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Sufficient Inequalities for Univalent Functions

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Abstract. In this work, applying a lemma due to Nunokawa et al. (J. Ineq. Appl. 2015. 1(2015) 7), we obtain some sufficient inequalities for some certain subclasses of univalent functions.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in Δ is denoted by $\mathcal{S}$. A function $f \in \mathcal{S}$ is called starlike (with respect to 0), denoted by $f \in \mathcal{S}^*$, if $tw \in f(\Delta)$ whenever $w \in f(\Delta)$ and $t \in [0, 1]$. A function $f \in \mathcal{S}$ that maps Δ onto a convex

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domain, denoted by $f \in \mathcal{K}$, is called a convex function. A function $f(z)$ in $\mathcal{A}$ is said to be starlike of order $0 \leq \gamma < 1$ if it satisfies

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \qquad z \in \Delta.$$

We denote by $\mathcal{S}^*(\gamma)$ the subclass of $\mathcal{A}$ consisting of all starlike functions of order γ in Δ. Put $\mathcal{S}^*(0) \equiv \mathcal{S}^*$. A function $f(z)$ in $\mathcal{A}$ is said to be convex of order $0 \leq \gamma < 1$ if it satisfies

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma \qquad z \in \Delta.$$

We denote by $\mathcal{K}(\gamma)$ the subclass of $\mathcal{A}$ consisting of all convex functions of order γ in Δ. Let $\mathcal{R}_\gamma$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\mathfrak{Re}\{f'(z)\} > \gamma \qquad z \in \Delta,$$

for some $0 \leq \gamma < 1$. The class $\mathcal{R}_\gamma$ is considered in [2]. We remark that the classes $\mathcal{S}^*(\gamma)$, $\mathcal{K}(\gamma)$ and $\mathcal{R}_\gamma$ $(0 \leq \gamma < 1)$ are subclasses of univalent functions.

Nunokawa, Cho, Kwon and Sokół [4], obtained the following result.

Lemma 1.1. *Let $B(z)$ and $C(z)$ be analytic in Δ with*

$$|\mathfrak{Im}\{C(z)\}| < \mathfrak{Re}\{B(z)\}.$$

If $p(z)$ is analytic in Δ with $p(0) = 1$, and if

$$|\arg\{B(z)zp'(z) + C(z)p(z)\}| < \pi/2 + t(z), \tag{1.2}$$

where

$$t(z) = \begin{cases} \arg\{C(z) + iB(z)\} & \text{when } \arg\{C(z) + iB(z)\} \in [0, \pi/2], \\ \arg\{C(z) + iB(z)\} - \pi/2 & \text{when } \arg\{C(z) + iB(z)\} \in (\pi/2, \pi], \end{cases}$$

then we have

$$\mathfrak{Re}\{p(z)\} > 0 \qquad z \in \Delta.$$

In this paper, applying the Lemma 1.1, we obtain sufficient conditions for some certain subclasses of univalent and analytic functions. The study of analytic functions and such results, also for other subclass were investigated in [1, 3, 7].

2. Main Results

One of our main results is contained in the following.

Theorem 2.1. *Assume that $f \in \mathcal{A}$, $f(z)/z$ is bounded and not vanishing in the unit disc. If*

$$\left|\arg\left\{\sqrt[n]{\left(\frac{z}{f(z)}\right)^{n-1}} f'(z) - \gamma\right\}\right| < \frac{\pi}{2} + \arctan n, \tag{2.1}$$

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then

$$\mathfrak{Re}\left\{\sqrt[n]{\frac{f(z)}{z}}\right\} > \gamma \qquad z \in \Delta, \tag{2.2}$$

where $\sqrt{1} = 1$, $\gamma < 1$ *and* $n = 1, 2, 3, \ldots$.

Proof. Let $f(z) \neq 0$ for $z \neq 0$ and let $p(z)$ be defined by

$$p(z) = \frac{1}{1-\gamma}\left(\sqrt[n]{\frac{f(z)}{z}} - \gamma\right) \qquad z \in \Delta,$$

where $\gamma < 1$. Then $p(z)$ is analytic in Δ, $p(0) = 1$ and

$$(1-\gamma)p(z) + n(1-\gamma)zp'(z) = \sqrt[n]{\left(\frac{z}{f(z)}\right)^{n-1}} f'(z) - \gamma.$$

If we put $B(z) = n(1-\gamma)$ and $C(z) = 1-\gamma$, from (2.1) and applying Lemma 1.1, we obtain (2.2) immediately. □

Putting $n = 1$, in Theorem 2.1, we have:

Corollary 2.2. *(see* [5]*) Let* $f \in \mathcal{A}$, $\gamma < 1$ *and* $f(z)/z$ *is bounded and not vanishing in the unit disc. If*

$$|\arg\{f'(z) - \gamma\}| < \frac{3\pi}{4},$$

then

$$\mathfrak{Re}\left\{\frac{f(z)}{z}\right\} > \gamma \qquad z \in \Delta.$$

If we take $n = 2$, in Theorem 2.1, we have:

Corollary 2.3. *Let* $f \in \mathcal{A}$, $\gamma < 1$ *and* $f(z)/z$ *is bounded and not vanishing in the unit disc. If*

$$\left|\arg\left\{\sqrt{\frac{z}{f(z)}} f'(z) - \gamma\right\}\right| < \frac{\pi}{2} + \arctan 2 \approx 153.4349...,$$

then

$$\mathfrak{Re}\left\{\sqrt{\frac{f(z)}{z}}\right\} > \gamma \qquad z \in \Delta.$$

In [6], Obradović, Ponnusamy and Tuneski introduced the class $\mathcal{P}(1/2)$ as follows:

$$\mathcal{P}(1/2) := \{f \in \mathcal{A} : \mathfrak{Re}(f(z)/z) > 1/2 \text{ for } z \in \Delta\}.$$

We remark that $\mathcal{K}(0) \subset \mathcal{P}(1/2)$ (see [6]).

Setting $n = 1$ and $\gamma = 1/2$, in Theorem 2.1, we have:

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Corollary 2.4. *Let $f \in \mathcal{A}$, $f(z)/z$ is bounded and not vanishing in the unit disc. If*

$$\left|\arg\left\{f'(z)-\frac{1}{2}\right\}\right| < \frac{3\pi}{4} \qquad z \in \Delta,$$

then $f \in \mathcal{P}(1/2)$.

Next, we derive the following.

Theorem 2.5. *Suppose that $f \in \mathcal{A}$, $f(z)/z$ is bounded and not vanishing in the unit disc. If*

$$\left|\arg\left\{\left[\frac{z}{f(z)}\right]^{\frac{n-1}{n(1-\gamma)}}\left(\frac{f(z)}{z^{\gamma}}\right)^{\frac{1}{1-\gamma}}\left(\frac{f'(z)}{f(z)}-\frac{\gamma}{z}\right)-\gamma(1-\gamma)\right\}\right| < \frac{\pi}{2}+\arctan n, \tag{2.3}$$

then

$$\mathfrak{Re}\left\{\left(\frac{f(z)}{z}\right)^{\frac{1}{n(1-\gamma)}}\right\} > \gamma \qquad z \in \Delta,$$

where $\gamma < 1$ and $n = 1, 2, 3, ...$

Proof. Let us define the function $p(z)$ by

$$p(z) = \left(\frac{f(z)}{z^{\gamma}}\right)^{\frac{1}{1-\gamma}} = z + \cdots \qquad z \in \Delta, \tag{2.4}$$

for $f \in \mathcal{A}$. Then $p(z) \in \mathcal{A}$ and

$$p'(z) = \frac{1}{1-\gamma}\left(\frac{f(z)}{z^{\gamma}}\right)^{\frac{1}{1-\gamma}}\left(\frac{f'(z)}{f(z)}-\frac{\gamma}{z}\right) \qquad z \in \Delta, \tag{2.5}$$

which gives

$$\left(\frac{z}{p(z)}\right)^{\frac{n-1}{n}} p'(z) - \gamma \tag{2.6}$$

$$= \frac{1}{1-\gamma}\left\{\left[\frac{z}{f(z)}\right]^{\frac{n-1}{n(1-\gamma)}}\left(\frac{f(z)}{z^{\gamma}}\right)^{\frac{1}{1-\gamma}}\left(\frac{f'(z)}{f(z)}-\frac{\gamma}{z}\right)-\gamma(1-\gamma)\right\} \qquad z \in \Delta.$$

Applying Theorem 2.1, we have

$$\mathfrak{Re}\left\{\sqrt[n]{\frac{p(z)}{z}}\right\} > \gamma \qquad z \in \Delta,$$

that is

$$\mathfrak{Re}\left\{\left(\frac{f(z)}{z}\right)^{\frac{1}{n(1-\gamma)}}\right\} > \gamma \qquad z \in \Delta.$$

So the proof is completed. □

If in Theorem 2.5 we take $n = 2$ and $\gamma = 1/2$, then we get:

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Corollary 2.6. *Suppose that $f \in \mathcal{A}$, $f(z)/z$ is bounded and not vanishing in the unit disc. If*

$$\left|\arg\left\{f'(z) - \frac{f(z)}{2z} - \frac{1}{4}\right\}\right| < \frac{\pi}{2} + \arctan 2 \approx 153.4349... \qquad z \in \Delta, \tag{2.7}$$

then

$$\mathfrak{Re}\left\{\frac{f(z)}{z}\right\} > \frac{1}{2} \qquad z \in \Delta,$$

i.e. $f \in \mathcal{P}(1/2)$.

In the following, we will provide some sufficient inequalities for the functions belonging to the classes $\mathcal{S}^*(\gamma)$, $\mathcal{K}(\gamma)$ and $\mathcal{R}(\gamma)$.

Theorem 2.7. *Assume that $f \in \mathcal{A}$ and $0 \leq \gamma < 1$. If*

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\left(2 - \frac{zf'(z)}{f(z)}\right) + \frac{z^2 f''(z)}{f(z)} - \gamma\right\}\right| < \frac{3\pi}{4}, \tag{2.8}$$

then $f \in \mathcal{S}^*(\gamma)$.

Proof. For $0 \leq \gamma < 1$ let $p(z)$ be defined by

$$p(z) = \frac{1}{1-\gamma}\left(\frac{zf'(z)}{f(z)} - \gamma\right) \qquad z \in \Delta.$$

Then $p(z)$ is analytic in Δ, $p(0) = 1$ and

$$(1-\gamma)p(z) + (1-\gamma)zp'(z) = \frac{zf'(z)}{f(z)}\left(2 - \frac{zf'(z)}{f(z)}\right) + \frac{z^2 f''(z)}{f(z)} - \gamma.$$

Put $B(z) = 1 - \gamma = C(z)$. Now, from (2.8) and applying Lemma 1.1, we can obtain the result. □

Corollary 2.8. *Let $f \in \mathcal{A}$. If*

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\left(2 - \frac{zf'(z)}{f(z)}\right) + \frac{z^2 f''(z)}{f(z)}\right\}\right| < \frac{3\pi}{4},$$

then f is starlike function.

Theorem 2.9. *Assume that $f \in \mathcal{A}$. If*

$$\left|\arg\left\{\frac{2zf''(z) + z^2 f'''(z)}{f'(z)} + 1 - \left(\frac{zf''(z)}{f'(z)}\right)^2 - \gamma\right\}\right| < \frac{3\pi}{4}, \tag{2.9}$$

then $f \in \mathcal{K}(\gamma)$, where $0 \leq \gamma < 1$.

Proof. It is enough that $p(z)$ to be defined as follows:

$$p(z) = \frac{1}{1-\gamma}\left(1 + \frac{zf''(z)}{f'(z)} - \gamma\right) \qquad z \in \Delta.$$

Then applying Lemma 1.1, the result is obtained. □

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Corollary 2.10. *Let $f \in \mathcal{A}$. If*

$$\left|\arg\left\{\frac{2zf''(z)+z^2f'''(z)}{f'(z)}+1-\left(\frac{zf''(z)}{f'(z)}\right)^2\right\}\right|<\frac{3\pi}{4}, \tag{2.10}$$

then f is convex function.

Theorem 2.11. *Assume that $f \in \mathcal{A}$. If*

$$|\arg\{f'(z)+zf''(z)-\gamma\}|<\frac{3\pi}{4}, \tag{2.11}$$

then $f \in \mathcal{R}_\gamma$ where $0 \leq \gamma < 1$.

Proof. Suppose that $p(z)$ be defined by

$$(1-\gamma)p(z)+\gamma = f'(z) \qquad z \in \Delta.$$

Then applying Lemma 1.1, the result is obtained. □

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