

## Left Annihilator of Identities Involving Generalized Derivations in Prime Rings

Basudeb Dhara<sup>a,\*</sup>, Krishna Gopal Pradhan<sup>b</sup> and Shailesh Kumar Tiwari<sup>c</sup>

<sup>a</sup>Department of Mathematics, Belda College, Belda, Paschim Medinipur,  
 721424, W.B., India.

<sup>b</sup>Department of Mathematics, Midnapore City College, Bhadutala, Paschim  
 Medinipur, W.B.- 721129 India.

<sup>c</sup>Department of Mathematics, Indian Institute of Technology Delhi, Hauz  
 Khas, New Delhi-110016, India.

E-mail: basu.dhara@yahoo.com

E-mail: kgp.math@gmail.com

E-mail: shaileshiitd84@gmail.com

ABSTRACT. Let  $R$  be a prime ring with its Utumi ring of quotients  $U$ ,  $C = Z(U)$  the extended centroid of  $R$ ,  $L$  a non-central Lie ideal of  $R$  and  $0 \neq a \in R$ . If  $R$  admits a generalized derivation  $F$  such that  $a(F(u^2) \pm F(u)^2) = 0$  for all  $u \in L$ , then one of the following holds:

- (1) there exists  $b \in U$  such that  $F(x) = bx$  for all  $x \in R$ , with  $ab = 0$ ;
- (2)  $F(x) = \mp x$  for all  $x \in R$ ;
- (3)  $\text{char}(R) = 2$  and  $R$  satisfies  $s_4$ ;
- (4)  $\text{char}(R) \neq 2$ ,  $R$  satisfies  $s_4$  and there exists  $b \in U$  such that  $F(x) = bx$  for all  $x \in R$ .

We also study the situations (i)  $a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0$  for all  $x, y \in R$ , and (ii)  $a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0$  for all  $x, y \in R$  in prime and semiprime rings.

**Keywords:** Prime ring, Generalized derivation, Utumi quotient ring.

**2010 Mathematics subject classification:** 16W25, 16R50, 16N60.

---

\*Corresponding Author

Received 30 January 2016; Accepted 15 May 2017

©2017 Academic Center for Education, Culture and Research TMU

## 1. INTRODUCTION

Let  $R$  be an associative prime ring with center  $Z(R)$  and  $U$  the Utumi quotient ring of  $R$ . The center of  $U$ , denoted by  $C$ , is called the extended centroid of  $R$  (we refer the reader to [2] for these objects). For given  $x, y \in R$ , the Lie commutator of  $x, y$  is denoted by  $[x, y] = xy - yx$ . An additive mapping  $d : R \rightarrow R$  is called a derivation, if it satisfies the rule  $d(xy) = d(x)y + xd(y)$  for all  $x, y \in R$ . In particular,  $d$  is said to be an inner derivation induced by an element  $a \in R$ , if  $d(x) = [a, x]$  for all  $x \in R$ . In [5], Bresar introduced the definition of generalized derivation: An additive mapping  $F : R \rightarrow R$  is called generalized derivation, if there exists a derivation  $d : R \rightarrow R$  such that  $F(xy) = F(x)y + xd(y)$  holds for all  $x, y \in R$ .

Let  $S$  be a nonempty subset of  $R$  and  $F : R \rightarrow R$  be an additive mapping. Then we say that  $F$  acts as homomorphism or anti-homomorphism on  $S$  if  $F(xy) = F(x)F(y)$  or  $F(xy) = F(y)F(x)$  holds for all  $x, y \in S$  respectively. The additive mapping  $F$  acts as a Jordan homomorphism on  $S$  if  $F(x^2) = F(x)^2$  holds for all  $x \in S$ .

Many results in literature indicate that global structure of a prime ring  $R$  is often tightly connected to the behavior of additive mappings defined on  $R$ . Asma, Rehman, Shakir in [1] proved that if  $d$  is a derivation of a 2-torsion free prime ring  $R$  which acts as a homomorphism or anti-homomorphism on a non-central Lie ideal of  $R$  such that  $u^2 \in L$ , for all  $u \in L$ , then  $d = 0$ . At this point the natural question is what happens in case the derivation is replaced by generalized derivation. Some papers have investigated, when generalized derivation  $F$  acts as homomorphism or anti-homomorphism on some subsets of  $R$  and then determined the structure of ring  $R$  as well as associated map  $F$  (see [1, 3, 8, 9, 11, 12, 13, 14, 15, 16, 18, 19, 26, 27]). In [18] Golbasi and Kaya proved the following: Let  $R$  be a prime ring of characteristic different from 2,  $F$  a generalized derivation of  $R$  associated to a derivation  $d$ ,  $L$  a Lie ideal of  $R$  such that  $u^2 \in L$  for all  $u \in L$ . If  $F$  acts as a homomorphism or anti-homomorphism on  $L$ , then either  $d = 0$  or  $L$  is central in  $R$ . More recently in [9], Filippis studied the situation when generalized derivation  $F$  acts as a Jordan homomorphism on a non-central Lie ideal  $L$  of  $R$ .

Recently in [26], Rehman and Raza proved the following: Let  $R$  be a prime ring of char  $(R) \neq 2$ ,  $Z$  the center of  $R$ , and  $L$  a nonzero Lie ideal of  $R$ . If  $R$  admits a generalized derivation  $F$  associated with a derivation  $d$  which acts as a homomorphism or as anti-homomorphism on  $L$ , then either  $d = 0$  or  $L \subseteq Z$ .

In the above result, Rehman and Raza [26] did not give the complete structure of the map  $F$ .

In the present article, we investigate the situations with left annihilator condition and we determine the structure of generalized derivation map  $F$ . The main results of this paper are as follows:

**Theorem 1.1.** *Let  $R$  be a prime ring with its Utumi ring of quotients  $U$ ,  $C = Z(U)$  the extended centroid of  $R$ ,  $L$  a non-central Lie ideal of  $R$  and  $0 \neq a \in R$ . If  $R$  admits a generalized derivation  $F$  such that  $a(F(u^2) \pm F(u)^2) = 0$  for all  $u \in L$ , then one of the following holds:*

- (1) *there exists  $b \in U$  such that  $F(x) = bx$  for all  $x \in R$ , with  $ab = 0$ ;*
- (2)  *$F(x) = \mp x$  for all  $x \in R$ ;*
- (3)  *$\text{char}(R) = 2$  and  $R$  satisfies  $s_4$ ;*
- (4)  *$\text{char}(R) \neq 2$ ,  $R$  satisfies  $s_4$  and there exists  $b \in U$  such that  $F(x) = bx$  for all  $x \in R$ .*

**Theorem 1.2.** *Let  $R$  be a noncommutative prime ring of characteristic different from 2 with its Utumi ring of quotients  $U$ ,  $C = Z(U)$  the extended centroid of  $R$ ,  $F$  a generalized derivation on  $R$  and  $0 \neq a \in R$ .*

- (1) *If  $a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0$  for all  $x, y \in R$ , then there exists  $b \in U$  such that  $F(x) = bx$  for all  $x \in R$ , with  $ab = 0$  or  $F(x) = \mp x$  for all  $x \in R$ .*
- (2) *If  $a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0$  for all  $x, y \in R$ , then there exists  $b \in U$  such that  $F(x) = bx$  for all  $x \in R$ , with  $ab = 0$ .*

**Theorem 1.3.** *Let  $R$  be a noncommutative 2-torsion free semiprime ring,  $U$  the left Utumi quotient ring of  $R$ ,  $C = Z(U)$  the extended centroid of  $R$ ,  $F(x) = bx + d(x)$  a generalized derivation on  $R$  associated to the derivation  $d$  and  $0 \neq a \in R$ . If any one of the following holds:*

- (1)  *$a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0$  for all  $x, y \in R$ ,*
- (2)  *$a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0$  for all  $x, y \in R$ ,*

*then there exist orthogonal central idempotents  $e_1, e_2, e_3 \in U$  with  $e_1 + e_2 + e_3 = 1$  such that  $d(e_1 U) = 0$ ,  $e_2 a = 0$ , and  $e_3 U$  is commutative.*

The following remarks are useful tools for the proof of main results.

**Remark 1.4.** Let  $R$  be a prime ring and  $L$  a noncentral Lie ideal of  $R$ . If  $\text{char}(R) \neq 2$ , by [4, Lemma 1] there exists a nonzero ideal  $I$  of  $R$  such that  $0 \neq [I, R] \subseteq L$ . If  $\text{char}(R) = 2$  and  $\dim_C RC > 4$ , i.e.,  $\text{char}(R) = 2$  and  $R$  does not satisfy  $s_4$ , then by [22, Theorem 13] there exists a nonzero ideal  $I$  of  $R$  such that  $0 \neq [I, R] \subseteq L$ . Thus if either  $\text{char}(R) \neq 2$  or  $R$  does not satisfy  $s_4$ , then we may conclude that there exists a nonzero ideal  $I$  of  $R$  such that  $[I, I] \subseteq L$ .

**Remark 1.5.** We denote by  $\text{Der}(U)$  the set of all derivations on  $U$ . By a derivation word  $\Delta$  of  $R$  we mean  $\Delta = d_1 d_2 d_3 \dots d_m$  for some derivations  $d_i \in \text{Der}(U)$ .

Let  $D_{\text{int}}$  be the  $C$ -subspace of  $\text{Der}(U)$  consisting of all inner derivations on  $U$  and let  $d$  be a non-zero derivation on  $R$ . By [21, Theorem 2] we have the following result:

If  $\Phi(x_1, x_2, \dots, x_n, d(x_1), d(x_2), \dots, d(x_n))$  is a differential identity on  $R$ , then one of the following holds:

- (1)  $d \in D_{int}$ ;
- (2)  $R$  satisfies the generalized polynomial identity  $\Phi(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n)$ .

*Remark 1.6.* In [23], Lee extended the definition of generalized derivation as follows: by a generalized derivation we mean an additive mapping  $F : I \rightarrow U$  such that  $F(xy) = F(x)y + xd(y)$  holds for all  $x, y \in I$ , where  $I$  is a dense left ideal of  $R$  and  $d$  is a derivation from  $I$  into  $U$ . Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of  $U$ , and thus all generalized derivations of  $R$  will be implicitly assumed to be defined on the whole of  $U$ . Lee obtained the following: every generalized derivation  $F$  on a dense left ideal of  $R$  can be uniquely extended to  $U$  and assumes the form  $F(x) = ax + d(x)$  for some  $a \in U$  and a derivation  $d$  on  $U$ .

## 2. PROOF OF THE MAIN RESULTS

Now we begin with the following Lemmas:

**Lemma 2.1.** *Let  $R = M_2(C)$  be the ring of all  $2 \times 2$  matrices over the field  $C$  of characteristic different from 2 and  $b, c \in R$ . Suppose that there exists  $0 \neq a \in R$  such that*

$$a\{b[x, y]^2 + [x, y]^2c - (b[x, y] + [x, y]c)^2\} = 0,$$

*for all  $x, y \in R$ . Then  $c \in C \cdot I_2$ .*

*Proof.* If  $c \in C \cdot I_2$ , then nothing to prove. Let  $c \notin C \cdot I_2$ . In this case  $R$  is a dense ring of  $C$ -linear transformations over a vector space  $V$ . Assume that there exists  $0 \neq v \in V$  such that  $\{v, cv\}$  is linearly  $C$ -independent. By density, there exist  $x, y \in R$  such that  $xv = v, xcv = 0; yv = 0, ycv = v$ . Then  $[x, y]v = 0$ ,  $[x, y]cv = v$  and hence  $a\{b[x, y]^2 + [x, y]^2c - (b[x, y] + [x, y]c)^2\}v = av$ .

Of course for any  $u \in V$ ,  $\{u, v\}$  linearly  $C$ -dependent implies  $au = 0$ . Since  $a \neq 0$ , there exists  $w \in V$  such that  $aw \neq 0$  and so  $\{w, v\}$  are linearly  $C$ -independent. Also  $a(w + v) = aw \neq 0$  and  $a(w - v) = aw \neq 0$ . By the above argument, it follows that  $w$  and  $cw$  are linearly  $C$ -dependent, as are  $\{w + v, c(w + v)\}$  and  $\{w - v, c(w - v)\}$ . Therefore there exist  $\alpha_w, \alpha_{w+v}, \alpha_{w-v} \in C$  such that

$$cw = \alpha_w w, \quad c(w + v) = \alpha_{w+v}(w + v), \quad c(w - v) = \alpha_{w-v}(w - v).$$

In other words we have

$$\alpha_w w + cv = \alpha_{w+v} w + \alpha_{w+v} v \tag{2.1}$$

and

$$\alpha_w w - cv = \alpha_{w-v} w - \alpha_{w-v} v. \tag{2.2}$$

By comparing (2.1) with (2.2) we get both

$$(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0 \quad (2.3)$$

and

$$2cv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v. \quad (2.4)$$

By (2.3), and since  $\{w, v\}$  are  $C$ -independent and  $\text{char}(R) \neq 2$ , we have  $\alpha_w = \alpha_{w+v} = \alpha_{w-v}$ . Thus by (2.4) it follows  $2cv = 2\alpha_w v$ . This leads a contradiction with the fact that  $\{v, cv\}$  is linear  $C$ -independent.

In light of this, we may assume that for any  $v \in V$  there exists a suitable  $\alpha_v \in C$  such that  $cv = \alpha_v v$ , and standard argument shows that there is  $\alpha \in C$  such that  $cv = \alpha v$  for all  $v \in V$ . Now let  $r \in R$ ,  $v \in V$ . Since  $cv = \alpha v$ ,

$$[c, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0.$$

Thus  $[c, r]v = 0$  for all  $v \in V$  i.e.,  $[c, r]V = 0$ . Since  $[c, r]$  acts faithfully as a linear transformation on the vector space  $V$ ,  $[c, r] = 0$  for all  $r \in R$ . Therefore,  $c \in Z(R)$ , a contradiction.  $\square$

**Lemma 2.2.** *Let  $R = M_2(C)$  be the ring of all  $2 \times 2$  matrices over the field  $C$  of characteristic different from 2 and  $0 \neq p \in R$ . Suppose that there exists  $0 \neq a \in R$  such that*

$$a(px^m y^n - px^m py^n) = 0,$$

*for all  $x, y \in R$ . Then either  $ap = 0$  or  $p = 1$ .*

*Proof.* Putting  $x = y = I_2$ , we get  $ap = ap^2$ . In this case  $R$  is a dense ring of  $C$ -linear transformations over a vector space  $V$ . Assume that there exists  $0 \neq v \in V$  such that  $\{v, pv\}$  is linearly  $C$ -independent. By density, there exist  $x, y \in R$  such that  $xv = v, xpv = 0; yv = v, ypv = 0$ . Then we get  $0 = a(px^m y^n - px^m py^n)v = apv$ . Then by same argument as in Lemma 2.1, we get either  $ap = 0$  or  $p \in C \cdot I_2$ . When  $0 \neq p \in C \cdot I_2$ , from  $ap = ap^2$ , we get  $0 = a(p - 1)$ . Since  $a \neq 0$ , we conclude  $p = 1$ .  $\square$

**Lemma 2.3.** *Let  $R = M_2(C)$  be the ring of all  $2 \times 2$  matrices over the field  $C$  of characteristic different from 2 and  $0 \neq p \in R$ . Suppose that there exists  $0 \neq a \in R$  such that*

$$a(px^m y^n - py^n px^m) = 0,$$

*for all  $x, y \in R$ . Then  $ap = 0$ .*

*Proof.* Putting  $x = y = I_2$ , we get  $ap = ap^2$ . Here  $R$  is a dense ring of  $C$ -linear transformations over a vector space  $V$ . Assume that there exists  $0 \neq v \in V$  such that  $\{v, pv\}$  is linearly  $C$ -independent. By density, there exist  $x, y \in R$  such that  $xv = v, xpv = 0; yv = 0, ypv = pv$ . Then we have  $0 = a(px^m y^n - py^n px^m)v = -ap^2 v = -apv$ . Then by same argument as in Lemma 2.1, we get either  $ap = 0$  or  $p \in C \cdot I_2$ . When  $0 \neq p \in C \cdot I_2$ , by hypothesis, we get  $0 = a[x^m, y^n]$ . Then for  $x = e_{11}$  and  $y = e_{11} + e_{12}$ , we have

$0 = a[x^m, y^n] = a[e_{11}, e_{11} + e_{12}] = ae_{12}$ . Again, for  $x = e_{22}$  and  $y = e_{22} + e_{21}$ , we have  $0 = a[x^m, y^n] = a[e_{22}, e_{22} + e_{21}] = ae_{21}$ . These imply  $a = 0$ , a contradiction.  $\square$

**Lemma 2.4.** *Let  $R$  be a noncommutative prime ring with extended centroid  $C$  and  $b, c \in R$ . Suppose that  $0 \neq a \in R$  such that*

$$a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\} = 0$$

*for all  $x, y \in R$ . Then one of the following holds:*

- (1)  $c \in C$  and  $a(b + c) = 0$ ;
- (2)  $b, c \in C$  and  $b + c = 1$ ;
- (3)  $\text{char}(R) = 2$  and  $R$  satisfies  $s_4$ ;
- (4)  $\text{char}(R) \neq 2$ ,  $R$  satisfies  $s_4$  and  $c \in C$ .

*Proof.* By assumption,  $R$  satisfies the generalized polynomial identity (GPI)

$$f(x, y) = a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\}.$$

By Chuang [6, Theorem 2], this generalized polynomial identity (GPI) is also satisfied by  $U$ . Now we consider the following two cases:

*Case-I.  $U$  does not satisfy any nontrivial GPI.*

Let  $T = U *_C C\{x, y\}$ , the free product of  $U$  and  $C\{x, y\}$ , the free  $C$ -algebra in noncommuting indeterminates  $x$  and  $y$ . Thus

$$a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\}$$

is zero element in  $T = U *_C C\{x, y\}$ . Let  $c \notin C$ . Then  $\{1, c\}$  is  $C$ -independent. Then from above

$$a\{[x, y]^2c - (b[x, y] + [x, y]c)[x, y]c\}$$

which is

$$a\{[x, y] - b[x, y] - [x, y]c\}[x, y]c,$$

is zero in  $T$ . Again, since  $c \notin C$ , we have that  $a[x, y]c[x, y]c$  is zero element in  $T$ , implying  $a = 0$  or  $c = 0$ , a contradiction. Thus we conclude that  $c \in C$ . Then the identity reduces to

$$a\{(b + c)[x, y] - (b + c)[x, y](b + c)\}[x, y],$$

is zero element in  $T$ . Again, if  $b + c \notin C$ , then  $a(b + c)[x, y]^2$  becomes zero element in  $T$ , implying  $a(b + c) = 0$ . If  $b + c \in C$ , then  $a(b + c)(b + c - 1)[x, y]^2$  becomes zero element in  $T$ , implying  $b + c = 0$  or  $b + c = 1$ . When  $b + c = 0$ , then  $a(b + c) = 0$ , which is our conclusion (1). When  $b + c = 1$ , then  $b = 1 - c \in C$ , which is our conclusion (2).

*Case-II.  $U$  satisfies a nontrivial GPI.*

Thus we assume that

$$a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\} = 0,$$

is a nontrivial GPI for  $U$ . In case  $C$  is infinite, we have  $f(x, y) = 0$  for all  $x, y \in U \otimes_C \overline{C}$ , where  $\overline{C}$  is the algebraic closure of  $C$ . Since both  $U$  and  $U \otimes_C \overline{C}$  are prime and centrally closed [17], we may replace  $R$  by  $U$  or  $U \otimes_C \overline{C}$  according to  $C$  finite or infinite. Thus we may assume that  $R$  centrally closed over  $C$  which either finite or algebraically closed and  $f(x, y) = 0$  for all  $x, y \in R$ . By Martindale's Theorem [25],  $R$  is then primitive ring having non-zero socle  $\text{soc}(R)$  with  $C$  as the associated division ring. Hence by Jacobson's Theorem [20],  $R$  is isomorphic to a dense ring of linear transformations of a vector space  $V$  over  $C$ . Since  $R$  is noncommutative,  $\dim_C V \geq 2$ . If  $\dim_C V = 2$ , then  $R \cong M_2(C)$ . In this case by Lemma 2.1, either  $c \in C$  or  $\text{char}(R) = 2$ . This gives conclusions (3) and (4).

Let  $\dim_C V \geq 3$ . Let for some  $v \in V$ ,  $v$  and  $cv$  are linearly independent over  $C$ . By density there exist  $x, y \in R$  such that

$$xv = v, \quad xcv = 0;$$

$$yv = 0, \quad ycv = v.$$

Then  $[x, y]v = 0$ ,  $[x, y]cv = v$  and hence  $a\{(b[x, y]^2 + [x, y]^2c) - (b[x, y] + [x, y]c)^2\}v = av$ .

This implies that if  $av \neq 0$ , then by contradiction we may conclude that  $v$  and  $cv$  are linearly  $C$ -dependent. Now choose  $v \in V$  such that  $v$  and  $cv$  are linearly  $C$ -independent. Set  $W = \text{Span}_C\{v, cv\}$ . Then  $av = 0$ . Since  $a \neq 0$ , there exists  $w \in V$  such that  $aw \neq 0$  and then  $a(v - w) = aw \neq 0$ . By the previous argument we have that  $w, cw$  are linearly  $C$ -dependent and  $(v - w), c(v - w)$  too. Thus there exist  $\alpha, \beta \in C$  such that  $cw = \alpha w$  and  $c(v - w) = \beta(v - w)$ . Then  $cv = \beta(v - w) + cw = \beta(v - w) + \alpha w$  i.e.,  $(\alpha - \beta)w = cv - \beta v \in W$ . Now  $\alpha = \beta$  implies that  $cv = \beta v$ , a contradiction. Hence  $\alpha \neq \beta$  and so  $w \in W$ . Again, if  $u \in V$  with  $au = 0$  then  $a(w + u) \neq 0$ . So,  $w + u \in W$  forcing  $u \in W$ . Thus it is observed that  $w \in V$  with  $aw \neq 0$  implies  $w \in W$  and  $u \in V$  with  $au = 0$  implies  $u \in W$ . This implies that  $V = W$  i.e.,  $\dim_C V = 2$ , a contradiction.

Hence, in any case,  $v$  and  $cv$  are linearly  $C$ -dependent for all  $v \in V$ . Thus for each  $v \in V$ ,  $cv = \alpha_v v$  for some  $\alpha_v \in C$ . It is very easy to prove that  $\alpha_v$  is independent of the choice of  $v \in V$ . Thus we can write  $cv = \alpha v$  for all  $v \in V$  and  $\alpha \in C$  fixed. Now let  $r \in R$ ,  $v \in V$ . Since  $cv = \alpha v$ ,

$$[c, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0.$$

Thus  $[c, r]v = 0$  for all  $v \in V$  i.e.,  $[c, r]V = 0$ . Since  $[c, r]$  acts faithfully as a linear transformation on the vector space  $V$ ,  $[c, r] = 0$  for all  $r \in R$ . Therefore,  $c \in Z(R)$ .

Thus our identity reduces to

$$a\{(b'[x, y]^2) - (b'[x, y])^2\} = 0,$$

for all  $x, y \in R$ , where  $b' = b + c$ .

Let for some  $v \in V$ ,  $v$  and  $b'v$  are linearly independent over  $C$ . Since  $\dim_C V \geq 3$ , there exists  $u \in V$  such that  $v, b'v, u$  are linearly independent over  $C$ . By density there exist  $x, y \in R$  such that

$$xv = v, \quad xb'v = 0, \quad xu = v;$$

$$yv = 0, \quad yb'v = u, \quad yu = v.$$

Then  $[x, y]v = 0$ ,  $[x, y]b'v = v$ ,  $[x, y]u = v$  and hence  $0 = a\{(b'[x, y]^2) - (b'[x, y])^2\}u = ab'v$ . Then by same argument as before, we have either  $ab' = 0$  or  $v$  and  $b'v$  are linearly  $C$ -dependent for all  $v \in V$ . In the first case,  $0 = ab' = a(b + c)$ , which is conclusion (1). In the last case, again by standard argument, we have that  $b' \in C$ . If  $b' = 0$ , then also  $ab' = a(b + c) = 0$  which gives conclusion (1). So assume that  $0 \neq b' \in C$ . Then our identity reduces to

$$ab'(b' - 1)[x, y]^2 = 0,$$

for all  $x, y \in R$ . This gives  $0 = ab'(b' - 1) = a(b' - 1)$ . Since  $a \neq 0$ , we get  $b' = 1$ . This gives conclusion (2).  $\square$

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** First we consider the case when

$$a(F(u^2) - F(u)^2) = 0,$$

for all  $u \in L$ . If  $\text{char}(R) = 2$  and  $R$  satisfies  $s_4$ , then we have our conclusion (3). So we assume that either  $\text{char}(R) \neq 2$  or  $R$  does not satisfy  $s_4$ . Since  $L$  is a noncentral by Remark 1.4, there exists a nonzero ideal  $I$  of  $R$  such that  $[I, I] \subseteq L$ . Thus by assumption  $I$  satisfies the differential identity

$$a(F([x, y]^2) - F([x, y])^2) = 0.$$

Now since  $R$  is a prime ring and  $F$  is a generalized derivation of  $R$ , by Lee [23, Theorem 3],  $F(x) = bx + d(x)$  for some  $b \in U$  and derivation  $d$  on  $U$ . Since  $I, R$  and  $U$  satisfy the same differential identities [24], without loss of generality,  $U$  satisfies

$$a(b[x, y]^2 + d([x, y]^2) - (b[x, y] + d([x, y]))^2) = 0. \quad (2.5)$$

Here we divide the proof into two cases:



*Case 1.* Let  $d$  be inner derivation induced by element  $c \in U$ , that is,  $d(x) = [c, x]$  for all  $x \in U$ . It follows that

$$a(b[x, y]^2 + [c, [x, y]^2] - (b[x, y] + [c, [x, y]]^2) = 0,$$

that is

$$a((b+c)[x, y]^2 - [x, y]^2 c - ((b+c)[x, y] - [x, y]c)^2) = 0,$$

for all  $x, y \in U$ . Now by Lemma 2.4, one of the following holds:

- (1)  $c \in C$  and  $0 = a(b+c-c) = ab$ . Thus  $F(x) = bx$  for all  $x \in R$ , with  $ab = 0$ .
- (2)  $b+c, c \in C$  and  $b+c-c = 1$ . Thus  $F(x) = x$  for all  $x \in R$ .
- (3)  $\text{char}(R) \neq 2$ ,  $R$  satisfies  $s_4$  and  $c \in C$ . Thus  $F(x) = bx$  for all  $x \in R$ .

*Case 2.* Assume that  $d$  is not inner derivation of  $U$ . We have from (2.5) that  $U$  satisfies

$$a(b[x, y]^2 + d([x, y])[x, y] + [x, y]d([x, y]) - (b[x, y] + d([x, y]))^2) = 0,$$

that is

$$a(b[x, y]^2 + ([d(x), y] + [x, d(y)])[x, y] + [x, y]([d(x), y] + [x, d(y)]) - (b[x, y] + [d(x), y] + [x, d(y)])^2) = 0.$$

Then by Kharchenko's Theorem [21],  $U$  satisfies

$$a(b[x, y]^2 + ([u, y] + [x, z])[x, y] + [x, y]([u, y] + [x, z]) - (b[x, y] + [u, y] + [x, z])^2) = 0. \quad (2.6)$$

Since  $R$  is noncommutative, we may choose  $q \in U$  such that  $q \notin C$ . Then replacing  $u$  by  $[q, x]$  and  $z$  by  $[q, y]$  in (2.6), we get

$$a(b[x, y]^2 + ([q, x], y] + [x, [q, y]])[x, y] + [x, y]([q, x], y] + [x, [q, y]]) - (b[x, y] + ([q, x], y] + [x, [q, y]]))^2) = 0,$$

which is

$$a(b[x, y]^2 + [q, [x, y]^2]) - (b[x, y] + [q, [x, y]])^2) = 0.$$

Then by Lemma 2.4, we have  $q \in C$ , a contradiction.

Now replacing  $F$  with  $-F$  in the above result, we obtain the conclusion for the situation  $a(F(u^2) + F(u)^2) = 0$  for all  $u \in L$ .

**Corollary 2.5.** *Let  $R$  be a prime ring with extended centroid  $C$ ,  $L$  a non-central Lie ideal of  $R$  and  $0 \neq a \in R$ . If  $R$  admits the generalized derivation  $F$  such that either  $a(F(XY) \pm F(X)F(Y)) = 0$  for all  $X, Y \in L$  or  $a(F(XY) \pm F(Y)F(X)) = 0$  for all  $X, Y \in L$ , then one of the following holds:*

- (1) *there exists  $b \in U$  such that  $F(x) = bx$  for all  $x \in R$ , with  $ab = 0$ ;*
- (2)  *$F(x) = \mp x$  for all  $x \in R$ ;*
- (3)  *$\text{char}(R) = 2$  and  $R$  satisfies  $s_4$ ;*
- (4)  *$\text{char}(R) \neq 2$ ,  $R$  satisfies  $s_4$  and there exists  $b \in U$  such that  $F(x) = bx$  for all  $x \in R$ .*

**Proof of Theorem 1.2.** First consider the case when  $a(F(x^m y^n) - F(x^m)F(y^n)) = 0$  for all  $x, y \in R$ . Let  $G_1$  be the additive subgroup of  $R$  generated by the set  $S_1 = \{x^m | x \in R\}$  and  $G_2$  be the additive subgroup of  $R$  generated by the set  $S_2 = \{x^n | x \in R\}$ . Then by assumption

$$a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in G_1, \quad \forall y \in G_2.$$

Then by [7], either  $G_1 \subseteq Z(R)$  or  $\text{char}(R) = 2$  and  $R$  satisfies  $s_4$ , except when  $G_1$  contains a noncentral Lie ideal  $L_1$  of  $R$ .  $G_1 \subseteq Z(R)$  implies that  $x^m \in Z(R)$  for all  $x \in R$ . It is well known that in this case  $R$  must be commutative, which is a contradiction. Since  $\text{char}(R) \neq 2$ , we are to consider the case when  $G_1$  contains a noncentral Lie ideal  $L_1$  of  $R$ . In this case by [4, Lemma 1], there exists a nonzero ideal  $I_1$  of  $R$  such that  $[I_1, I_1] \subseteq L_1$ .

Thus we have

$$a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in [I_1, I_1], \quad \forall y \in G_2.$$

Analogously, we see that there exists a nonzero ideal  $I_2$  of  $R$  such that

$$a(F(xy) - F(x)F(y)) = 0 \quad \forall x \in [I_1, I_1], \quad \forall y \in [I_2, I_2].$$

By Lee [23, Theorem 3],  $F(x) = bx + d(x)$  for some  $b \in U$  and derivations  $d$  on  $U$ . Since  $I_1, I_2, R$  and  $U$  satisfy the same differential identities [24], without loss of generality,

$$a(F(xy) - F(x)F(y)) = 0 \quad \forall x, y \in [R, R],$$

and in particular

$$a(F(x^2) - F(x)^2) = 0 \quad \forall x \in [R, R].$$

Then by Theorem 1.1, we get

- (1) there exists  $b \in U$  such that  $F(x) = bx$  for all  $x \in R$ , with  $ab = 0$ ;
- (2)  $F(x) = x$  for all  $x \in R$ ;
- (3)  $R$  satisfies  $s_4$  and there exists  $b \in U$  such that  $F(x) = bx$  for all  $x \in R$ .

In the last conclusion,  $R$  satisfies polynomial identity and hence  $R \subseteq M_2(C)$  for some field  $C$  and  $M_2(C)$  satisfies  $a(bx^m y^n - bx^m b y^n) = 0$ . By lemma 2.2, we get either  $ab = 0$  or  $b = 1$ . If  $ab = 0$ , then  $F(x) = bx$  for all  $x \in R$ , with  $ab = 0$ , which is our conclusion (1). If  $b = 1$  then  $F(x) = x$  for all  $x \in R$ , which is our conclusion (2).

Now replacing  $F$  with  $-F$  in the hypothesis  $a(F(x^m y^n) - F(x^m)F(y^n)) = 0$ , we get  $0 = a((-F)(x^m y^n) - (-F)(x^m)(-F)(y^n))$ , that is  $0 = a(F(x^m y^n) + F(x^m)F(y^n))$  for all  $x, y \in R$  implies one of the following:

- (1) there exists  $b \in U$  such that  $F(x) = bx$  for all  $x \in R$ , with  $ab = 0$ ;
- (2)  $F(x) = -x$  for all  $x \in R$ ;

Now consider the case when  $a(F(x^m y^n) - F(y^n)F(x^m)) = 0$  for all  $x, y \in R$ . By similar argument as above we get

$$a(F(xy) - F(y)F(x)) = 0 \quad \forall x, y \in [R, R],$$

and in particular

$$a(F(x^2) - F(x)^2) = 0 \quad \forall x \in [R, R].$$

Then by Theorem 1.1, we get

- (1) there exists  $b \in U$  such that  $F(x) = bx$  for all  $x \in R$ , with  $ab = 0$ ;
- (2)  $F(x) = x$  for all  $x \in R$ ;
- (3)  $R$  satisfies  $s_4$  and there exists  $b \in U$  such that  $F(x) = bx$  for all  $x \in R$ .

In the conclusion (3),  $R$  satisfies polynomial identity and hence  $R \subseteq M_2(C)$  for some field  $C$  and  $M_2(C)$  satisfies  $a(bx^m y^n - by^n bx^m) = 0$ . Then by Lemma 2.3, we have  $ab = 0$ , which is our conclusion (1).

Now replacing  $F$  with  $-F$  in the hypothesis  $a(F(x^m y^n) - F(y^n)F(x^m)) = 0$ , we get  $0 = a((-F)(x^m y^n) - (-F)(y^n)(-F)(x^m))$ . That is,  $0 = a(F(x^m y^n) + F(y^n)F(x^m))$  for all  $x, y \in R$ . This implies that there exists  $b \in U$  such that  $F(x) = bx$  for all  $x \in R$  with  $ab = 0$  or  $F(x) = -x$ . This completes the proof.

In particular, we have the following corollary.

**Corollary 2.6.** *Let  $R$  be a prime ring of characteristic different from 2 and  $0 \neq a \in R$ . Suppose that  $R$  admits the generalized derivation  $F$  associated with a nonzero derivation  $d$  of  $R$ . If any one of the following conditions is satisfied:*

- (1)  $a(F(x^m y^n) \pm F(x^m)F(y^n)) = 0$  for all  $x, y \in R$ ;
- (2)  $a(F(x^m y^n) \pm F(y^n)F(x^m)) = 0$  for all  $x, y \in R$ ,

*then  $R$  is commutative.*

**Proof of Theorem 1.3.** First we consider the case  $a(F(x^m y^n) + F(x^m)F(y^n)) = 0$  for all  $x, y \in R$ . Other cases are similar. We know the fact that any derivation of a semiprime ring  $R$  can be uniquely extended to a derivation of its left Utumi quotient ring  $U$  and so any derivation of  $R$  can be defined on the whole of  $U$  [24, Lemma 2]. Moreover  $R$  and  $U$  satisfy the same GPIs as well as same differential identities. Thus

$$a(bx^m y^n + d(x^m y^n) + (bx^m + d(x^m))(by^n + d(y^n))) = 0$$

for all  $x, y \in U$ . Let  $M(C)$  be the set of all maximal ideals of  $C$  and  $P \in M(C)$ . Now by the standard theory of orthogonal completions for semiprime rings (see [24, p.31-32]), we have  $PU$  is a prime ideal of  $U$  invariant under all derivations of  $U$ . Moreover,  $\bigcap \{PU \mid P \in M(C)\} = 0$ . Set  $\bar{U} = U/PU$ . Then derivation  $d$  canonically induces a derivation  $\bar{d}$  on  $\bar{U}$  defined by  $\bar{d}(\bar{x}) = \overline{d(x)}$  for all  $x \in U$ . Therefore,

$$\bar{a}(\bar{b}\bar{x}^m \bar{y}^n + \bar{d}(\bar{x}^m \bar{y}^n) + (\bar{b}\bar{x}^m + \bar{d}(\bar{x}^m))(\bar{b}\bar{y}^n + \bar{d}(\bar{y}^n))) = 0$$

for all  $\bar{x}, \bar{y} \in \bar{U}$ . By the prime ring case of Corollary 2.6, we have either  $\bar{d} = 0$  or  $[\bar{U}, \bar{U}] = 0$  or  $\bar{a} = 0$ . In any case we have  $ad(U)[U, U] \subseteq PU$  for all  $P \in M(C)$ . Since  $\bigcap \{PU \mid P \in M(C)\} = 0$ ,  $ad(U)[U, U] = 0$ . In particular,  $ad(R)[R, R] = 0$ . This implies  $0 = ad(R)[R^2, R] = ad(R)R[R, R] + ad(R)[R, R]R = ad(R)R[R, R]$ . In particular,  $ad(R)R[R, ad(R)] = 0$ . Therefore,  $[ad(R), R]R[ad(R), R] = 0$ . Since  $R$  is semiprime, we obtain that  $ad(R) \subseteq Z(R)$ . By Theorem 3.2 in [10], there exist orthogonal central idempotents  $e_1, e_2, e_3 \in U$  with  $e_1 + e_2 + e_3 = 1$  such that  $d(e_1U) = 0$ ,  $e_2a = 0$ , and  $e_3U$  is commutative. Hence the theorem is proved.

#### ACKNOWLEDGMENTS

The authors would like to thank the referee for providing very helpful comments and suggestions.

#### REFERENCES

1. A. Ali, N. ur Rehman, S. Ali, On Lie Ideals with Derivations as Homomorphisms and Anti-Homomorphisms, *Acta Math. Hung.*, **101**(1–2), (2003), 79–82.
2. K. I. Beidar, W. S. Martindale III, A. V. Mikhalev, Rings with Generalized Identities, *Monographs and Textbooks in Pure and Applied Math.*, **196**, (1996). New York: Marcel Dekker, Inc.
3. H. E. Bell, L. C. Kappe, Rings in Which Derivations Satisfy Certain Algebraic Conditions, *Acta Math. Hung.*, **53**, (1989), 339–346.
4. J. Bergen, I. N. Herstein, J. W. Kerr, Lie Ideals and Derivations of Prime Rings, *J. Algebra*, **71**, (1981), 259–267.
5. M. Bresar, On the Distance of the Composition of Two Derivations to be the Generalized Derivations, *Glasgow Math. J.*, **33**, (1991), 89–93.
6. C. L. Chuang, GPIs Having Coefficients in Utumi Quotient Rings, *Proc. Amer. Math. Soc.*, **103**, (1988), 723–728.
7. C. L. Chuang, The Additive Subgroup Generated by a Polynomial, *Israel J. Math.*, **59**(1), (1987), 98–106.
8. V. De Filippis, N. ur Rehman, A. Z. Ansari, Lie Ideals and Generalized Derivations in Semiprime Rings, *Iranian Journal of Mathematical Sciences and Informatics*, **10**(2), (2015), 45–54.
9. V. De Filippis, Generalized Derivations as Jordan Homomorphisms on Lie Ideals and Right Ideals, *Acta Math. Sinica, English Series*, **25**(2), (2009), 1965–1974.
10. B. Dhara, N. Argac, K. G. Pradhan, Annihilator Condition of a Pair of Derivations in Prime and Semiprime Rings, *Indian J. Pure Appl. Math.*, **47**(1), (2016), 111–124.
11. B. Dhara, S. Kar, K. G. Pradhan, Generalized Derivations Acting as Homomorphisms or anti-Homomorphisms with Central Values in Semiprime Rings, *Miskolc Math. Notes*, **16**(2), (2015), 781–791.
12. B. Dhara, Sh. Sahebi, V. Rahmani, Generalized Derivations as a Generalization of Jordan Homomorphisms on Lie Ideals and Right Ideals, *Math. Slovaca*, **65**(5), (2015), 963–974.
13. B. Dhara, Sh. Sahebi, V. Rahmani, Generalized Derivations as a Generalization of Jordan Homomorphisms Acting on Lie Ideals, *Matematiki Vesnik*, **67**(2), (2015), 92–101.
14. B. Dhara, S. Kar, D. Das, A Multiplicative (generalized)- $(\sigma, \sigma)$ -Derivation Acting as (anti-)Homomorphism in Semiprime Rings, *Palestine J. Mathematics*, **3**(2), (2014), 240–246.

15. B. Dhara, Generalized Derivations Acting as Homomorphism or anti-Homomorphism in Semiprime Rings, *Beitr. Algebra Geom.*, **53**, (2012), 203–209.
16. A. Ebadian, M. E. Gordji, Left Jordan Derivations on Banach Algebras, *Iranian Journal of Mathematical Science and Informatics*, **6**(1), (2011), 1–6.
17. T. S. Erickson, W. S. Martindale III, J. M. Osborn, Prime Nonassociative Algebras, *Pacific J. Math.*, **60**, (1975), 49–63.
18. O. Golbasi, K. Kaya, On Lie Ideals with Generalized Derivations, *Siberian Math. J.*, **47**(5), (2006), 862–866.
19. I. Gusic, A Note on Generalized Derivations of Prime Rings, *Glasnik Mate.*, **40**(60), (2005), 47–49.
20. N. Jacobson, *Structure of Rings*, Amer. Math. Soc. Colloq. Pub., 37. Providence, RI: Amer. Math. Soc., 1964.
21. V. K. Kharchenko, Differential Identity of Prime Rings, *Algebra and Logic*, **17**, (1978), 155–168.
22. C. Lanski, S. Montgomery, Lie Structure of Prime Rings of Characteristic 2, *Pacific J. Math.*, **42**(1), (1972), 117–136.
23. T. K. Lee, Generalized Derivations of Left Faithful Rings, *Comm. Algebra*, **27**(8), (1999), 4057–4073.
24. T. K. Lee, Semiprime Rings with Differential Identities, *Bull. Inst. Math. Acad. Sinica*, **20**(1), (1992), 27–38.
25. W. S. Martindale III, Prime Rings Satisfying a Generalized Polynomial Identity, *J. Algebra*, **12**, (1969), 576–584.
26. N. ur Rehman, M. A. Raza, Generalized Derivations as Homomorphisms or anti-Homomorphisms on Lie Ideals, *Arab J. Math. Sc.*, **22**(1), (2016), 22–28.
27. Y. Wang, H. You, Derivations as Homomorphisms or anti-Homomorphisms on Lie Ideals, *Acta Math. Sinica, English Series*, **23**(6), (2007), 1149–1152.