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On Almost n-Layered $QTAG$-Modules

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Abstract. We define the notion of almost n-layered $QTAG$-modules and study their basic properties. One of the main result is that almost 1-layered modules are almost $(\omega + 1)$-projective exactly when they are almost direct sum of countably generated modules of length less than or equal to $(\omega + 1)$. Some other characterizations of this new class are also established.

Keywords: QTAG-modules, Almost Σ-uniserial modules, Almost $(\omega + n)$-projective modules, Almost 1-layered modules.

2000 Mathematics subject classification: 16K20, 13C12, 13C13.

1. Introduction and Background Material

Following [10], a unital module M_R is called a $QTAG$-module if it satisfies the following condition: Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

Through a number of papers it has been seen that the structure theory of these modules is similar to that of torsion abelian groups and that these modules occur over any ring. Here the rings are almost restriction free and the $QTAG$-modules satisfy a simple condition. Several authors worked extensively on these modules. Many interesting results have been obtained, but still there

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remains a lot to explore.

All rings examined in the current paper contain unity ($1 \neq 0$) and modules are unital $QTAG$-modules. A uniserial module M is a module over a ring R, whose submodules are totally ordered by inclusion. This means simply that for any two submodules N_1 and N_2 of M, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. A module is called a serial module if it is a direct of uniserial modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module, and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\left\{d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_n(M)$ denotes the submodule of M generated by the elements of height at least n and $H^n(M)$ is the submodule of M generated by the elements of exponents at most n. M is h-divisible if $M = M^1 = \bigcap_{n=0}^{\infty} H_n(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height. M is called separable if $M^1 = 0$. M is said to be n-bounded, if there exists an integer n such that $H_M(x) \leq n$ for every uniform element $x \in M$.

A submodule N of M is h-pure in M if $N \cap H_n(M) = H_n(N)$, for every integer $n \geq 0$. A submodule $B \subseteq M$ is a basic submodule of M, if B is h-pure in M, $B = \oplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h-divisible. A submodule N of M is said to be $(\omega + n - 1)$-high if $N \cap H_{\omega+n-1}(M) = \{0\}$ and N is maximal with respect to this intersection, that is, it is not properly contained in any different submodule of M having the same property. In particular, if $n = 1$, N is called ω-high in M or just for shortness high in M. A submodule N of M is said to be essential in M if $N \cap K \neq 0$ for every non-zero submodule K of M.

For an ordinal σ, a submodule N of M is said to be σ-pure, if $H_\beta(M) \cap N = H_\beta(N)$ for all $\beta \leq \sigma$ and a submodule N of M is said to be isotype in M, if it is σ-pure for every ordinal σ [7]. A submodule $N \subset M$ is nice [4] in M, if $H_\sigma(M/N) = (H_\sigma(M) + N)/N$ for all ordinals σ, i.e. every coset of M modulo N may be represented by an element of the same height.

A $QTAG$-module M is $(\omega + 1)$-projective if there exists submodule $N \subset H^1(M)$ such that M/N is a direct sum of uniserial modules and a $QTAG$-module M is $(\omega + n)$-projective if there exists a submodule $N \subset H^n(M)$ such that M/N is a direct sum of uniserial modules [5]. If M is a $QTAG$-module, then M is called Σ-uniserial [1], if it is isomorphic to a direct sum of uniserial

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modules.

The sum of all simple submodules of M is called the socle of M and is denoted by $Soc(M)$. The cardinality of a minimal generating set of M is denoted by $g(M)$. For all ordinals α, $f_M(\alpha)$ is the α^{th}-Ulm-Kaplansky invariant of M and it is equal to $g\big(Soc(H_\alpha(M))/Soc(H_{\alpha+1}(M))\big)$.

It is interesting to note that almost all the results which hold for TAG-modules are also valid for $QTAG$-modules [7]. The terminologies and notations are well-known and followed by [8, 9].

2. Main Concepts and Results

We begin with the following crucial concept.

Definition 2.1. The separable $QTAG$-module M is said to be almost Σ-uniserial if it possesses a collection $\mathcal{N}$ consisting of nice submodules of M which satisfies the following three conditions:

(i) $\{0\} \in \mathcal{N}$;

(ii) $\mathcal{N}$ is closed with respect to ascending unions, i.e., if $N_i \in \mathcal{N}$ with $N_i \subseteq N_j$ whenever $i \leq j$ $(i, j \in I)$ then $\cup_{i\in I} N_i \in \mathcal{N}$;

(iii) If K is a countably generated submodule of M, then there is $L \in \mathcal{N}$ (that is, a nice submodule L of M) such that $K \subseteq L$ and L is countably generated.

When M is h-reduced and satisfies clauses (i), (ii) and (iii), it is called almost totally projective, and when M has length not exceeding ω_1, it is called almost direct sum of countably generated modules.

Example 2.2. Let M/N is almost Σ-uniserial for some module M and its countably generated submodule N. We claim that M is almost totally projective.

We see that $H_\omega(M) \subseteq N$ is countably generated and

$$M/N \cong (M/H_\omega(M))/(N/H_\omega(M))$$

is almost Σ-uniserial where $N/H_\omega(M)$ remains countably generated. Infact,

$$N_M \cong (N \oplus H_\omega(M))/H_\omega(M) \subseteq M/H_\omega(M)$$

is almost Σ-uniserial where N_M is a high submodule of M. Thus M must be almost totally projective.

The last concept can be generalized to the following one.

Definition 2.3. The $QTAG$-module M is said to be almost $(\omega+n)$-projective if there exists $N \subseteq H^n(M)$ such that M/N is almost Σ-uniserial.

Clearly, $H_{\omega+n}(M) = \{0\}$.

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EXAMPLE 2.4. There is a module M of length $(\omega+n)$ which is totally projective and almost totally projective, but is not almost $(\omega+n)$-projective.

Let M' be any separable module which is $(\omega+n)$-projective , but not Σ-uniserial. If B is a basic submodule of M', we can let $M = M'/Soc(B)$. It is readily checked that $Soc(M)$ is isometric to $H_\omega(M) \oplus Soc(H_1(B))$, so that M is almost totally projective. Since $M \cong H_1(M')$ is $(\omega+n)$-projective, M is totally projective. Since $M'/Soc(M)$ is not almost Σ-uniserial, M can not be an almost $(\omega+n)$-projective.

In [6, 2, 3], respectively, a $QTAG$-module M is called an n-layered module if some (and hence each) its $(\omega+n-1)$-high submodule is a direct sum of countably generated module. Using the above terminology, one may state the following notion.

Definition 2.5. The $QTAG$-module M is said to be an almost n-layered module if every its $(\omega+n-1)$-high submodule is an almost direct sum of countably generated module.

In particular, when $n = 1$, we have that a $QTAG$-module M is an almost 1-layered module (or for simpleness just an almost layered module) if one (and hence every) its high submodule is an almost Σ-uniserial module.

EXAMPLE 2.6. There exists an n-layered module which is not almost n-layered.

Let M be a $(\omega+n)$-projective module, and let N be a countably generated submodule of M. We claim that $H_1(\bar{N})$ is countably generated. If K is bounded submodule of M such that M/K is Σ-uniserial, then there is a submodule L of M containing K and N such that L/K is a countably generated submodule of M/K. It follows that L is closed in M, so that $\bar{N} \subseteq L$. Since $L = K + P$ for some countably generated submodule P, We have $H_1(\bar{N}) \subseteq H_1(L) = H_1(P)$ is countably generated.

Suppose M is a module containing K such that M/K is Σ-uniserial. Then M is totally projective of length $(\omega+n)$, and hence it is n-layered module of some length, but M is not almost totally projective; even more $M \oplus P$ is not almost totally projective for every almost Σ-uniserial module P. This means that M is not almost n-layered module.

The objective of the present article is to improve some of the results in [6] in the context of the new Definition 2.5. In some cases we will restrict our attention only to $n = 1$. This work is organized thus: in the first section, we have studied the basic notation as well as the terminology necessary for applicable purposes. In the second section, i.e. here, we proceed by proving the chief results, and in the third one we pose some unanswered questions that seem to be interesting.

Before stating our main result, we will first discover some elementary but helpful properties of the so-defined class of almost n-layered modules.

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Lemma 2.7. (i) *Isotype submodules of almost 1-layered modules are again almost 1-layered modules.*
(ii) *Let $n \geq 0$ be an integer. Any $(\omega + n)$-high submodule of an almost direct sum of countably generated module is also an almost direct sum of countably generated module. In particular, almost direct sum of countably generated modules are almost n-layered modules.*

Proof. (i) Suppose K is an isotype submodule of the almost 1-layered module M, and let N be a high submodule of K. Therefore $N \cap H_\omega(M) = N \cap H_\omega(K) = \{0\}$. Thus N can be embedded in a high submodule of M, which submodule must be almost Σ-uniserial. We finally conclude that N is also an almost Σ uniserial module, as desired.
(ii) Assume that L is a $(\omega + n)$-high submodule of the almost direct sum of countably generated module M. Since L is h-pure in M (see, for instance, [8, 9]), we derive that

$$L/H_\omega(L) = L/(L \cap H_\omega(M)) \cong (L + H_\omega(M))/H_\omega(M) \subseteq M/H_\omega(M)$$

is almost Σ-uniserial. But $H_\omega(L)$ is bounded by n, whence we get that N is an almost direct sum of countably generated module, as stated.

The second half is now immediate. □

Corollary 2.8. *Suppose M is a $(\omega + n - 1)$-bounded $QTAG$-module. Then M is an almost n-layered module if and only if M is an almost direct sum of countably generated module.*

In particular, a separable module is an almost 1-layered module if and only if it is an almost Σ-uniserial module.

Proof. The"if" part follows directly from Lemma 2.7.

As for the"and only if" part, since $H_{\omega+n-1}(M) = \{0\}$, the module M is $(\omega + n - 1)$-high of itself, so that Definition 2.5 works.

Finally, to deduce the second claim, just take $n = 1$, and we are done. □

The above assertion can be extended to the following.

Proposition 2.9. *Let M be a $QTAG$-module for which $H_{\omega+n-1}(M)$ is countably generated. Then M is an almost n-layered module if and only if it is an almost direct sum of countably generated module.*

Proof. " $\Rightarrow$ ". Letting N be a $(\omega + n - 1)$-high submodule of M, we observe that $H_{\omega+n-1}(M)$ embeds isomorphically in an essential submodule of M/N, that is , $H_{\omega+n-1}(M) \cong (N \oplus H_{\omega+n-1}(M))/N$. In fact, we will show that $[(N \oplus H_{\omega+n-1}(M))/N] \cap (M'/N) \neq \{0\}$ for any module M' such that $N \subset M' \subset M$. To that aim, assuming the contrary $[(N \oplus H_{\omega+n-1}(M))/N] \cap (M'/N) = \{0\}$, one may see that

$$[(N \oplus H_{\omega+n-1}(M)) \cap M']/N = [(N + H_{\omega+n-1}(M)) \cap M']/N = \{0\}$$

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or equivalently $H_{\omega+n-1}(M) \cap M' \subseteq N$. This means that $H_{\omega+n-1}(M) \cap M' = H_{\omega+n-1}(M) \cap N = \{0\}$ which contradicts the maximality of N. This substantiates our claim.

Furthermore, it follows that M/N is countably generated and hence so will be its epimorphic image $M/(N \oplus H_{\omega+n-1}(M)) \cong M/H_{\omega+n-1}(M)/(N \oplus H_{\omega+n-1}(M))/H_{\omega+n-1}(M)$. However, $N \cong (N \oplus H_{\omega+n-1}(M))/H_{\omega+n-1}(M)$ is an almost direct sum of countably generated module, and so we conclude that $M/H_{\omega+n-1}(M)$ is an almost direct sum of countably generated modules. Thus, M is an almost direct sum of countably generated modules, as desired.

" $\Leftarrow$ ". Follows immediately from Lemma 2.7. □

The following statement is useful for further applications.

Proposition 2.10. *Suppose $\beta \geq \omega$ is an ordinal. Then M is an almost 1-layered module if and only if $M/H_{\beta+1}(M)$ is an almost 1-layered module.*

Proof. If N is high submodule of M, then $(N \oplus H_{\beta+1}(M))/H_{\beta+1}(M) \cong N$ should be high in $M/H_{\beta+1}(M)$. Given N is a high submodule of M, we derive that N is isotype in M and hence it plainly follows that $(N \oplus H_\beta(M))/H_\beta(M) \cong N$ is isotype in $M/H_\beta(M)$. Thus Lemma 2.7 enables us that N is an almost 1-layered module, which means by Corollary 2.8 that it is almost Σ-uniserial, as expected. □

Theorem 2.11. (i) *Suppose that M is an almost n-layered QTAG-module. Then each almost n-layered module is almost $(\omega+n)$-projective if and only if it is an almost direct sum of countably generated module of length not exceeding $(\omega + n)$.*
(ii) *For any $n \geq 1$, every almost n-layered module is almost $(\omega+n-1)$-projective if and only if it is an almost direct sum of countably generated module of length not exceeding $(\omega + n - 1)$.*

Proof. (i) "Necessity". By Definition 2.3, we write that M/K is almost Σ-uniserial for some $K \subseteq Soc(M)$. Hence $H_\omega(M) \subseteq K$ and thus $M/K \cong M/H_\omega(M)/K/H_\omega(M)$. But $K/H_\omega(M) \subseteq (Soc(M) + H_\omega(M))/H_\omega(M) = (Soc(N) \oplus H_\omega(M))/H_\omega(M) \subseteq (N \oplus H_\omega(M))/H_\omega(M) \cong N$ is almost Σ-uniserial, because $Soc(M) = Soc(N) \oplus Soc(H_\omega(M))$ for some high submodule N of M. Besides, it is not hard to verify that $(N \oplus H_\omega(M))/H_\omega(M)$ is h-pure in $M/H_\omega(M)$ since N is necessarily h-pure in M. Therefore, the quotient $M/H_\omega(M)$ is almost Σ-uniserial. Since $H_1(H_\omega(M)) = \{0\}$, which allows us to infer that M is an almost direct sum of countably generated module, as stated.
"Sufficiency". The quotient $M/H_\omega(M)$ is almost Σ-uniserial and $H_n(H_\omega(M)) = \{0\}$, so that with Definition 2.3 at hand we are done.

As for the second part for $n \geq 2$, one may infer that $H^n(M) = H^n(N) \oplus H^n(H_\omega(M))$, and therefore we observe that the given above idea is workable to conclude the claim.

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(ii) "Necessity". Since $H_{\omega+n-1}(M) = \{0\}$, we simply apply Corollary 2.8. "Sufficiency". Since the quotient $M/H_\omega(M)$ is almost Σ-uniserial, and $H_{n-1}(H_\omega(M)) = \{0\}$, Definition 2.3 is applied. □

Note that part (i) of the last assertion can be slightly strengthened like this:

Proposition 2.12. *If M is an almost 1-layered module such that $M/H_{\omega+1}(M)$ is almost $(\omega+1)$-projective, then $M/H_\omega(M)$ is an almost Σ-uniserial module.*

Proof. Observe that

$$\begin{aligned} M/H_\omega(M) &\cong M/H_{\omega+1}(M)/H_\omega(M)/H_{\omega+1}(M) \\ &= M/H_{\omega+1}(M)/H_\omega(M/H_{\omega+1}(M)). \end{aligned}$$

But, by virtue of Proposition 2.10, the quotient $M/H_{\omega+1}(M)$ is also an almost 1-layered module. Hence, in view of Theorem 2.11 (i) the quotient $M/H_{\omega+1}(M)$ should be an almost direct sum of countably generated module. Thus the above isomorphism sequence assure that $M/H_\omega(M)$ is an almost Σ-uniserial module, as claimed. □

Theorem 2.13. *Suppose M is a $QTAG$-module. Then the following conditions are equivalent:*

(i) *M is an almost 1-layered module and $H_\omega(M)$ is countably generated;*
(ii) *$M/H_\omega(M)$ is almost Σ-uniserial and $H_\omega(M)$ is countably generated;*
(iii) *$M \cong K+L$ where K is countably generated and L is almost Σ-uniserial.*

Proof. The equivalence $(ii) \Leftrightarrow (iii)$ is obvious.
$(i) \Rightarrow (ii)$. Supposing N is a high submodule of M, we know that it is almost Σ-uniserial. Since $H_\omega(M)$ embeds isomorphically in an essential submodule of M/N, especially $H_\omega(M) \cong (N \oplus H_\omega(M))/N$ where, as demonstrated in Proposition 2.9, the latter obviously has a non-trivial intersection with any proper submodule of M/N, it follows that M/N is countably generated whence so is $M/(N \oplus H_\omega(M))$ being its epimorphic image. Furthermore, one may write that

$$M/(N \oplus H_\omega(M)) \cong M/H_\omega(M)/(N \oplus H_\omega(M))/H_\omega(M)$$

is countably generated, where $(N \oplus H_\omega(M))/H_\omega(M) \cong N$ is almost Σ-uniserial. Consequently, we derive that $M/H_\omega(M)$ is almost Σ-uniserial, as asserted.
$(i) \Leftarrow (ii)$. Letting N be a high submodule of M, we deduce that $N \cong (N \oplus H_\omega(M))/H_\omega(M) \subseteq M/H_\omega(M)$ is almost Σ-uniserial, as needed. □

So, we are ready to proceed by proving our main result.

Theorem 2.14. *Suppose that M is an almost 1-layered module of length $\leq \omega+1$. Then M is an almost direct sum of countably generated module if and only if there exists a nice submodule $K \subseteq M$ such that $K \cap H_\omega(M) = \{0\}$ and M/K is an almost direct sum of countably generated module.*

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Proof. "Necessity". Just take $K = H_{\omega+1}(M) = \{0\}$.
"Sufficiency". We observe that

$$M/K/H_\omega(M/K) \cong M/(H_\omega(M) \oplus K) \cong M/H_\omega(M)/(H_\omega(M) \oplus K)/H_\omega(M)$$

is an almost Σ-uniserial module. However, $K \cap H_\omega(M) = \{0\}$ implies that K can be embedded in a high submodule N of M. But N is h-pure in M as well as by assumption it has to be also almost Σ-uniserial module.

On the other hand, $(H_\omega(M) \oplus K)/H_\omega(M) \subseteq (H_\omega(M) \oplus N)/H_\omega(M) \cong N$ where the latter quotient is obviously checked to be h-pure in $M/H_\omega(M)$. Consequently, we find that $M/H_\omega(M)$ is almost Σ-uniserial. And since $H_\omega(M)$ is bounded, we get finally that M is an almost direct sum of countably generated module, as promised. □

3. Open Problems

In closing, we pose the following questions of interest:

Problem 3.1. *Does it follow that isotype submodules of countable length of almost direct sum of countably generated modules are again an almost direct sum of countably generated modules?*

Problem 3.2. *Let $n > 1$ be a natural. Is it true that if one $(\omega + n - 1)$-high submodule of a QTAG-module is an almost direct sum of countably generated module, then all $(\omega + n - 1)$-high submodules are also almost direct sum of countably generated modules?*

Problem 3.3. *Suppose n is a positive integer. Is M an almost n-layered module if and only if M/K is an almost n-layered module, provided K is countably generated nice submodule of M?*

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