Iranian Journal of Mathematical Sciences and Informatics
Vol. 13, No. 2 (2018), pp 93-99
DOI: 10.7508/ijmsi.2018.13.008

Fractal Dimension of Graphs of Typical Continuous Functions on Manifolds

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Abstract. If M is a compact Riemannian manifold and $C(M,R)$ is the set of all real valued continuous functions defined on M, then we show that for a typical element $f \in C(M,R)$, $\overline{dim}_B(graph(f))$ is as big as possible and for a typical $f \in C(M,R)$, $\underline{dim}_B(graph(f))$ is as small as possible.

Keywords: Manifold, Fractal, Box dimension.

2000 Mathematics subject classification: 54E50, 54E52, 54F45, 57N40.

1. Introduction

A subset A of a topological space X is called to be *comeagre*, if there is a countable collection $\{W_i\}$ of open and dense subsets of X such that $\bigcap_i W_i \subset A$. Complement of a comeagre subset is called a meagre subset. A meagre subset can be considered as subset of a countable union of nowhere dense subsets and they are negligible in some sense. So, we say that some property holds for *typical* elements of X, if it holds on a comeagre subset. Study of properties of typical elements in X is a classic and interesting problem. One can find many papers dealing with typical elements when X is supposed to be the space $C(W,R)$ of all continuous functions defined on a compact topological space W, endowed with the metric topology defined by the metric $d(f,g) = sup_{x\in W}|f(x)-g(x)|$. A well known theorem due to Banach [1], states that typical elements of $C([0,1],R)$ are nowhere differentiable, so the image or graph of a typical f in $C([0,1],R)$ is a

Received 13 January 2016; Accepted 14 January 2017

fractal set. Calculating fractal dimensions (including box dimension, Hausdorff dimension, packing dimension, etc) of the image of f or $graph(f)$ is a well known problem and one can find many results in the literature. It is proved in [6] that for a typical $g \in C([0,1],R)$, $dim_H(graph(g)) = 1$. It is proved in [3] that if $W \subset R$ is bounded with only finitely many isolated points and $X = \{f \in C(W,R) : f$ is uniformly countinuous $\}$, then for a typical $f \in X$, $\overline{dim}_B(graph(f))$ is as big as possible and $\underline{dim}_B(graph(f))$ is as small as possible. In the previous paper [7] we generalized Banach's theorem to the set $C(M,R)$, where M is a compact Riemannian manifold. Now, we show in the present paper that the main results of [3] about upper and lower box dimensions are also true when W is replaced by a compact Riemannian manifold M.

2. Preliminaries

In what follows, M is a compact Riemannian manifold with the Riemannian metric d, and $C(M,R)$ will denote the collection of all continuous functions defined on M endowed with the metric d defined by $d(f,g) = max_{x\in M}|f(x) - g(x)|$.

If (X,d_1) and (Y,d_2) are metric spaces then we will consider the usual product metric d on $X\times Y$ defined by $d((x_1,y_1),(x_2,y_2)) = \sqrt{d_1^2(x_1,x_2)+d_2^2(y_1,y_2)}$.

If E is a bounded subset of M then the upper box dimension of E is defined by

$$\overline{dim}_B(E) = limsup_{\delta\to 0}\frac{N_\delta(E)}{-log\delta}.$$

Where, $N_\delta(E)$ is the minimum number of balls of radius δ (or minimum number of sets of diameter at most δ) covering E (The lower box dimension $\underline{dim}_B(E)$ is defined in similar way). Another definition for dimension, which is widely used in fractal geometry is Hausdorff dimension (see [4]).

Now, we mention some facts which we need in the proofs of theorems.

Remark 2.1. If E is a bounded subset of R^m then $\overline{dim}_B(E\times I^n) = \overline{dim}_B(E)+n$. The similar result is true if we replace $\overline{dim}_B$ by $\underline{dim}_B$ or dim_H.

Proof. We give the proof for $\overline{dim}_B(E\times I) = \overline{dim}_B(E)+1$. The general case comes by induction. If $\delta > 0$ then the smallest number of intervals of length δ covering I is equal to $[\frac{1}{\delta}]$ or $[\frac{1}{\delta}]+1$. If U_δ (I_δ) is a bounded subset of R^m (I) with diameter δ, then the diameter of $U_\delta \times I_\delta$ is equal to $\sqrt{2}\delta$. So,

$$N_{\sqrt{2}\delta}(E\times I) \le ([\frac{1}{\delta}]+1)N_\delta(E)$$

Then we have

$$\overline{dim}_B(E \times I) = limsup_{\delta \to 0} \frac{log(N_{\sqrt{2}\delta}(E \times I))}{-log(\sqrt{2}\delta)}$$

$$\leq limsup_{\delta \to 0} \frac{log([\frac{1}{\delta}] + 1)N_\delta(E))}{-log(\sqrt{2}\delta)}$$

$$= 1 + limsup_{\delta \to 0} \frac{N_\delta(E)}{-log\delta} = 1 + \overline{dim}_B(E)$$

Also we know that $\overline{dim}_B(E \times I^n) \geq \overline{dim}_B(E) + n$ (see [4]). So we get the equality. □

Remark 2.2. If M is a compact metric space and $f : M \to R$ is a locally lipschitz function, then f is globally lipschitz.

Proof. Since f is locally lischitz and M is compact, then there is a finite collection of open cover of balls $B_i, 1 \leq i \leq m$, and constants L_i such that

$$d(f(x), f(y)) \leq L_i d(x, y), \quad x, y \in B_i$$

Since M is compact then the function $F : M \times M \to R$, defined by $F(x, y) = d(f(x), f(y))$ has a maximum which we denote it by N. Let δ be the lebesgue's number related to the covering B_i of M, and put $L = max\{\frac{N}{\delta}, L_i : i\}$. Then for given $x, y \in M$, either there is a B_i such that $x, y \in B_i$ or $d(x, y) \geq \delta$. In the first case we have $d(f(x), f(y)) \leq Ld(x, y)$. In the second case we have

$$d(f(x), f(y)) \leq N \leq \frac{N}{\delta} d(x, y) \leq Ld(x, y)$$

□

If M and N are compact differentiable manifolds and $f : M \to N$ is continuously differentiable, then f is a lipschitz function. So, we get the following remark easily.

Remark 2.3. If M and N are compact Riemannian manifolds and $\phi : M \to N$ is a map such that ϕ and its inverse are continuously differentiable, then the map $\psi : M \times R \to N \times R$ defined by $\psi(x, y) = (\phi(x), y)$ is bilipschitz.

Remark 2.4. If M is a compact Riemannian manifold, $f : M \to R$ is continuously differentiable, $g : M \to R$ is continuous and $k = f + g$, then $\overline{dim}_B(graph(k)) = \overline{dim}_B(graph(g))$. The same result is true for $\underline{dim}_B$.

Proof. Consider the map $\psi : graph(g) \to graph(k)$, defined by $\psi(x, g(x)) = (x, k(x))$. We show that ψ and ψ^{-1} are Lipschitz functions. We have

$$d(\psi(x, g(x)), \psi(y, g(y))) = d((x, k(x)), (y, k(y))) = \sqrt{d^2(x, y) + (k(x) - k(y))^2}$$

Since f is continuously differentiable, it is locally Lischitz and by Remark 2.2, it must be Lischitz. Then, there exist a positive number N such that $|f(x) - f(y)| \leq Nd(x, y)$, $x, y \in M$. Thus

$$\begin{aligned}(k(x) - k(y))^2 &= (f(x) - f(y) + g(x) - g(y))^2 \leq (Nd(x, y) + |g(x) - g(y)|)^2 \\ &= N^2 d^2(x, y) + 2Nd(x, y)|g(x) - g(y)| + |g(x) - g(y)|^2 \\ &\leq N^2 d^2(x, y) + N^2 d^2(x, y) + |g(x) - g(y)|^2 + |g(x) - g(y)|^2 \\ &= 2N^2 d^2(x, y) + 2|g(x) - g(y)|^2\end{aligned}$$

Then

$$\begin{aligned}d(\psi(x, g(x)), \psi(y, g(y))) &\leq \sqrt{d^2(x, y) + 2N^2 d^2(x, y) + 2|g(x) - g(y)|^2} \\ \leq \sqrt{2(N^2 + 1)}\sqrt{d^2(x, y) + (g(x) - g(y))^2} &= \sqrt{2(N^2 + 1)}d((x, g(x)), (y, g(y))).\end{aligned}$$

Therefore, ψ is Lipschitz. In a similar way we can show that ψ^{-1} is Lipschitz. □

Remark 2.5. (generalized StoneWeierstrass Theorem) . Suppose X is a compact Hausdorff space and A is a subalgebra of C(X, R) which contains a non-zero constant function. Then A is dense in $C(X, R)$ if and only if it separates points.

3. Results

Lemma 3.1. *If $f : M \to R$ is continuously differentiable and $\epsilon > 0$, then there exists $g \in C(M, R)$ such that $d(f, g) < \epsilon$ and $\overline{dim}_B(graph(g)) = n + 1$, $n = dimM$.*

Proof. Let N be a compact Riemannian manifold. Consider a function $g_1 \in C(I, R^+)$ such that $\overline{dim}_B(graph(g_1)) = 2$ and put

$$g_2 : I^n = I \times I^{n-1} \to R^+, \quad g_2(t_1, t_2) = g_1(t_1).$$

Then

$$\begin{aligned}graph(g_2) = \{((t_1, t_2), g_1(t_1)), (t_1, t_2) \in I \times I^{n-1}\} \simeq \\ \{((t_1, g_1(t_1)), t_2), (t_1, t_2) \in I \times I^{n-1}\} = graph(g_1) \times I^{n-1}.\end{aligned}$$

So, by Remark 2.1

$$\overline{dim}_B(graph(g_2)) = 2 + n - 1 = n + 1.$$

Consider a chart (U, ϕ) on N such that $I^n \subset \phi(U)$ and put $W = \phi^{-1}(I^n)$. Now, put $g_3 = g_2 o\phi : W \to R$. By Remark 2.3, the function $\psi : W \times R \to I^n \times R$, defined by $\psi(x, y) = (\phi(x), y)$ is bilipschitz. Since $\psi(graph(g_3)) = graph(g_2)$, then $\overline{dim}_B(graph(g_3)) = n+1$. Extend the function g_3 to a continuous function $g_4 : N \to R$. Since $graph(g_3) \subset graph(g_4)$ then $\overline{dim}_B(graph(g_4)) = n+1$. Now put $N = graph(f)$. We know that N is a submanifold of $M \times R$, which with the induced metric is a riemannian manifold. Given $\delta > 0$, the function $g_5 = \delta g_4 : N \to R$ is a positive function such that $\overline{dim}(graph(g_5)) = \overline{dim}(graph(g_4) =$

$n+1$. By compactness condition we can choose δ small enough such that for all $y=(x,f(x))\in N$, $g_5(y)<\epsilon$.
Now, consider the function $g_6 : M \to R$, defined by $g_6(x) = g_5(x, f(x))$ and put $\psi : M\times R \to N\times R$, $\psi(x,y) = ((x,f(x)),y)$. We have

$$\psi : graph(g_6) = graph(g_5)$$

By Remark 2.3, ψ is bilipshitze, so

$$\overline{dim}_B(graph(g_6)) = \overline{dim}_B(graph(g_5)) = n+1$$

Put $g : M \to R$, $g(x) = f(x)+g_6(x)$. Since f is differentiable, then by Remark 2.4, $\overline{dim}_B(graph(g)) = \overline{dim}_B(graph(g_6) = n+1$. Also, we have $d(f,g) = max_{x\in M}|g(x)-f(x)| = max_{x\in M}|g_6(x)| = max_{x\in M} g_5(x,f(x)) < \epsilon$. □

Theorem 3.2. *Let M be a compact Riemannian manifold, $dim(M)=n$, and $C(M,R)$ be the set of all continuous functions defined on M. Then for typical members f in $C(M,R)$, $\underline{dim}_B(graph(f)) = n$.*

Proof. Put
$$A = \{f\in C(M,R) : \underline{dim}_B(graph(f)) = n\}.$$
Let $f\in A$ and consider a positive number $\epsilon>0$ and $g\in C(M,R)$ such that $d(f,g)<\epsilon$. If a collection of balls of radius δ in $M\times R$ covers $graph(f)$ and $\epsilon<\delta$, then the same number of balls with radius 2δ covers $graph(g)$. Since each ball of radius 2δ can be covered by 4^{n+1} balls of radius δ, then

$$N_\delta(graph(g)) \le 4^{n+1} N_\delta(graph(f))$$

If $\delta<1$ then
$$\frac{logN_\delta(graph(g))}{-log(\delta)} \le (n+1)\frac{lo4}{-log\delta} + \frac{logN_\delta(graph(f))}{-log\delta}$$

Since $\underline{dim}_B(graph(f)) = n$ and $lim_{\delta\to 0}\frac{log4}{-log\delta} = 0$, then for each $k\in N$ there exists $\delta=\delta(f,k)>0$ such that

$$\frac{logN_\delta(graph(g))}{-log(\delta)} \le (n+1)\frac{log4}{-log\delta} + \frac{logN_\delta(graph(f))}{-log\delta} < n+\frac{1}{k}$$

Put
$$U_{f,k} = \{g\in C(M,R) : d(f,g) < \delta(f,k)\}$$
and
$$W_k = \bigcup_{(f\in A)} U_{f,k}$$
$W_{f,k}$ is an open set in $C(M,R)$ such that for each $g\in W_k$,

$$\underline{dim}_B(graph(g) < n+\frac{1}{k}.$$

Clearly $A\subset \bigcap_k W_k$. If $g\in\bigcap_k W_k$ then $\underline{dim}_B(graph(g)) \le n$, and since for all $g\in C(M,R)$, $n\le \underline{dim}_B(graph(g))$ then $\underline{dim}_B(graph(g)) = n$. Thus

$\bigcap_k W_k = A$. Now, we show that W_k is dense for all k, then the proof will be complete. Given $g \in C(M,R)$ and $\epsilon > 0$. By Remark 2.5, collection of differentiable functions is dense, so there exists a differentiable function $f : M \to R$ such that $d(f,g) < \epsilon$. But for a differentiable function f, $\underline{dim}_B(graph(f)) = \overline{dim}_B(graph(f)) = n$. So $f \in A \subset W_k$. $\square$

Lemma 3.3. *If $g \in C(M,R)$ and $\epsilon > 0$, then there exists $k \in C(M,R)$ such that $d(g,k) < \epsilon$ and $\overline{dim}_B(graph(k)) = n+1$.*

Proof. By Remark 2.5, for a given $\delta > 0$ there exists a differentiable function $f \in C(M,R)$ such that $d(f,g) < \delta$. Consider a function $f_1 \in C(M,R)$ such that $\overline{dim}_B(graph(f_1)) = n+1$. Since M is compact, for a given number $\delta_2 > 0$ there is a positive number δ_1 such that $|\delta_1 f_1(x)| < \delta_2$ for all $x \in M$. Now, put $k = f + \delta_1 f_1$. By Remark 2.4, we have

$$\overline{dim}_B(graph(k) = \overline{dim}_B(graph(\delta_1 f_1)) = \overline{dim}_B(graph(f_1)) = n+1.$$

If we choose δ and δ_2 smaller than $\frac{\epsilon}{2}$, then

$$d(g,k) \le d(g,f) + d(f,k) \le \delta + \delta_1||f_1|| \le \delta + \delta_2 < \epsilon.$$

$\square$

Theorem 3.4. *Let M be a compact Riemannian manifold, $dim(M) = n$, and $C(M,R)$ be the set of all continuous functions defined on M. Then for typical members f in $C(M,R)$, $\overline{dim}_B(graph(f)) = n+1$.*

Proof. Clearly for all $f \in C(M,R)$, $\overline{dim}_B(graph(f)) \le n+1$. Put

$$A = \{f \in C(M,R) : \overline{dim}_B(graph(f)) = n+1\}.$$

Consider $f \in A$, a positive number $\epsilon > 0$ and $g \in C(M,R)$ such that $d(f,g) < \epsilon$. If a collection of balls of radius δ in $M \times R$ covers $graph(g)$ and $\epsilon < \delta$, then the same number of balls with radius 2δ covers $graph(f)$. Since each ball of radius 2δ can be covered by 4^{n+1} balls of radius δ, then

$$N_\delta(graph(f)) < 4^{n+1} N_\delta(graph(g))$$

So, if $\delta < 1$ then

$$\frac{logN_\delta(graph(f))}{-log(\delta)} < (n+1)\frac{lo4}{-log\delta} + \frac{logN_\delta(graph(g))}{-log\delta}$$

Since $\overline{dim}_B(graph(f)) = n+1$, then for each $k \in N$ there is $\delta(k) = \delta(f,k) > 0$ such that

$$n+1-\frac{1}{k} < \frac{logN_{\delta(k)}(graph(f))}{-log(\delta(k))} - (n+1)\frac{log4}{-log\delta(k)} < \frac{logN_{\delta(k)}(graph(g))}{-log\delta(k)}$$

Put

$$U_{f,k} = \{g \in C(M,R) : d(f,g) < \delta(f,k)\}$$

and

$$W_k = \bigcup_{(f \in A)} U_{f,k}$$

W_k is an open set in $C(M, R)$ such that for each $g \in W_k$,

$$\overline{dim}_B(graph(g) > n + 1 - \frac{1}{k}$$

Clearly

$$\bigcap_k W_k = A$$

Now it remains to show that W_k is dense for all k. Let $h \in C(M, R)$ and $\epsilon > 0$ we show that there exists $g \in W_k$ such that $d(h, g) < \epsilon$. Since by Remark 2.5, the collection of all differentiable functions is dense in $C(M, R)$ then there exists a differentiable function $g_1 \in C(M, R)$ such that $d(h, g_1) < \frac{\epsilon}{2}$. Consider a function $f \in A \subset W_k$. Since f is continuous and M is compact then there exists $\delta > 0$ such that $|\delta f(x)| < \frac{\epsilon}{2}$ for all $x \in M$. Now, put $g = g_1 + \delta f$. Since g_1 is differentiable then $\overline{dim}_B(graph(g) = \overline{dim}_B(graph \delta f) = \overline{dim}_B(graph(f)) = n + 1$. So, $g \in A \subset W_k$ and we have

$$d(h, g) \leq d(h, g_1) + d(g_1, g) \leq \frac{\epsilon}{2} + max_{x \in M}|\delta f| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Acknowledgments

The author wish to thank the referee for his/her helpful comments.

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