \le M(t) + \overline{M}(s), \quad \forall s, t \ge 0.$$

Given two N-functions, we write $P \ll Q$ to indicate P grows essentially less rapidly than Q; i. e. for each $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to +\infty$. This is the case if and only if

$$\lim_{t\to\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0.$$

Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $k_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$ is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that

$$\int_\Omega M(|u(x)|)\,dx < +\infty \quad \text{(resp. } \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right)dx < +\infty \text{ for some } \lambda > 0).$$

The set $L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\Big\{\lambda > 0 : \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1\Big\},$$

and $k_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uv\,dx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M},\Omega}$. We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ [resp. $W^1E_M(\Omega)$] is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm

$$\|u\|_{1,M,\Omega} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_{M,\Omega}.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$. We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$, $\int_\Omega M\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right) dx \to 0$ for all $|\alpha| \leq 1$. This implies convergence for $\sigma(\prod L_M, \prod L_{\overline{M}})$. If M satisfies the Δ_2 condition on $\mathbb{R}^+$ (near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence.

Let $W^{-1}L_{\overline{M}}(\Omega)$ [resp. $W^{-1}E_{\overline{M}}(\Omega)$] denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ [resp. $E_{\overline{M}}(\Omega)$]. It is a Banach space under the usual quotient norm (for more details see [1]).

A domain Ω has the segment property if for every $x \in \partial\Omega$ there exists an open set G_x and a nonzero vector y_x such that $x \in G_x$ and if $z \in \overline{\Omega} \cap G_x$, then $z + ty_x \in \Omega$ for all $0 < t < 1$. The following lemmas can be found in [6].

Lemma 2.1. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let M be an N-function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then $F(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial}{\partial x_i}u & a.e. \text{ in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & a.e. \text{ in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 2.2. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. We suppose that the set of discontinuity points of F' is finite. Let M be an N-function, then the mapping $F : W^1L_M(\Omega) \to W^1L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\prod L_M, \prod E_{\overline{M}})$.*

Lemma 2.3. *([21]) Let Ω have the segment property. Then for each $\nu \in W_0^1L_M(\Omega)$, there exists a sequence $\nu_n \in \mathcal{D}(\Omega)$ such that ν_n converges to ν for the modular convergence in $W_0^1L_M(\Omega)$. Furthermore, if $\nu \in W_0^1L_M(\Omega) \cap L^\infty(\Omega)$, then*

$$||\nu_n||_{L^\infty(\Omega)} \leq (N+1)||\nu||_{L^\infty(\Omega)}.$$

We give now the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [8]).

Lemma 2.4. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Let M, P, Q be N-functions such that $Q \ll P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:*

$$|f(x,s)| \leq c(x) + k_1 P^{-1}M(k_2|s|),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$.

Then the Nemytskii operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is strongly continuous from $\mathcal{P}(E_M(\Omega), \frac{1}{k_2}) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2}\}$ into $E_Q(\Omega)$.

We will also use the following technical lemma.

Lemma 2.5. *([26]) If $\{f_n\} \subset L^1(\Omega)$ with $f_n \to f \in L^1(\Omega)$ a.e. in Ω, $f_n, f \geq 0$ a.e. in Ω and $\int_\Omega f_n(x)\,dx \to \int_\Omega f(x)\,dx$, then*

$$f_n \to f \text{ in } L^1(\Omega).$$

3. Structural Assumptions and Main Result

Throughout the paper Ω will be a bounded subset of $\mathbb{R}^N$, $N \geq 2$, satisfying the segment property. Let M and P be two N-functions such that $P \ll M$. Let A be the non everywhere defined operator defined from its domain $\mathcal{D}(\Omega) \subset W_0^1L_M(\Omega)$ into $W^{-1}L_{\overline{M}}(\Omega)$ given by

$$A(u) := -\operatorname{div} a(\cdot, u, \nabla u),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function. We assume that there exist a nonnegative function $c(x)$ in $E_{\overline{M}}(\Omega)$, $\alpha > 0$ and positive real constants k_1, k_2, k_3 and k_4, such that for every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, $\xi' \in \mathbb{R}^N$ $(\xi \neq \xi')$ and for almost every $x \in \Omega$

$$|a(x,s,\xi)| \leq c(x) + k_1\overline{P}^{-1}M(k_2|s|) + k_3\overline{M}^{-1}M(k_4|\xi|), \tag{3.1}$$

$$(a(x,s,\xi)-a(x,s,\xi'))(\xi-\xi')>0, \tag{3.2}$$

$$a(x,s,\xi)\xi \geq \alpha M(|\xi|). \tag{3.3}$$

Here, $g(x,s,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory function satisfying for almost every $x\in\Omega$ and for all $s\in\mathbb{R}$, $\xi\in\mathbb{R}^N$,

$$|g(x,s,\xi)|\leq b(|s|)\left(d(x)+M(|\xi|)\right), \tag{3.4}$$

$$g(x,s,\xi)s\geq 0, \tag{3.5}$$

where $b:\mathbb{R}\to\mathbb{R}^+$ is a continuous and increasing function while d is a given nonnegative function in $L^1(\Omega)$.

The right-hand side of (1.1) and $\Phi:\mathbb{R}\to\mathbb{R}^N$, are assumed to satisfy

$$f\in L^1(\Omega) \text{ and } |F|\in E_{\overline{M}}(\Omega), \tag{3.6}$$

$$\Phi\in\mathcal{C}^0(\mathbb{R},\mathbb{R}^N). \tag{3.7}$$

Our aim in this paper is to give a meaning to a possible solution of (1.1). In view of assumptions (3.1), (3.2), (3.3) and (3.6), the natural space in which one can seek for a solution u of problem (1.1) is the Orlicz-Sobolev space $W_0^1L_M(\Omega)$. But when u is only in $W_0^1L_M(\Omega)$ there is no reason for $\Phi(u)$ to be in $(L^1(\Omega))^N$ since no growth hypothesis is assumed on the function Φ. Thus, the term div $(\Phi(u))$ may be ill-defined even as a distribution. This hindrance is bypassed by solving some weaker problem obtained formally trough a pointwise multiplication of equation (1.1) by $h(u)$ where h belongs to $C_c^1(\mathbb{R})$, the class of $C^1(\mathbb{R})$ functions with compact support.

Definition 3.1. A measurable function $u:\Omega\to\mathbb{R}$ is called a renormalized solution of (1.1) if $u\in W_0^1L_M(\Omega)$, $a(x,u,\nabla u)\in(L_{\overline{M}}(\Omega))^N$, $g(x,u,\nabla u)\in L^1(\Omega)$, $g(x,u,\nabla u)u\in L^1(\Omega)$,

$$\lim_{m\to+\infty}\int_{\{x\in\Omega\,:\,m\leq|u(x)|\leq m+1\}} a(x,u,\nabla u)\nabla u\,dx=0,$$

and

$$\begin{cases} -\text{div } a(x,u,\nabla u)h(u)-\text{div }(\Phi(u)h(u))+h'(u)\Phi(u)\nabla u \\ +g(x,u,\nabla u)h(u)=fh(u)-\text{div }(Fh(u))+h'(u)F\nabla u \text{ in } \mathcal{D}'(\Omega), \end{cases} \tag{3.8}$$

for every $h\in C_c^1(\mathbb{R})$.

Remark 3.2. Every term in the problem (3.8) is meaningful in the distributional sense. Indeed, for h in $C_c^1(\mathbb{R})$ and u in $W_0^1L_M(\Omega)$, $h(u)$ belongs to $W^1L_M(\Omega)$ and for φ in $\mathcal{D}(\Omega)$ the function $\varphi h(u)$ belongs to $W_0^1L_M(\Omega)$. Since $(-\text{div } a(x,u,\nabla u))\in W^{-1}L_{\overline{M}}(\Omega)$, we also have

$$\begin{aligned}&\langle -\text{div } a(x,u,\nabla u)h(u),\varphi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}\\ &\qquad=\langle -\text{div } a(x,u,\nabla u),\varphi h(u)\rangle_{W^{-1}L_{\overline{M}}(\Omega),W_0^1L_M(\Omega)}\\ &\quad\forall\varphi\in\mathcal{D}(\Omega).\end{aligned}$$

Finally, since Φh and $\Phi h' \in (C_c^0(\mathbb{R}))^N$, for any measurable function u we have $\Phi(u)h(u)$ and $\Phi(u)h'(u) \in (L^\infty\Omega))^N$ and then div $(\Phi(u)h(u)) \in W^{-1,\infty}(\Omega)$ and $\Phi(u)h'(u) \in L_M(\Omega)$.

Our main result is the following

Theorem 3.3. *Suppose that assumptions* (3.1)–(3.7) *are fulfilled. Then, problem* (1.1) *has at least one renormalized solution.*

Remark 3.4. The condition (3.4) can be replaced by the weaker one

$$|g(x,s,\xi)| \leq d(x) + b(|s|)M(|\xi|),$$

with $b : \mathbb{R} \to \mathbb{R}^+$ a continuous function belonging to $L^1(\mathbb{R})$ and $d(x) \in L^1(\Omega)$.

Actually the original equation (1.1) will be recovered whenever $h(u) \equiv 1$, but unfortunately this cannot happen in general strong additional requirements on u. Therefore, (3.8) is to be viewed as a weaker form of (1.1).

4. Proof of the Main Result

From now on, we will use the standard truncation function T_k, $k > 0$, defined for all $s \in \mathbb{R}$ by $T_k(s) = \max\{-k, \min\{k, s\}\}$.

Step 1: Approximate problems. Let f_n be a sequence of $L^\infty(\Omega)$ functions that converge strongly to f in $L^1(\Omega)$. For $n \in \mathbb{N}$, $n \geq 1$, let us consider the following sequence of approximate equations

$$-\text{div } a(x,u_n,\nabla u_n) + \text{ div } \Phi_n(u_n) + g_n(x,u_n,\nabla u_n) = f_n - \text{ div } F \text{ in } \mathcal{D}'(\Omega), \tag{4.1}$$

where we have set $\Phi_n(s) = \Phi(T_n(s))$ and $g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}$. For fixed $n > 0$, it's obvious to observe that

$$g_n(x,s,\xi)s \geq 0, \quad |g_n(x,s,\xi)| \leq |g(x,s,\xi)| \text{ and } |g_n(x,s,\xi)| \leq n.$$

Moreover, since Φ is continuous one has $|\Phi_n(t)| \leq \max_{|t|\leq n} |\Phi(t)|$. Therefore, applying both Proposition 1, Proposition 5 and Remark 2 of [22] one can deduces that there exists at least one solution u_n of the approximate Dirichlet problem (4.1) in the sense

$$\begin{cases} u_n \in W_0^1L_M(\Omega), a(x,u_n,\nabla u_n) \in (L_{\overline{M}}(\Omega))^N \text{ and} \\ \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\nabla v dx + \int_\Omega \Phi_n(u_n)\nabla v dx \\ \displaystyle + \int_\Omega g_n(x,u_n,\nabla u_n)v dx = \langle f_n, v\rangle + \int_\Omega F\nabla v dx, \text{ for every } v \in W_0^1L_M(\Omega). \end{cases} \tag{4.2}$$

Step 2: Estimation in $W_0^1L_M(\Omega)$. Taking u_n as function test in problem (4.2), we obtain

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n dx+\int_\Omega \Phi_n(u_n)\nabla u_n dx\\&+\int_\Omega g_n(x,u_n,\nabla u_n)u_n dx=\langle f_n,u_n\rangle+\int_\Omega F\nabla u_n dx.\end{aligned}\tag{4.3}$$

Define $\widetilde{\Phi}_n\in(C^1(\mathbb{R}))^N$ as $\widetilde{\Phi}_n(t)=\int_0^t\Phi_n(\tau)d\tau$. Then formally $\operatorname{div}(\widetilde{\Phi}_n(u_n))=\Phi_n(u_n)\nabla u_n$, $u_n=0$ on $\partial\Omega$, $\widetilde{\Phi}_n(0)=0$ and by the Divergence theorem

$$\int_\Omega \Phi_n(u_n)\nabla u_n dx=\int_\Omega \text{ div }(\widetilde{\Phi}_n(u_n))dx=\int_{\partial\Omega}\widetilde{\Phi}_n(u_n)\overrightarrow{n}\,ds=0,$$

where $\overrightarrow{n}$ is the outward pointing unit normal field of the boundary $\partial\Omega$ (ds may be used as a shorthand for $\overrightarrow{n}ds$). Thus, by virtue of (3.5) and Young's inequality, we get

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n dx\le C_1+\frac{\alpha}{2}\int_\Omega M(|\nabla u_n|)dx,\tag{4.4}$$

which, together with (3.3) give

$$\int_\Omega M(|\nabla u_n|)dx\le C_2.\tag{4.5}$$

Moreover, we also have

$$\int_\Omega g_n(x,u_n,\nabla u_n)u_n dx\le C_3.\tag{4.6}$$

As a consequence of (4.5) there exist a subsequence of $\{u_n\}_n$, still indexed by n, and a function $u\in W_0^1L_M(\Omega)$ such that

$$\begin{aligned}&u_n\rightharpoonup u \text{ weakly in } W_0^1L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega),\Pi E_{\overline{M}}(\Omega)),\\&u_n\to u \text{ strongly in } E_M(\Omega) \text{ and a. e. in } \Omega.\end{aligned}\tag{4.7}$$

Step 3: Boundedness of $(a(x,u_n,\nabla u_n))_n$ in $(L_{\overline{M}}(\Omega))^N$. Let $w\in(E_M(\Omega))^N$ with $\|w\|_M\le 1$. Thanks to (3.2), we can write

$$\Big(a(x,u_n,\nabla u_n)-(a(x,u_n,\frac{w}{k_4})\Big)\Big(\nabla u_n-\frac{w}{k_4}\Big)\ge 0,$$

which implies

$$\begin{aligned}\frac{1}{k_4}\int_\Omega a(x,u_n,\nabla u_n)w dx\quad\le\quad&\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n dx\\&+\int_\Omega a\Big(x,u_n,\frac{w}{k_4}\Big)\Big(\frac{w}{k_4}-\nabla u_n\Big)dx.\end{aligned}$$

Thanks to (4.4) and (4.5), one has

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n dx\le C_5.$$

Define $\lambda = 1 + k_1 + k_3$. By the growth condition (3.1) and Young's inequality, one can write

$$\begin{aligned}
&\Big| \int_\Omega a\Big(x, u_n, \frac{w}{k_4}\Big)\Big(\frac{w}{k_4} - \nabla u_n\Big)dx \Big| \\
&\leq \Big(1 + \frac{1}{k_4}\Big)\Big(\int_\Omega \overline{M}(c(x))dx + k_1 \int_\Omega \overline{M}\,\overline{P}^{-1} M(k_2|u_n|)dx \\
&\quad + k_3 \int_\Omega M(|w|)dx \Big) + \frac{\lambda}{k_4}\int_\Omega M(|w|)dx + \lambda \int_\Omega M(|\nabla u_n|)dx.
\end{aligned}$$

By virtue of [18] and Lemma 4.14 of [20], there exists an N-function Q such that $M \ll Q$ and the space $W_0^1 L_M(\Omega)$ is continuously embedded into $L_Q(\Omega)$. Thus, by (4.5) there exists a constant $c_0 > 0$, not depending on n, satisfying $\|u_n\|_Q \leq c_0$. Since $M \ll Q$, we can write $M(k_2 t) \leq Q(\frac{t}{c_0})$, for $t > 0$ large enough. As $P \ll M$, we can find a constant c_1, not depending on n, such that $\int_\Omega \overline{M}\,\overline{P}^{-1} M(k_2|u_n|)dx \leq \int_\Omega Q\Big(\frac{|u_n|}{c_0}\Big) + c_1$. Hence, we conclude that the quantity $\Big|\int_\Omega a(x, u_n, \nabla u_n w dx\Big|$ is bounded from above for all $w \in (E_M(\Omega))^N$ with $\|w\|_M \leq 1$. Using the Orlicz norm we deduce that

$$\Big(a(x, u_n, \nabla u_n)\Big)_n \text{ is bounded in } (L_{\overline{M}}(\Omega))^N. \tag{4.8}$$

Step 4: Renormalization identity for the approximate solutions. For any $m \geq 1$, define $\theta_m(r) = T_{m+1}(r) - T_m(r)$. Observe that by [19, Lemma2] one has $\theta_m(u_n) \in W_0^1 L_M(\Omega)$. The use of $\theta_m(u_n)$ as test function in (4.2) yields

$$\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n)\nabla u_n dx \leq \langle f_n, \theta_m(u_n)\rangle + \int_{\{m \leq |u_n| \leq m+1\}} F\nabla u_n dx,$$

By Hölder's inequality and 4.5 we have

$$\begin{aligned}
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n)\nabla u_n dx &\leq \langle f_n, \theta_m(u_n)\rangle \\
&\quad + C_6 \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|)dx.
\end{aligned}$$

It's not hard to see that

$$\|\nabla \theta_m(u_n)\|_M \leq \|\nabla u_n\|_M.$$

So that by (4.5) and (4.7) one can deduce that

$$\theta_m(u_n) \rightharpoonup \theta_m(u) \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega)).$$

Note that as m goes to ∞, $\theta_m(u) \rightharpoonup 0$ weakly in $W_0^1 L_M(\Omega)$ for $\sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega))$, and since f_n converges strongly in $L^1(\Omega)$, by Lebesgue's theorem we have

$$\lim_{m\to\infty}\lim_{n\to\infty} \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|)dx = \lim_{m\to\infty}\lim_{n\to\infty} \langle f_n, \theta_m(u_n)\rangle = 0.$$

By (3.3) we finally have

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,u_n,\nabla u_n)\nabla u_n dx = 0. \tag{4.9}$$

Step 5: Almost everywhere convergence of the gradients. Define $\phi(s) = se^{\lambda s^2}$ with $\lambda = \left(\frac{b(k)}{2\alpha}\right)^2$. One can easily verify that for all $s \in \mathbb{R}$

$$\phi'(s) - \frac{b(k)}{\alpha}|\phi(s)| \ge \frac{1}{2}. \tag{4.10}$$

For $m \ge k$, we define the function ψ_m by

$$\begin{cases} \psi_m(s) = 1 & \text{if} \quad |s| \le m, \\ \psi_m(s) = m+1-|s| & \text{if} \quad m \le |s| \le m+1, \\ \psi_m(s) = 0 & \text{if} \quad |s| \ge m+1. \end{cases}$$

By virtue of [21, Theorem 4] there exists a sequence $\{v_j\}_j \subset D(\Omega)$ such that $v_j \to u$ in $W_0^1L_M(\Omega)$ for the modular convergence and a.e. in Ω. Let us define the following functions $\theta_n^j = T_k(u_n) - T_k(v_j)$, $\theta^j = T_k(u) - T_k(v_j)$ and $z_{n,m}^j = \phi(\theta_n^j)\psi_m(u_n)$. Using $z_{n,m}^j \in W_0^1L_M(\Omega)$ as test function in (4.2) we get

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla z_{n,m}^j dx + \int_\Omega \Phi_n(u_n)\nabla\phi\big(T_k(u_n) - T_k(v_j)\big)\psi_m(u_n)dx \\ + \int_{\{m\le|u_n|\le m+1\}} \Phi_n(u_n)\nabla u_n \psi_m'(u_n)\phi\big(T_k(u_n) - T_k(v_j)\big)dx \\ + \int_\Omega g_n(x,u_n,\nabla u_n)z_{n,m}^j dx = \int_\Omega f_n z_{n,m}^j dx + \int_\Omega F\nabla z_{n,m}^j dx.\end{aligned} \tag{4.11}$$

From now on we denote by $\epsilon_i(n,j)$, $i = 0,1,2,...$, various sequences of real numbers which tend to zero, when n and $j \to +\infty$, i. e.

$$\lim_{j\to+\infty}\lim_{n\to+\infty}\epsilon_i(n,j) = 0.$$

In view of (4.7), we have $z_{n,m}^j \rightharpoonup \phi(\theta^j)\psi_m(u)$ weakly in $L^\infty(\Omega)$ for $\sigma^*(L^\infty, L^1)$ as $n \to +\infty$, which yields

$$\lim_{n\to+\infty}\int_\Omega f_n z_{n,m}^j dx = \int_\Omega f\phi(\theta^j)\psi_m(u)dx,$$

and since $\phi(\theta^j) \rightharpoonup 0$ weakly in $L^\infty(\Omega)$ for $\sigma(L^\infty, L^1)$ as $j \to +\infty$, we have

$$\lim_{j\to+\infty}\int_\Omega f\phi(\theta^j)\psi_m(u)dx = 0.$$

Thus, we write

$$\int_\Omega f_n z_{n,m}^j dx = \epsilon_0(n,j).$$

Thanks to (4.5) and (4.7), we have as $n \to +\infty$,

$$z_{n,m}^j \rightharpoonup \phi(\theta^j)\psi_m(u) \text{ in } W_0^1L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega)),$$

which implies that

$$\lim_{n\to+\infty}\int_\Omega F\nabla z^j_{n,m}dx=\int_\Omega F\nabla\theta^j\phi'(\theta^j)\psi_m(u)dx+\int_\Omega F\nabla u\phi(\theta^j)\psi'_m(u)dx$$

On the one hand, by Lebesgue's theorem we get

$$\lim_{j\to+\infty}\int_\Omega F\nabla u\phi(\theta^j)\psi'_m(u)dx=0,$$

on the other hand, we write

$$\begin{aligned}\int_\Omega F\nabla\theta^j\phi'(\theta^j)\psi_m(u)dx &= \int_\Omega F\nabla T_k(u)\phi'(\theta^j)\psi_m(u)dx\\ &\quad-\int_\Omega F\nabla T_k(v_j)\phi'(\theta^j)\psi_m(u)dx,\end{aligned}$$

so that, by Lebesgue's theorem one has

$$\lim_{j\to+\infty}\int_\Omega F\nabla T_k(u)\phi'(\theta^j)\psi_m(u)dx = \int_\Omega F\nabla T_k(u)\psi_m(u)dx.$$

Let $\lambda>0$ such that $M\Big(\frac{|\nabla v_j-\nabla u|}{\lambda}\Big)\to 0$ strongly in $L^1(\Omega)$ as $j\to+\infty$ and $M\Big(\frac{|\nabla u|}{\lambda}\Big)\in L^1(\Omega)$, the convexity of the N-function M allows us to have

$$\begin{aligned}&M\Big(\tfrac{|\nabla T_k(v_j)\phi'(\theta^j)\psi_m(u)-\nabla T_k(u)\psi_m(u)|}{4\lambda\phi'(2k)}\Big)\\ &\qquad=\tfrac14 M\Big(\tfrac{|\nabla v_j-\nabla u|}{\lambda}\Big)+\tfrac14\Big(1+\tfrac{1}{\phi'(2k)}\Big)M\Big(\tfrac{|\nabla u|}{\lambda}\Big).\end{aligned}$$

Then, by using the modular convergence of $\{\nabla v_j\}$ in $(L_M(\Omega))^N$ and Vitali's theorem, we obtain

$$\nabla T_k(v_j)\phi'(\theta^j)\psi_m(u)\to\nabla T_k(u)\psi_m(u)\text{ in }(L_M(\Omega))^N,\ \text{ as } j \text{ tends to } +\infty,$$

for the modular convergence, and then

$$\lim_{j\to+\infty}\int_\Omega F\nabla T_k(u)\phi'(\theta^j)\psi_m(u)dx = \int_\Omega F\nabla T_k(u)\psi_m(u)dx.$$

We have proved that

$$\int_\Omega F\nabla z^j_{n,m}dx=\epsilon_1(n,j).$$

It's easy to see that by the modular convergence of the sequence $\{v_j\}_j$, one has

$$\lim_{j\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}}\Phi_n(u_n)\nabla u_n\psi'_m(u_n)\phi\big(T_k(u_n)-T_k(v_j)\big)dx=0,$$

while for the third term in the left-hand side of (4.11) we can write

$$\begin{aligned}&\int_\Omega\Phi_n(u_n)\nabla\phi\big(T_k(u_n)-T_k(v_j)\big)\psi_m(u_n)dx\\ &=\int_\Omega\Phi_n(u_n)\nabla T_k(u_n)\phi'(\theta^j_n)\psi_m(u_n)dx-\int_\Omega\Phi_n(u_n)\nabla T_k(v_j)\phi'(\theta^j_n)\psi_m(u_n)dx.\end{aligned}$$

Firstly, we have

$$\lim_{j\to+\infty}\lim_{n\to+\infty}\int_\Omega \Phi_n(u_n)\nabla T_k(u_n)\phi'(\theta_n^j)\psi_m(u_n)dx=0.$$

In view of (4.7), one has

$$\Phi_n(u_n)\phi'(\theta_n^j)\psi_m(u_n)\to\Phi(u)\phi'(\theta^j)\psi_m(u),$$

almost everywhere in Ω as n tends to $+\infty$. Furthermore, we can check that

$$\|\Phi_n(u_n)\phi'(\theta_n^j)\psi_m(u_n)\|_{\overline{M}}\le\overline{M}(c_m\phi'(2k))|\Omega|+1,$$

where $c_m=\max_{|t|\le m+1}\Phi(t)$. Applying [27, Theorem 14.6] we get

$$\lim_{n\to+\infty}\int_\Omega \Phi_n(u_n)\nabla T_k(v_j)\phi'(\theta_n^j)\psi_m(u_n)dx=\int_\Omega \Phi(u)\nabla T_k(v_j)\phi'(\theta^j)\psi_m(u)dx.$$

Using the modular convergence of the sequence $\{v_j\}_j$, we obtain

$$\lim_{j\to+\infty}\lim_{n\to+\infty}\int_\Omega \Phi_n(u_n)\nabla T_k(v_j)\phi'(\theta_n^j)\psi_m(u_n)dx=\int_\Omega \Phi(u)\nabla T_k(u)\psi_m(u)dx.$$

Then, using again the Divergence theorem we get

$$\int_\Omega \Phi(u)\nabla T_k(u)\psi_m(u)dx=0.$$

Therefore, we write

$$\int_\Omega \Phi_n(u_n)\nabla\phi\big(T_k(u_n)-T_k(v_j)\big)\psi_m(u_n)dx=\epsilon_2(n,j).$$

Since $g_n(x,u_n,\nabla u_n)z_{n,m}^j\ge 0$ on the set $\{|\,u_n\,|>k\}$ and $\psi_m(u_n)=1$ on the set $\{|\,u_n\,|\le k\}$, from (4.11) we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla z_{n,m}^j dx+\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\phi(\theta_n^j)dx\le\epsilon_3(n,j). \quad (4.12)$$

We now evaluate the first term of the left-hand side of (4.12) by writing

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla z_{n,m}^j dx\\
&\quad=\int_\Omega a(x,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla T_k(v_j))\phi'(\theta_n^j)\psi_m(u_n)dx\\
&\qquad+\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\phi(\theta_n^j)\psi_m'(u_n)dx\\
&\quad=\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(v_j))\phi'(\theta_n^j)dx\\
&\qquad-\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(v_j)\phi'(\theta_n^j)\psi_m(u_n)dx\\
&\qquad+\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\phi(\theta_n^j)\psi_m'(u_n)dx,
\end{aligned}$$

and then

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla z^j_{n,m}dx\\
&=\int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi^s_j)\big)\\
&\qquad\qquad \big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j\big)\phi'(\theta^j_n)\,dx\\
&\qquad +\int_\Omega a\big(x,T_k(u_n),\nabla T_k(v_j)\chi^s_j\big)\big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j\big)\phi'(\theta^j_n)\,dx\\
&\qquad -\int_{\Omega\setminus\Omega^s_j} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\phi'(\theta^j_n)dx\\
&\qquad -\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(v_j)\phi'(\theta^j_n)\psi_m(u_n)dx\\
&\qquad +\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\phi(\theta^j_n)\psi'_m(u_n)dx,
\end{aligned}\tag{4.13}$$

where by χ^s_j, $s>0$, we denote the characteristic function of the subset

$$\Omega^s_j=\{x\in\Omega\,:\,|\nabla T_k(v_j)|\le s\}.$$

For fixed m and s, we will pass to the limit in n and then in j in the second, third, fourth and fifth terms in the right side of (4.13). Starting with the second term, we have

$$\begin{aligned}
&\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi^s_j)\big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j\big)\phi'(\theta^j_n)dx\\
&\to\int_\Omega a(x,T_k(u),\nabla T_k(v_j)\chi^s_j)\big(\nabla T_k(u)-\nabla T_k(v_j)\chi^s_j\big)\phi'(\theta^j)dx,
\end{aligned}$$

as $n\to+\infty$. Since by lemma (2.4) one has

$$a(x,T_k(u_n),\nabla T_k(v_j)\chi^s_j)\phi'(\theta^j_n)\to a(x,T_k(u),\nabla T_k(v_j)\chi^s_j)\phi'(\theta^j),$$

strongly in $(E_{\overline{M}}(\Omega))^N$ as $n\to\infty$, while by (4.5)

$$\nabla T_k(u_n)\rightharpoonup\nabla T_k(u),$$

weakly in $(L_M(\Omega))^N$. Let χ^s denote the characteristic function of the subset

$$\Omega^s=\{x\in\Omega\,:\,|\nabla T_k(u)|\le s\}.$$

As $\nabla T_k(v_j)\chi^s_j\to\nabla T_k(u)\chi^s$ strongly in $(E_M(\Omega))^N$ as $j\to+\infty$, one has

$$\int_\Omega a\big(x,T_k(u),\nabla T_k(v_j)\chi^s_j\big)\cdot\big(\nabla T_k(u)-\nabla T_k(v_j)\chi^s_j\big)\phi'(\theta^j)dx\to 0,$$

as $j\to\infty$. Then

$$\int_\Omega a\big(x,T_k(u_n),\nabla T_k(v_j)\chi^s_j\big)\big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j\big)\phi'(\theta^j_n)dx=\epsilon_4(n,j).\tag{4.14}$$

We now estimate the third term of (4.13). It's easy to see that by (3.3), $a(x,s,0)=0$ for almost everywhere $x\in\Omega$ and for all $s\in\mathbb{R}$. Thus, from (4.8) we have that $\big(a(x,T_k(u_n),\nabla T_k(u_n))\big)_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$ for all $k\ge 0$.

Therefore, there exist a subsequence still indexed by n and a function l_k in $(L_{\overline{M}}(\Omega))^N$ such that

$$a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup l_k \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}},\Pi E_M). \quad (4.15)$$

Then, since $\nabla T_k(v_j)\chi_{\Omega\setminus\Omega_j^s} \in (E_{\overline{M}}(\Omega))^N$, we obtain

$$\int_{\Omega\setminus\Omega_j^s} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\phi'(\theta_n^j)dx \to \int_{\Omega\setminus\Omega_j^s} l_k\nabla T_k(v_j)\phi'(\theta^j)dx,$$

as $n\to+\infty$. The modular convergence of $\{v_j\}$ allows us to get

$$-\int_{\Omega\setminus\Omega_j^s} l_k\nabla T_k(v_j)\phi'(\theta^j)dx \to -\int_{\Omega\setminus\Omega^s} l_k\nabla T_k(u)dx,$$

as $j\to+\infty$. This, proves

$$-\int_{\Omega\setminus\Omega_j^s} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\phi'(\theta_n^j)dx = -\int_{\Omega\setminus\Omega^s} l_k\nabla T_k(u)dx + \epsilon_5(n,j). \quad (4.16)$$

As regards the fourth term, observe that $\psi_m(u_n)=0$ on the subset $\{|u_n|\ge m+1\}$, so we have

$$\begin{aligned}-\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(v_j))\phi'(\theta_n^j)\psi_m(u_n)dx = \\ -\int_{\{|u_n|>k\}} a(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_k(v_j)\phi'(\theta_n^j)\psi_m(u_n)dx.\end{aligned}$$

Since

$$\begin{aligned}-\int_{\{|u_n|>k\}} a(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_k(v_j)\phi'(\theta_n^j)\psi_m(u_n)dx = \\ -\int_{\{|u|>k\}} l_{m+1}\nabla T_k(u)\psi_m(u)dx + \epsilon_5(n,j),\end{aligned}$$

observing that $\nabla T_k(u)=0$ on the subset $\{|u|>k\}$, one has

$$-\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(v_j)\phi'(\theta_n^j)\psi_m(u_n)dx = \epsilon_6(n,j). \quad (4.17)$$

For the last term of (4.13), we have

$$\begin{aligned}&\Big|\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\phi(\theta_n^j)\psi'_m(u_n)dx\Big| \\ &\quad = \Big|\int_{\{m\le|u_n|\le m+1\}} a(x,u_n,\nabla u_n)\nabla u_n\phi(\theta_n^j)\psi'_m(u_n)\,dx\Big| \\ &\quad \le \phi(2k)\int_{\{m\le|u_n|\le m+1\}} a(x,u_n,\nabla u_n)\nabla u_n dx.\end{aligned}$$

To estimate the last term of the previous inequality, we use $(T_1(u_n - T_m(u_n)) \in W_0^1 L_M(\Omega))$ as test function in (4.2), to get

$$\int_{\{m\le|u_n|\le m+1\}} a(x,u_n,\nabla u_n)\nabla u_n dx + \int_{\{m\le|u_n|\le m+1\}} \Phi_n(u_n)\nabla u_n dx$$
$$+\int_{\{|u_n|\ge m\}} g_n(x,u_n,\nabla u_n)T_1(u_n-T_m(u_n))dx = \langle f_n, T_1(u_n - T_m(u_n))\rangle$$
$$+\int_{\{m\le|u_n|\le m+1\}} F\nabla u_n dx.$$

By Divergence theorem, we have

$$\int_{\{m\le|u_n|\le m+1\}} \Phi_n(u_n)\nabla u_n dx = 0.$$

Using the fact that $g_n(x,u_n,\nabla u_n)T_1(u_n - T_m(u_n)) \ge 0$ on the subset $\{|u_n| \ge m\}$ and Young's inequality, we get

$$\int_{\{m\le|u_n|\le m+1\}} a(x,u_n,\nabla u_n)\nabla u_n dx$$
$$\le \langle f_n, T_1(u_n - T_m(u_n))\rangle + \int_{\{m\le|u_n|\le m+1\}} \overline{M}(|F|)dx.$$

It follows that

$$\Big|\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \phi(\theta_n^j)\psi'_m(u_n)dx\Big|$$
$$\le 2\phi(2k)\Big(\int_{\{m\le|u_n|\}} |f_n|dx + \int_{\{m\le|u_n|\le m+1\}} \overline{M}(|F|)dx\Big). \tag{4.18}$$

From (4.14), (4.16), (4.17) and (4.18) we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla z_{n,m}^j dx$$
$$\ge \int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big)$$
$$\big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\big)\phi'(\theta_n^j)dx$$
$$-\alpha\phi(2k)\Big(\int_{\{m\le|u_n|\}} |f_n|dx + \int_{\{m\le|u_n|\le m+1\}} \overline{M}(|F|)dx\Big)$$
$$-\int_{\Omega\setminus\Omega^s} l_k\cdot\nabla T_k(u)dx + \epsilon_7(n,j). \tag{4.19}$$

Now, we turn to second term in the left-hand side of (4.12). We have

$$\Big|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\phi(\theta_n^j)dx\Big|$$
$$= \Big|\int_{\{|u_n|\le k\}} g_n(x,T_k(u_n),\nabla T_k(u_n))\phi(\theta_n^j)dx\Big|$$
$$\le b(k)\int_\Omega M(|\nabla T_k(u_n)|)|\phi(\theta_n^j)|dx + b(k)\int_\Omega d(x)|\phi(\theta_n^j)|dx$$
$$\le \frac{b(k)}{\alpha}\int_\Omega a_n(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\phi(\theta_n^j)|dx + \epsilon_8(n,j).$$

Then

$$
\begin{aligned}
&\Big|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\phi(\theta_n^j)dx\Big| \\
&\le \frac{b(k)}{\alpha}\int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big) \\
&\qquad \big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\big)|\phi(\theta_n^j)|dx \\
&\qquad + \frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\big)|\phi(\theta_n^j)|dx \\
&\qquad + \frac{b(k)}{\alpha}\int_\Omega a_n(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi_j^s|\phi(\theta_n^j)|dx + \epsilon_9(n,j).
\end{aligned} \tag{4.20}
$$

We proceed as above to get

$$
\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\big)|\phi(\theta_n^j)|dx = \epsilon_9(n,j)
$$

and

$$
\frac{b(k)}{\alpha}\int_\Omega a_n(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi_j^s|\phi(\theta_n^j)|dx = \epsilon_{10}(n,j).
$$

Hence, we have

$$
\begin{aligned}
&\Big|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\phi(\theta_n^j)dx\Big| \\
&\le \frac{b(k)}{\alpha}\int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big) \\
&\qquad \big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\big)|\phi(\theta_n^j)|dx + \epsilon_{11}(n,j).
\end{aligned} \tag{4.21}
$$

Combining (4.12), (4.19) and (4.21), we get

$$
\begin{aligned}
&\int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big)\big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\big) \\
&\qquad \Big(\phi'(\theta_n^j) - \tfrac{b(k)}{\alpha}|\phi(\theta_n^j)|\Big)dx \\
&\qquad \le \int_{\Omega\setminus\Omega^s} l_k\nabla T_k(u)\,dx + \alpha\phi(2k)\Big(\int_{\{m\le|u_n|\}}|f_n|dx + \int_{\{m\le|u_n|\le m+1\}}\overline{M}(|F|)dx\Big) \\
&\qquad\qquad + \epsilon_{12}(n,j).
\end{aligned}
$$

By (4.10), we have

$$
\begin{aligned}
&\int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big)\big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\big)dx \\
&\le 2\int_{\Omega\setminus\Omega^s} l_k\nabla T_k(u)dx + 4\alpha\phi(2k)\Big(\int_{\{m\le|u_n|\}}|f_n|dx + \int_{\{m\le|u_n|\le m+1\}}\overline{M}(|F|)dx\Big) \\
&+\epsilon_{12}(n,j).
\end{aligned} \tag{4.22}
$$

On the other hand we can write

$$\begin{aligned}
&\int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)\big)\big(\nabla T_k(u_n) - \nabla T_k(u)\chi^s\big)dx \\
&\quad = \int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big)\big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\big)dx \\
&\qquad + \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\big(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s\big)dx \\
&\qquad - \int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi^s)\big(\nabla T_k(u_n) - \nabla T_k(u)\chi^s\big)dx \\
&\qquad + \int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\big)dx
\end{aligned}$$

We shall pass to the limit in n and then in j in the last three terms of the right hand side of the above equality. In a similar way as done in (4.13) and (4.20), we obtain

$$\begin{aligned}
&\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\big(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s\big)dx = \epsilon_{13}(n,j), \\
&\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi^s)\big(\nabla T_k(u_n) - \nabla T_k(u)\chi^s\big)dx = \epsilon_{14}(n,j), \\
&\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\big)dx \\
&= \epsilon_{15}(n,j).
\end{aligned} \tag{4.23}$$

So that

$$\begin{aligned}
&\int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)\big)\big(\nabla T_k(u_n) - \nabla T_k(u)\chi^s\big)dx \\
&= \int_\Omega \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big)\big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\big)dx \\
&+\epsilon_{16}(n,j).
\end{aligned} \tag{4.24}$$

Let $r \leq s$. Using (3.2), (4.22) and (4.24) we can write

$$\begin{aligned}
&0 \leq \int_{\Omega^r} \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\big)\big(\nabla T_k(u_n) - \nabla T_k(u)\big)dx \\
&\leq \int_{\Omega^s} \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\big)\big(\nabla T_k(u_n) - \nabla T_k(u)\big)dx \\
&= \int_{\Omega^s} \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)\big)\big(\nabla T_k(u_n) - \nabla T_k(u)\chi^s\big)dx \\
&\leq \int_{\Omega} \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)\big)\big(\nabla T_k(u_n) - \nabla T_k(u)\chi^s\big)dx \\
&= \int_{\Omega} \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\big)\big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\big)dx \\
&+\epsilon_{15}(n,j) \\
&\leq 2\int_{\Omega\setminus\Omega^s} l_k \nabla T_k(u)dx + 2\alpha\phi(2k)\Big(\int_{\{m\leq|u_n|\}} |f_n|dx + \int_{\{m\leq|u_n|\leq m+1\}} \overline{M}(|F|)dx\Big) \\
&+\epsilon_{17}(n,j).
\end{aligned}$$

By passing to the superior limit over n and then over j

$$\begin{aligned} 0 &\leq \limsup_{n\to+\infty} \int_{\Omega^r} \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\big)\big(\nabla T_k(u_n) - \nabla T_k(u)\big)dx \\ &\leq 2\int_{\Omega\backslash\Omega^s} l_k \nabla T_k(u)dx + 4\alpha\phi(2k)\Big(\int_{\{m\leq|u_n|\}}|f|dx + \int_{\{m\leq|u_n|\leq m+1\}}\overline{M}(|F|)dx\Big). \end{aligned}$$

Letting $s\to+\infty$ and then $m\to+\infty$, taking into account that $l_k\nabla T_k(u)\in L^1(\Omega)$, $f\in L^1(\Omega)$, $|F|\in(E_{\overline{M}}(\Omega))^N$, $|\Omega\backslash\Omega^s|\to 0$, and $|\{m\leq|u|\leq m+1\}|\to 0$, one has

$$\int_{\Omega^r} \big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\big)\big(\nabla T_k(u_n) - \nabla T_k(u)\big)dx, \tag{4.25}$$

tends to 0 as $n\to+\infty$. As in [20], we deduce that there exists a subsequence of $\{u_n\}$ still indexed by n such that

$$\nabla u_n \to \nabla u \text{ a. e. in } \Omega. \tag{4.26}$$

Therefore, having in mind (4.8) and (4.7), we can apply [27, Theorem 14.6] to get

$$a(x,u,\nabla u)\in(L_{\overline{M}}(\Omega))^N$$

and

$$a(x,u_n,\nabla u_n)) \rightharpoonup a(x,u,\nabla u) \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}},\Pi E_M). \tag{4.27}$$

Step 6: Modular convergence of the truncations. Going back to equation (4.22), we can write

$$\begin{aligned} &\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dx \\ &\leq \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi_j^s dx \\ &\qquad + \int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)dx \\ &\qquad +2\alpha\phi(2k)\Big(\int_{\{m\leq|u_n|\}}|f_n|dx + \int_{\{m\leq|u_n|\leq m+1\}}\overline{M}(|F|)dx\Big) \\ &\qquad +2\int_{\Omega\backslash\Omega^s} a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)dx + \epsilon_{12}(n,j). \end{aligned}$$

By (4.23) we get

$$\begin{aligned} &\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dx \\ &\leq \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi_j^s dx \\ &\qquad +2\alpha\phi(2k)\Big(\int_{\{m\leq|u_n|\}}|f_n|dx + \int_{\{m\leq|u_n|\leq m+1\}}\overline{M}(|F|)dx\Big) \\ &\qquad +2\int_{\Omega\backslash\Omega^s} a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)dx + \epsilon_{18}(n,j). \end{aligned}$$

We now pass to the superior limit over n in both sides of this inequality using (4.27), to obtain

$$\begin{aligned}
&\limsup_{n\to+\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dx\\
&\le \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(v_j)\chi_j^s dx\\
&\qquad +2\alpha\phi(2k)\Big(\int_{\{m\le|u|\}}|f|dx+\int_{\{m\le|u|\le m+1\}}\overline{M}(|F|)dx\Big)\\
&\qquad +2\int_{\Omega\setminus\Omega^s} a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)dx.
\end{aligned}$$

We then pass to the limit in j to get

$$\begin{aligned}
&\limsup_{n\to+\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dx\\
&\le \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi^s dx\\
&\qquad +2\alpha\phi(2k)\Big(\int_{\{m\le|u|\}}|f|dx+\int_{\{m\le|u|\le m+1\}}\overline{M}(|F|)dx\Big)\\
&\qquad +2\int_{\Omega\setminus\Omega^s} a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)dx.
\end{aligned}$$

Letting s and then $m\to+\infty$, one has

$$\limsup_{n\to+\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dx \le \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)dx.$$

On the other hand, by (3.3), (4.5), (4.26) and Fatou's lemma, we have

$$\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)dx \le \liminf_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dx.$$

It follows that

$$\lim_{n\to+\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dx = \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)dx.$$

By Lemma 2.5 we conclude that for every $k>0$

$$a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\to a(x,T_k(u),\nabla T_k(u))\nabla T_k(u), \tag{4.28}$$

strongly in $L^1(\Omega)$. The convexity of the N-function M and (3.3) allow us to have

$$\begin{aligned}
&M\Big(\tfrac{|\nabla T_k(u_n)-\nabla T_k(u)|}{2}\Big)\\
&\le \tfrac{1}{2\alpha}a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)+\tfrac{1}{2\alpha}a(x,T_k(u),\nabla T_k(u))\nabla T_k(u).
\end{aligned}$$

From Vitali's theorem we deduce

$$\lim_{|E|\to 0}\sup_n\int_E M\Big(\frac{|\nabla T_k(u_n)-\nabla T_k(u)|}{2}\Big)dx=0.$$

Thus, for every $k>0$

$$T_k(u_n)\to T_k(u) \text{ in } W_0^1L_M(\Omega),$$

for the modular convergence.

Step 7: Compactness of the nonlinearities. We need to prove that

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \text{ strongly in } L^1(\Omega). \tag{4.29}$$

By virtue of (4.7) and (4.26) one has

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \quad \text{a. e. in } \Omega. \tag{4.30}$$

Let E be measurable subset of Ω and let $m > 0$. Using (3.3) and (3.4) we can write

$$\begin{aligned}
&\int_E |g_n(x,u_n,\nabla u_n)|dx \\
&= \int_{E\cap\{|u_n|\le m\}} |g_n(x,u_n,\nabla u_n)|dx + \int_{E\cap\{|u_n|>m\}} |g_n(x,u_n,\nabla u_n)|\,dx \\
&\quad \le b(m)\int_E d(x)dx + b(m)\int_E a(x,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dx \\
&\qquad\qquad + \frac{1}{m}\int_\Omega g_n(x,u_n,\nabla u_n)u_n\,dx.
\end{aligned}$$

From (3.5) and (4.6), we deduce that

$$0 \le \int_\Omega g_n(x,u_n,\nabla u_n)u_n dx \le C_3.$$

So

$$0 \le \frac{1}{m}\int_\Omega g_n(x,u_n,\nabla u_n)u_n dx \le \frac{C_3}{m}.$$

Then

$$\lim_{m\to+\infty} \frac{1}{m}\int_\Omega g_n(x,u_n,\nabla u_n)u_n dx = 0.$$

Thanks to (4.28) the sequence $\{a(x,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)\}_n$ is equi-integrable. This fact allows us to get

$$\lim_{|E|\to 0}\sup_n \int_E a(x,T_m(u_n),\nabla T_m(u_n))\cdot\nabla T_m(u_n)dx = 0.$$

This shows that $g_n(x,u_n,\nabla u_n)$ is equi-integrable. Thus, Vitali's theorem implies that $g(x,u,\nabla u) \in L^1(\Omega)$ and

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \text{ strongly in } L^1(\Omega).$$

Step 8: Renormalization identity for the solutions. In this step we prove that

$$\lim_{m\to+\infty}\int_{\{m\le|u|\le m+1\}} a(x,u,\nabla u)\nabla u dx = 0. \tag{4.31}$$

Indeed, for any $m \geq 0$ we can write

$$\begin{aligned}\int_{\{m\leq|u_n|\leq m+1\}} & a(x,u_n,\nabla u_n)\nabla u_n dx \\ &= \int_\Omega a(x,u_n,\nabla u_n)(\nabla T_{m+1}(u_n)-\nabla T_m(u_n))dx \\ &= \int_\Omega a(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n)dx \\ &\quad -\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dx.\end{aligned}$$

In view of (4.28), we can pass to the limit as n tends to $+\infty$ for fixed $m \geq 0$

$$\begin{aligned}\lim_{n\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}} & a(x,u_n,\nabla u_n)\nabla u_n dx \\ &= \int_\Omega a(x,T_{m+1}(u),\nabla T_{m+1}(u))\nabla T_{m+1}(u)dx \\ &\quad -\int_\Omega a(x,T_m(u),\nabla T_m(u))\nabla T_m(u)dx \\ &= \int_\Omega a(x,u,\nabla u)(\nabla T_{m+1}(u)-\nabla T_m(u))dx \\ &= \int_{\{m\leq|u|\leq m+1\}} a(x,u,\nabla u)\nabla u dx.\end{aligned}$$

Having in mind (4.9), we can pass to the limit as m tends to $+\infty$ to obtain (4.31).

Step 9: Passing to the limit. Thanks to (4.28) and Lemma (2.5), we obtain

$$a(x,u_n,\nabla u_n)\nabla u_n \to a(x,u,\nabla u)\nabla u \text{ strongly in } L^1(\Omega). \tag{4.32}$$

Let $h \in \mathcal{C}_c^1(\mathbb{R})$ and $\varphi \in \mathcal{D}(\Omega)$. Inserting $h(u_n)\varphi$ as test function in (4.2), we get

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n h'(u_n)\varphi dx + \int_\Omega a(x,u_n,\nabla u_n)\nabla\varphi h(u_n)dx \\ &+\int_\Omega \Phi_n(u_n)\nabla(h(u_n)\varphi)dx + \int_\Omega g_n(x,u_n,\nabla u_n)h(u_n)\varphi dx \\ &\qquad = \langle f_n, h(u_n)\varphi\rangle + \int_\Omega F\nabla(h(u_n)\varphi)dx.\end{aligned} \tag{4.33}$$

We shall pass to the limit as $n \to +\infty$ in each term of the equality (4.33). Since h and h' have compact support on $\mathbb{R}$, there exists a real number $\nu > 0$, such that supp $h \subset [-\nu,\nu]$ and supp $h' \subset [-\nu,\nu]$. For $n > \nu$, we can write

$$\Phi_n(t)h(t) = \Phi(T_\nu(t))h(t) \text{ and } \Phi_n(t)h'(t) = \Phi(T_\nu(t))h'(t).$$

Moreover, the functions Φh and $\Phi h'$ belong to $\left(\mathcal{C}^0(\mathbb{R})\cap L^\infty(\mathbb{R})\right)^N$. Observe first that the sequence $\{h(u_n)\varphi\}_n$ is bounded in $W_0^1L_M(\Omega)$. Indeed, let $\rho > 0$

be a positive constant such that $\|h(u_n)\nabla\varphi\|_\infty \leq \rho$ and $\|h'(u_n)\varphi\|_\infty \leq \rho$. Using the convexity of the $N-$function M and taking into account (4.5) we have

$$\begin{aligned}\int_\Omega M\left(\frac{|\nabla(h(u_n)\varphi)|}{2\rho}\right)dx &\leq \int_\Omega M\left(\frac{|h(u_n)\nabla\varphi| + |h'(u_n)\varphi||\nabla u_n|}{2\rho}\right)dx \\ &\leq \frac{1}{2}M(1)|\Omega| + \frac{1}{2}\int_\Omega M(|\nabla u_n|)dx \\ &\leq \frac{1}{2}M(1)|\Omega| + \frac{1}{2}C_2.\end{aligned}$$

This, together with (4.7), imply that

$$h(u_n)\varphi \rightharpoonup h(u)\varphi \text{ weakly in } W_0^1L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}). \tag{4.34}$$

This enables us to get

$$\langle f_n, h(u_n)\varphi\rangle \to \langle f, h(u)\varphi\rangle.$$

Let E be a measurable subset of Ω. Define $c_\nu = \max_{|t|\leq\nu}\Phi(t)$. Let us denote by $\|v\|_{(M)}$ the Orlicz norm of a function $v \in L_M(\Omega)$. Using strengthened Hölder inequality with both Orlicz and Luxemburg norms, we get

$$\begin{aligned}\|\Phi(T_\nu(u_n))\chi_E\|_{(\overline{M})} &= \sup_{\|v\|_M\leq 1}\left|\int_E \Phi(T_\nu(u_n))v dx\right| \\ &\leq c_\nu \sup_{\|v\|_M\leq 1}\|\chi_E\|_{(\overline{M})}\|v\|_M \\ &\leq c_\nu|E|M^{-1}\Big(\frac{1}{|E|}\Big).\end{aligned}$$

Thus, we get

$$\lim_{|E|\to 0}\sup_n \|\Phi(T_\nu(u_n))\chi_E\|_{(\overline{M})} = 0.$$

Therefore, thanks to (4.7) by applying [27, Lemma 11.2] we obtain

$$\Phi(T_\nu(u_n)) \to \Phi(T_\nu(u)) \text{ strongly in } (E_{\overline{M}})^N,$$

which jointly with (4.34) allow us to pass to the limit in the third term of (4.33) to have

$$\int_\Omega \Phi(T_\nu(u_n))\nabla(h(u_n)\varphi)dx \to \int_\Omega \Phi(T_\nu(u))\nabla(h(u)\varphi)dx.$$

We remark that

$$|a(x,u_n,\nabla u_n)\nabla u_n h'(u_n)\varphi| \leq \rho a(x,u_n,\nabla u_n)\nabla u_n.$$

Consequently, using (4.32) and Vitali's theorem, we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n h'(u_n)\varphi dx \to \int_\Omega a(x,u,\nabla u)\nabla u h'(u)\varphi dx.$$

and

$$\int_\Omega F\nabla u_n h'(u_n)\varphi dx \to \int_\Omega F\nabla u h'(u)\varphi dx.$$

For the second term of (4.33), as above we have

$$h(u_n)\nabla\varphi \to h(u)\nabla\varphi \text{ strongly in } (E_M(\Omega))^N,$$

which together with (4.27) give

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla\varphi h(u_n)dx \to \int_\Omega a(x,u,\nabla u)\nabla\varphi h(u)dx$$

and

$$\int_\Omega F\nabla\varphi h(u_n)dx \to \int_\Omega F\nabla\varphi h(u)dx.$$

The fact that $h(u_n)\varphi \rightharpoonup h(u)\varphi$ weakly in $L^\infty(\Omega)$ for $\sigma^*(L^\infty, L^1)$ and (4.29) enable us to pass to the limit in the fourth term of (4.33) to get

$$\int_\Omega g_n(x,u_n,\nabla u_n)h(u_n)\varphi\, dx \to \int_\Omega g(x,u,\nabla u)h(u)\varphi\, dx.$$

At this point we can pass to the limit in each term of (4.33) to get

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)(\nabla\varphi h(u) + h'(u)\varphi\nabla u)dx + \int_\Omega \Phi(u)h'(u)\varphi\nabla u dx\\ + \int_\Omega \Phi(u)h(u)\nabla\varphi\, dx + \int_\Omega g(x,u,\nabla u)h(u)\varphi\, dx\\ = \langle f, h(u)\varphi\rangle + \textstyle\int_\Omega F(\nabla\varphi h(u) + h'(u)\varphi\nabla u)dx,\end{aligned}$$

for all $h \in \mathcal{C}_c^1(\mathbb{R})$ and for all $\varphi \in \mathcal{D}(\Omega)$. Moreover, as we have (3.5), (4.6) and (4.30) we can use Fatou's lemma to get $g(x,u,\nabla u)u \in L^1(\Omega)$. By virtue of (4.7), (4.27), (4.29), (4.31), the function u is a renormalized solution of problem (1.1).

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