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A Third-degree B-spline Collocation Scheme for Solving a Class of the Nonlinear Lane-Emden Type Equations

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Abstract. In this paper, we use a numerical method involving collocation method with third B-splines as basis functions for solving a class of singular initial value problems (IVPs) of Lane–Emden type equation. The original differential equation is modified at the point of singularity. The modified problem is then treated by using B-spline approximation. In the case of non-linear problems, we first linearize the equation using quasi-linearization technique and solve the resulting problem by a third degree B-spline function. Some numerical examples are included to demonstrate the feasibility and the efficiency of the proposed technique. The method is easy to implement and produces accurate results. The numerical results are also found to be in good agreement with the exact solutions.

Keywords: B-spline, Collocation method, Lane–Emden equation, Singular IVPs.

2000 Mathematics subject classification: 65D07, 65N35, 34-XX.

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1. Introduction

Many problems in mathematical physics and astrophysics can be modelled by the so-called initial value problems (IVPs) of Lane-Emden type equation [42, 1]

$$y'' + \frac{2}{x}y' + f(y) = 0, \quad 0 < x \leqslant 1, \tag{1.1}$$

subject to the conditions

$$y(0) = A, \quad y'(0) = B, \tag{1.2}$$

where A and B are constants and $f(y)$ is a real-valued continuous function. Eq. (1.1) with specified $f(y)$ is used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere and theory of thermionic currents [42]. Another important class of singular IVPs of the Lane-Emden type is

$$y'' + \frac{2}{x}y' + f(x, y) = g(x), \quad 0 < x \leqslant 1, \tag{1.3}$$

subject to condition (1.2), where $f(x, y)$ is a continuous real-valued function and $g \in C[0, 1]$.

Recently, many analytical methods have been used to solve Lane-Emden type equations, the main difficulty arises in the singularity of the equations at $x = 0$. Currently, most techniques which were used in handling the Lane-Emden type problems are based on either series solutions or perturbation techniques.

Bender et al. [5] proposed a new perturbation technique based on an artificial parameter δ, the method is often called δ-method. The equation of second kind, governing isothermal gas spheres, was solved using delta-expansion later in [49] and [50].

Mandelzweig and Tabakin [28] used the quasilinearization approach to solve the standard Lane-Emden equation. This method approximates the solution of a nonlinear differential equation by treating the nonlinear terms as a perturbation about the linear ones, and unlike perturbation theories is not based on the existence of some small parameters.

Shawagfeh [43] applied a nonperturbative approximate analytical solution for the Lane-Emden equation using the Adomian decomposition method (ADM). His solution was in the form of a power series. Dehghan [13], [14] used Padé approximants method to accelerate the convergence of the power series.

In [51], Wazwaz employed the ADM [15], [16] with an alternate framework designed to overcome the difficulty of the singular point. It was applied to the differential equations of Lane-Emden type. Further more author of [52] used the modified decomposition method for solving the analytical treatment of nonlinear differential equations such as the Lane-Emden type equation.

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Liao [27] provided an analytical algorithm for Lane-Emden type equations. This algorithm logically contains the well–known ADM. Different from all other analytical techniques provide us with a convenient way to adjust convergence regions even without Padé technique. This work was later extended in [50] where solutions were shown to converge depending on the choice of convergence control parameter taken.

J. H. He [22] employed Ritz's method to obtain an analytical solution of the problem. By the semi-inverse method, a variational principle is obtained for the Lane-Emden type equation.

Parand et al. [32]-[34] presented two numerical techniques to solve higher ordinary differential equations such as Lane-Emden. Their approach was based on the rational Chebyshev and rational Legendre Tau methods.

Ramos [38]–[41] solved Lane-Emden equations through different methods. Author of [39] presented the linearization method for singular initial-value problems in second-order ordinary differential equations such as Lane-Emden. These methods result in linear constant–coefficients ordinary differential equations which can be integrated analytically, thus yielding piecewise analytical solutions and globally smooth solutions. Later this author [41] developed piecewise-adaptive decomposition methods for the solution of nonlinear ordinary differential equations. In [40], series solutions of the Lane-Emden type equation have been obtained by writing this equation as a Volterra integral equation and assuming that the nonlinearities are sufficiently differentiable. These series solutions have been obtained by either working with the original differential equation or transforming it into an ordinary differential equation that does not contain the first–order derivatives. Series solutions to the Lane-Emden type equation have also been obtained by working directly on the original differential equation or transforming it into a simpler one.

Yousefi [54] presented a numerical method for solving the Lane-Emden equations. He converted Lane-Emden equations to integral equations, using integral operator, and then he applied Legendre wavelet approximations.

Bataineh et al. [4] presented an algorithm based on homotopy analysis method (HAM) [17] to obtain the exact and/or approximate analytical solutions of the singular IVPs of the Emden-Fowler type equation.

In [9], Chowdhury and Hashim presented an algorithm based on the homotopy-perturbation method (HPM) [20, 44, 45] to solve singular IVPs of time-independent equations.

Aslanov [3] introduced a further development in the ADM to overcome the difficulty at the singular point of non-homogeneous, linear and non-linear Lane-Emden-like equations.

Dehghan and Shakeri [19] applied an exponential transformation to the Lane-Emden type equations to overcome the difficulty of a singular point at

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$x = 0$ and solved the resulting nonsingular problem by the variational iteration method (VIM) [21, 47].

Yildirim and Öziş [53] presented approximate solutions of a class of Lane-Emden type singular IVPs problems, by the VIM.

Marzban et al. [29] used a method based upon hybrid function approximations. They used the properties of hybrid of block-pulse functions and Lagrange interpolating polynomials together for solving the nonlinear second-order initial value problems and the Lane-Emden equation.

Singh et al. [46] provided an efficient analytic algorithm for Lane-Emden type equations using modified HAM, also they used some well-known Lane-Emden type equations as test examples.

We refer the interested readers to [24, 25] for analysis of the Lane-Emden equation based on the Lie symmetry approach.

Parand et al. [35] proposed a collocation method for solving some well–known classes of Lane-Emden type equations which is based on a Hermite function collocation (HFC) method. This method reduces the solution of a problem to the solution of a system of algebraic equations.

It is worth noting that a different collocation scheme, the Boubaker Polynomials Expansion Scheme (BPES), was recently considered for the Lane-Emden equations in [7].

In 2011, Yuzbasi et al. used a numerical method based on the Bessel polynomials and collocation points for solving the Lane-Emden equations [55]-[58]. By using Bessel polynomials and collocation points, this method transforms the Lane-Emden equations into the matrix equation. The matrix equation corresponds to a system of linear equations with the unknown Bessel coefficients. This method gives the analytic solution when the exact solution is polynomial.

From the above remarks, we see that the Lane-Emden equations are an area of active interest. Hence the readers are advised to include these and other modern references, to see where the present results sit with the available literature.

The theory of spline functions is very active field of approximation theory, boundary value problems and partial differential equations, when numerical aspects are considered. Among the various classes of splines, the polynomial spline has been received the greatest attention primarily because it admits a basis of B-splines [6, 8, 12, 23, 30, 36] which can be accurately and efficiently computed. As the piecewise polynomial, B-spline have also become a fundamental tool for numerical methods to get the solution of the differential equations.

In this paper, we apply the B-spline method to obtain approximate solutions of the Lane-Emden-type equations. In the case of non-linear problems, quasilinearization technique, originally developed by Bellman and Kalaba [6], has been used to reduce the given non-linear problem to a sequence of linear

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TABLE 1. Table of values of $B_i(x)$ and its derivatives at nodal points.

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$B_i(x)$	0	1	4	1	0
$B_i^{'}(x)$	0	$\frac{3}{h}$	0	$\frac{-3}{h}$	0
$B_i^{''}(x)$	0	$\frac{6}{h^2}$	$\frac{-12}{h^2}$	$\frac{6}{h^2}$	0

problems. The linear problem by L-Hospital rule [8], is modified at the singular point. The numerical experiments for the model problems have been given to illustrate the method.

It should be noted that the Lane-Emden-type equations have been solved by some B-spline functions in [26] and [31]. In these papers, some properties of the B-spline functions are presented and utilized to reduce the solutions of the Lane-Emden equations to the solution of algebraic equations. However, the proposed scheme in this manuscript is completely different from the methods in [26] and [31] in linear and nonlinear cases. In fact, in [26] and [31] authors have used some properties of operational matrices of the derivative and integration. Then, the solution of the Lane-Emden equation is converted to the solution of a system of algebraic equations.

2. The Third-degree B-splines for Linear Problems

Consider equally-spaced knots of a partition $\pi : a = x_0 < x_1 < ... < x_N = b$ on $[a, b]$ such that $h = 1/N$. Let $S_3[\pi]$ be the space of continuously-differentiable, piecewise, third-degree polynomials on π. That is, $S_3[\pi]$ is the space of third-degree splines on π. $S_3[\pi]$ is a linear space [36]. The third-degree B-splines for $i = 1, 2, ..., N$ are defined as [23]

$$B_i(x) = \frac{1}{h^3} \begin{cases} (x - x_{i-2})^3, & x \in [x_{i-2}, x_{i-1}], \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & x \in [x_{i-1}, x_i], \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & x \in [x_i, x_{i+1}], \\ (x_{i+2} - x)^3, & x \in [x_{i+1}, x_{i+2}], \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

We introduce four additional knots $x_{-2} < x_{-1} < x_0$ and $x_N < x_{N+1} < x_{N+2}$. See Table 1. Let $\Omega = \{B_{-1}, B_0, B_1, ..., B_{N+1}\}$ and let $B_3(\pi) = \text{span } \Omega$. The functions in Ω are linearly independent on $[0, 1]$, thus $B_3(\pi)$ is $(N + 3)$ dimensional. Also $B_3(\pi) = S_3(\pi)$ [36]. Let $S(x)$ be the B-spline interpolating function $y(x)$ at the nodal points and $S(x) \in B_3(\pi)$. Then $S(x)$ can be written as

$$S(x) = \sum_{k=-1}^{N+1} c_k B_k(x). \tag{2.2}$$

In linear case of Eq. (1.3), we take $f(x, y) = r(x)y$; thus Eq. (1.3) becomes

$$y'' + \frac{2}{x}y' + r(x)y = g(x), \quad 0 < x \leqslant 1, \tag{2.3}$$

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with conditions

$$y(0) = A, \quad y'(0) = 0. \tag{2.4}$$

Now putting Eq. (2.2) in (2.3) and applying interpolation conditions for any ith nodal point $x = x_i$, $i = 1, 2, 3, ..., N$, Eq. (2.3) reduces to

$$\begin{aligned} &c_{i-1}(B''_{i-1}(x_i) + (2/x_i)B'_{i-1}(x_i) + r_iB_{i-1}(x_i)) + c_i(B''_i(x_i) + \\ &(2/x_i)B'_i(x_i) + r_iB_i(x_i)) + c_{i+1}(B''_{i+1}(x_i) + \\ &(2/x_i)B'_{i+1}(x_i) + r_iB_{i+1}(x_i)) = g_i, \forall i = 1, 2, ..., N, \end{aligned} \tag{2.5}$$

where $r(x_i) = r_i$ and $g(x_i) = g_i$. By simplifying Eq. (2.5), we get

$$\begin{aligned} &(6 - 6h/x_i + r_ih^2)c_{i-1} + (-12 + 4r_ih^2)c_i + \\ &(6 + 6h/x_i + r_ih^2)c_{i+1} = g_ih^2, \forall i = 1, 2, ..., N. \end{aligned} \tag{2.6}$$

At the singular point $x = 0$, we modify Eq. (2.3) by L-Hospital rule as

$$3y'' + r(x)y = g(x), \quad x = 0. \tag{2.7}$$

We apply the above mentioned B-spline method to (2.7) at $x = 0$ and obtain

$$3 \sum_{i=-1,0,1} c_iB''_i(x_0) + r_0 \sum_{i=-1,0,1} c_iB_i(x_0) = g_0. \tag{2.8}$$

Also Eq. (2.4) gives

$$\sum_{i=-1,0,1} c_iB_i(x_0) = A, \tag{2.9}$$

$$\sum_{i=-1,0,1} c_iB'_i(x_0) = 0. \tag{2.10}$$

(2.10) gives $c_1 = c_{-1}$. Now eliminating c_{-1} from Eqs. (2.8)–(2.10), we find

$$c_0(4 - 36/h^2 + 4r_0) + c_1(2 + 36/h^2 + 2r_0) = A + g_0. \tag{2.11}$$

Eqs. (2.6)-(2.11) lead to the tridiagonal system of $(N + 1)$ linear equations $Tx_N = d_N$ in the $(N + 2)$ unknowns $x_N = (c_0, c_1, ..., c_{N+1})^T$ with right hand side

$$d_N = (A + g_0, g_1h^2, ..., g_Nh^2)^T,$$

and the coefficient matrix T given as

$$T = \begin{bmatrix} 4 - 36/h^2 + 4r_0 & 2 + 36/h^2 + 2r_0 & 0 & \cdots & 0 & 0 \\ \alpha_1 & \beta_1 & \gamma_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \beta_2 & \gamma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & \dots & 0 & \alpha_N & \beta_N & \gamma_N \end{bmatrix},$$

where

$$\alpha_i = 6 - 6h/x_i + r_ih^2, \quad \beta_i = -12 + 4r_ih^2, \quad \gamma_i = 6 + 6h/x_i + r_ih^2, \ (i = 1, 2, ..., N).$$

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We solve the system $Tx_N = d_N$ by means of a home-made program which is based on singular value decomposition (SVD) method [37] and obtain the B-spline solutions $S(x_i)$ $(i = 0, ..., N)$. The condition number of T

$$\kappa_s(T) = \|T\|_s \|T^{-1}\|_s, \ s = 1, 2, \infty,$$

depends on function $r(x)$ in Eq. (2.3) and distance of collocation points h. Therefore a small perturbation in initial data $r(x)$ may produce a large amount of perturbation in the solution. Also the condition number grows with N for fixed function $r(x)$.

3. B-spline for non-Linear Problems

In the quasilinearization technique, the non-linear differential equation is solved recursively by a sequence of linear differential equations. The main advantage of this method is that if the procedure converges, it converges quadratically to the solution of the original problem. Quadratic convergence means that the error in the $(n+1)$th iteration is proportional to the square of the error in the (n)th iteration. The linear equation is obtained by using the first and second terms of the Taylor's series expansion of the original non-linear differential equation.

We consider initial value problems

$$y'' + \frac{2}{x}y' = f(x, y), \tag{3.1}$$

subject to conditions (2.4). Suppose $f(x, y)$ in Eq. (3.1) is non-linear in y and at the $(n+1)$th step it can be written as

$$(y'')^{n+1} + \frac{2}{x}(y')^{n+1} = f(x, y^{n+1}), \quad n = 0, 1, 2, ... \tag{3.2}$$

The non-linear term $f(x, y^{n+1})$ can be expanded as

$$f(x, y^{n+1}) = f(x, y^n) + (y^{n+1} - y^n)f'(x, y^n) + ..., \quad n = 0, 1, 2, ... \tag{3.3}$$

We choose a reasonable initial approximation for the function y in $f(x, y)$, as y^0. Now Eq. (3.2) can be written as

$$(y'')^{n+1} + \frac{2}{x}(y')^{n+1} + p^{(n)}(x)(y)^{n+1} = q^{(n)}(x), \tag{3.4}$$

subject to the initial conditions

$$(y)^{n+1}(0) = A, \quad (y')^{n+1}(0) = 0, \tag{3.5}$$

where

$$p^{(n)}(x) = -f'(x, y^n), \ q^{(n)}(x) = f(x, y^n) - y^n f'(x, y^n). \tag{3.6}$$

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In order to solve the system of linear singular IVP given by Eq. (3.4) subject to Eq. (3.5), we use B-spline as described above. At the ith nodal point we get

$$c_{i-1}^{n+1}(B_{i-1}''(x_i)+(2/x_i)B_{i-1}'(x_i)+p_i^nB_{i-1}(x_i))+c_i^{n+1}(B_i''(x_i)+(2/x_i)B_i'(x_i)+p_i^nB_i(x_i))+c_{i+1}^{n+1}(B_{i+1}''(x_i)+(2/x_i)B_{i+1}'(x_i)+p_i^nB_{i+1}(x_i))=q_i^n,\forall i=1,2,...,N,\quad \forall n=0,1,2,..., \tag{3.7}$$

where $p_i^n=p^n(x_i)$ and $q_i^n=q^n(x_i)$. As discussed earlier, at the singular point $x=x_0$, the modified differential equation can be written as

$$3(y'')^{n+1}+p^{(n)}(x)(y)^{n+1}=q^{(n)}(x),\quad x=x_0,\quad \forall n=0,1,\cdots. \tag{3.8}$$

We will now employ the B-spline method at $x=x_0$ for this modified Eq. (3.8) as

$$3\sum_{i=-1,0,1}c_i^{n+1}B_i''(x_0)+p_0\sum_{i=-1,0,1}c_i^{n+1}B_i(x_0)=q_0^n\quad \forall n=0,1,2,.. \tag{3.9}$$

Initial conditions (3.5) can be written as

$$\sum_{i=-1,0,1}c_i^{n+1}B_i(x_0)=A, \tag{3.10}$$

and

$$\sum_{i=-1,0,1}c_i^{n+1}B_i'(x_0)=0. \tag{3.11}$$

Using Eqs. (3.7) and (3.9)-(3.11), we will get the tridiagonal system as obtained in the previous section. The approximate solution values, at each stage of the iteration ($n=0,1,2,...$) are computed untill the convergence criteria $|s_i^{n+1}-s_i^n|<\epsilon$, where ϵ the prescribed value, is satisfied.

4. Numerical Results

In this section, the numerical results of some model examples are presented. To compare the solutions, we define an error function

$$e(x_i)=|S(x_i)-y(x_i)|,\quad i=0,1,\cdots,N, \tag{4.1}$$

where we assume $S(x_i)=s_i$ be B-spline solutions and $y(x_i)=y_i$ be the exact solution.

Example 1:

We consider the linear, nonhomogeneous Lane-Emden equation,

$$y''+\frac{2}{x}y'+y=6+12x+x^2+x^3, \tag{4.2}$$

subject to the initial conditions

$$y(0)=0, \tag{4.3}$$

$$y'(0)=0, \tag{4.4}$$

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TABLE 2. Comparison of $s(x)$ and $y(x)$, between present method and exact solution for Example 1.

x_i	$s_i(h = 1/10)$	$s_i(h = 1/40)$	$s_i(h = 1/80)$	y_i
0.000	-0.0007371902	0.00318296172	0.0000813798	0
0.025	-	0.0038390921	0.000700475015	0.000640625000
0.050	-	0.00589113	0.0026598904	0.002625000
0.075	-	0.00879089700	0.0061403999	0.00604687500
0.100	0.010209798	0.01348570	0.011006400	0.01100000
0.200	0.05275542	0.05126506	0.04805612	0.04800000
0.300	0.12628751	0.11951704	0.11700966	0.1170000
0.400	0.2357222	0.227181926	0.224101022	0.224000000
0.500	0.393020877	0.3779410	0.37509730	0.37500000
0.600	0.580976605	0.57810760	0.57605819	0.57600000
0.700	0.838157713	0.836072132	0.833043800	0.83300000
0.800	1.1562238	1.15435219	1.152007730	1.15200000
0.900	1.538792788	1.541118932	1.5389952	1.53900000
1	2.005683660	2.0033839	2.00007706	2.00000

which has the following analytical solution:

$$y(x) = x^2 + x^3. \tag{4.5}$$

This equation has been solved by Ramos [39], Yildirim and Ö ziş [53], Chowdhury and Hashim [10], Zhang et al. [59], with linearization, VIM, HPM and TSADM methods respectively. Table 2 shows the comparison of $s(x)$ obtained by the method proposed in this paper with different values of h, and analytic solution (4.5). The resulting graph of Eq. (4.2) by our method in comparison to the analytic solution is shown in Figure 1.

Example 2 : (Isothermal gas spheres equation)

Isothermal gas spheres equation are modelled by

$$y'' + \frac{2}{x}y' + e^y = 0, \tag{4.6}$$

subject to the initial conditions

$$y(0) = 0, \tag{4.7}$$

$$y'(0) = 0. \tag{4.8}$$

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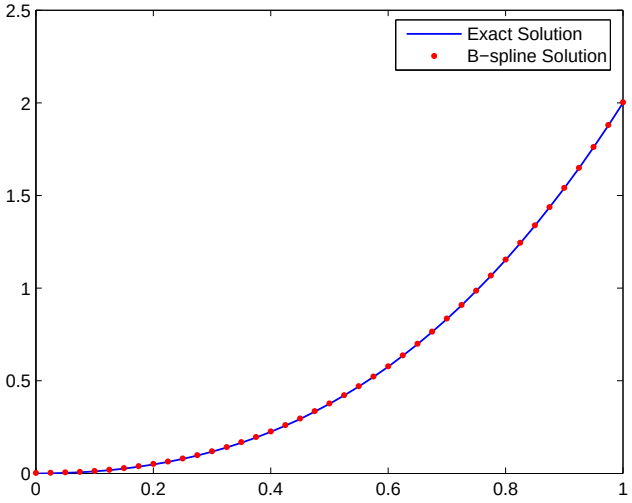


FIGURE 1. Graph of equation of Example 1 in comparing the presented method and analytic solution.

This model can be used to view the isothermal gas spheres, where the temperature remains constant. For a discussion of the formulation of Eq. (4.6), see [11].

A series solution obtained by Wazwaz [51], Liao [27], Singh et al. [46] and Ramos [40] by using ADM, ADM, MHAM and series expansion respectively:

$$y(x) = -\frac{1}{6}x^2 + \frac{1}{5.4!}x^4 - \frac{8}{21.6!}x^6 + \frac{122}{81.8!}x^8 - \frac{61.67}{495.10!}x^{10} + ... \qquad (4.9)$$

We recall that this equation has been solved by Chowdhury and Hashim [9], Aslanov [3], Aslanov [2], Bataineh, Noorani and Hashim, [4] with HPM, series solutions, ADM and HAM methods respectively. We intend to apply the B-spline method to solve the isothermal gas spheres equation (4.6) too.

Table 3 shows the comparison of $s(x)$ obtained by our method with $N = 400, n = 5$ and 4-term HPM solution of (4.9). The resulting graph of the isothermal gas spheres equation in comparison to the presented method with $n = 15$ and $N = 40$ and 4-term HPM solution of (4.9) is shown in Figure 2. The error function in (4.1) can be seen in Figure 3. This graph shows that the new method has an appropriate convergence rate.

Example 3: (Richardson's theory of thermionic currents)

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TABLE 3. Comparison of $s(x)$ and $y(x)$, between our method and 4-term HPM solution for isothermal gas spheres equation.

x_i	s_i	y_i	e_i
0.000	0.0000019	0.000000	1.9563×10^{-6}
0.100	-0.001662	-0.001665	3.2556×10^{-6}
0.200	-0.006648	-0.006653	5.2904×10^{-6}
0.300	-0.014926	-0.014932	6.3796×10^{-6}
0.400	-0.026452	-0.026455	2.7937×10^{-6}
0.500	-0.041154	-0.041153	5.2559×10^{-7}
0.600	-0.058943	-0.058944	8.3931×10^{-7}
0.700	-0.079724	-0.079725	1.8277×10^{-6}
0.800	-0.103387	-0.103385	1.4790×10^{-6}
0.900	-0.129802	-0.129797	5.1157×10^{-6}
1	-0.158831	-0.158825	6.4673×10^{-6}

Next we consider the nonlinear differential equation of Richardsons theory of thermionic currents

$$y'' + \frac{2}{x}y' + e^{-y} = 0, \tag{4.10}$$

subject to the initial conditions

$$y(0) = 0, \tag{4.11}$$

$$y'(0) = 0. \tag{4.12}$$

This model can be used when the density and electric force of an electron gas in the neighborhood of a hot body in thermal equilibrium is to be determined.

A series solution obtained by Wazwaz [52] by using ADM and series expansion is

$$y(x) = -\frac{1}{6}x^2 - \frac{1}{5.4!}x^4 - \frac{8}{21.6!}x^6 - \frac{122}{81.8!}x^8 - \frac{61.67}{495.10!}x^8 + ... \tag{4.13}$$

We recall that this equation has been solved by Chowdhury and Hashim [10] with HPM series solution. We intend to apply the B-spline method to solve the Richardson's theory of thermionic currents (4.10) too. Table 4 shows the comparison of $s(x)$ obtained by the method proposed in this paper with 4-term HPM solution in (4.13). In order to compare our method with 4-term HPM solution in (4.13), the resulting graph of Eq. (4.10) is shown in Figure 4. It should be noted that these solutions correspond to the iteration index $n = 5$ by taking $u^0 = 0$ as the initial approximation. The iterations were stopped by

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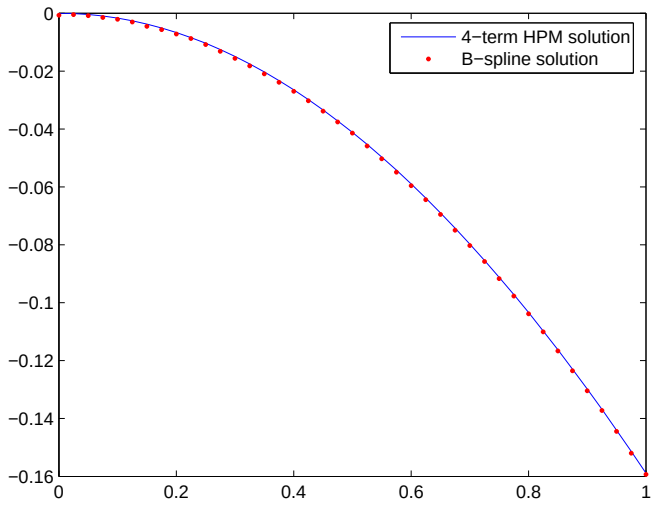


FIGURE 2. Graph of equation of Example 2 in comparing the presented method and the 4-term HPM solution.

satisfying the absolute error criterion $|s_i^{n+1} - s_i^n| \leqslant 10^{-4}$ for all i. The error function (4.1) can be also seen in Figure 5.

Example 4: (Homogeneous case)

Now we consider the nonlinear, homogeneous Lane-Emden-type equation,

$$y'' + \frac{2}{x}y' + 4(2e^y + e^{y/2}) = 0, \tag{4.14}$$

subject to the initial conditions

$$y(0) = 0, \tag{4.15}$$

$$y'(0) = 0. \tag{4.16}$$

which has the following analytical solution

$$y(x) = -2\ln(1 + x^2). \tag{4.17}$$

This type of equation has been solved by Yildirim and Ö ziş [53] and Chowdhury and Hashim [10] with VIM and HPM methods respectively. Table 5 shows the comparison of $s(x)$, correspond to the iteration index $n = 10$ and $y^0 = 0$ as the initial approximation, obtained by our method and the analytic solution (4.17). The error function (4.1) can be also seen in Figure 6.

Example 5: (Nonhomogeneous case)

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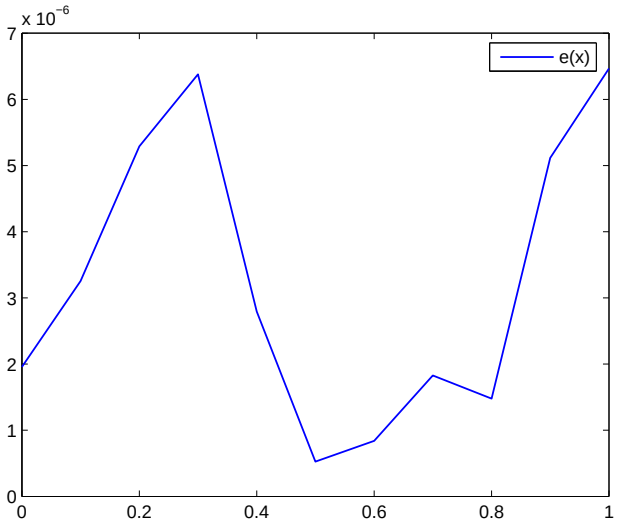


FIGURE 3. The error function of Example 2.

Finally we consider the nonlinear, nonhomogeneous Lane-Emden equation,

$$y'' + \frac{2}{x}y' + y^3 = 6 + x^6, \tag{4.18}$$

subject to the initial conditions

$$y(0) = 0, \tag{4.19}$$

$$y'(0) = 0, \tag{4.20}$$

which has the following analytical solution:

$$y(x) = x^2. \tag{4.21}$$

This type of equation has been solved by [10] with HPM method. Table 6 shows the comparison of $s(x)$, correspond to the iteration index $n = 15$ and $y^0 = 0$ as the initial approximation, obtained by our method and the analytic solution (4.17). In order to compare the present method with those obtained by Chowdhury and Hashim [10] the resulting graph of Eq. (4.18) is shown in Figure 7.

5. Conclusion

In this paper, we have applied the family of B-spline functions to obtain the numerical solution of singular IVPs of Lane–Emden-type equations. This

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TABLE 4. Comparison of $s(x)$ and $y(x)$, between present method and 4-term HPM solution for Richardsons theory of thermionic currents in Example 3.

x_i	s_i	y_i	e_i
0.000	0.000023	0.000000	2.3473×10^{-5}
0.100	-0.001645	-0.001667	2.1890×10^{-5}
0.200	-0.006660	-0.006680	1.9794×10^{-5}
0.300	-0.015047	-0.015067	1.9900×10^{-5}
0.400	-0.026861	-0.026882	2.0715×10^{-5}
0.500	-0.042177	-0.042195	1.8134×10^{-5}
0.600	-0.061088	-0.061105	1.6873×10^{-5}
0.700	-0.083713	-0.083731	1.8086×10^{-5}
0.800	-0.110206	-0.110224	1.8168×10^{-5}
0.900	-0.140753	-0.140764	1.1084×10^{-6}
1	-0.175553	-0.175566	1.2710×10^{-5}

TABLE 5. Comparison of $s(x)$ and $y(x)$, between present method and exact solution for Homogeneous case in Example 4.

x_i	s_i	y_i	e_i
0.100	-0.020161	-0.019900	2.608403×10^{-4}
0.200	-0.078361	-0.078441	7.950440×10^{-5}
0.300	-0.172637	-0.172355	2.823797×10^{-4}
0.400	-0.297300	-0.296840	4.602114×10^{-4}
0.500	-0.446798	-0.446287	5.111812×10^{-4}
0.600	-0.615341	-0.614969	3.717533×10^{-4}
0.700	-0.797702	-0.797552	1.503498×10^{-4}
0.800	-0.989376	-0.989392	1.633027×10^{-5}
0.900	-1.186689	-1.186653	3.57437×10^{-5}
1	-1.385376	-1.386294	9.181566×10^{-4}

method gives comparable results and is easy to compute. Also this method produces a spline function which may be used to obtain the solution at any

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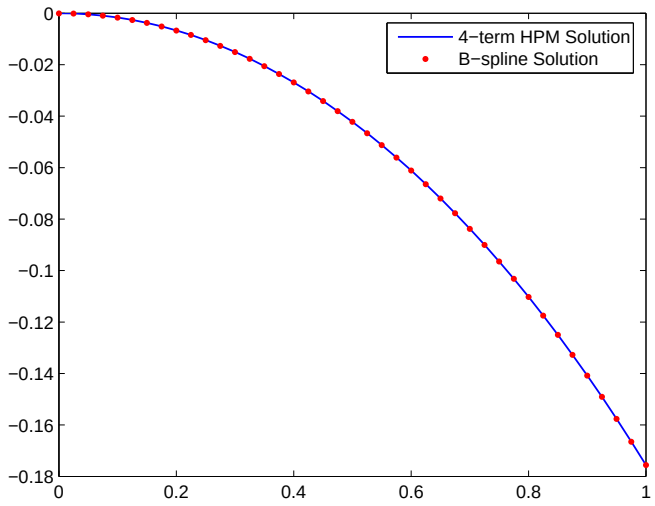


FIGURE 4. Graph of equation of Example 3 in comparing the presented method and the 4-term HPM solution.

point in the range. Most importantly, the third-degree B-spline methods are especially advisable for obtaining numerical solutions of differential equations when higher continuity of the solutions exist.

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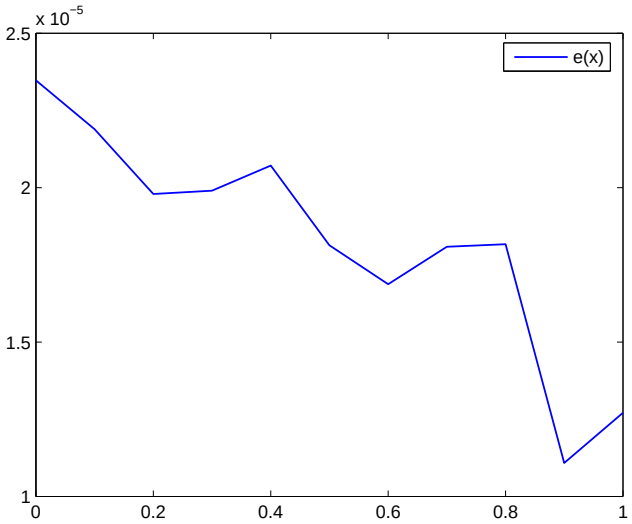


FIGURE 5. The error function of Example 3.

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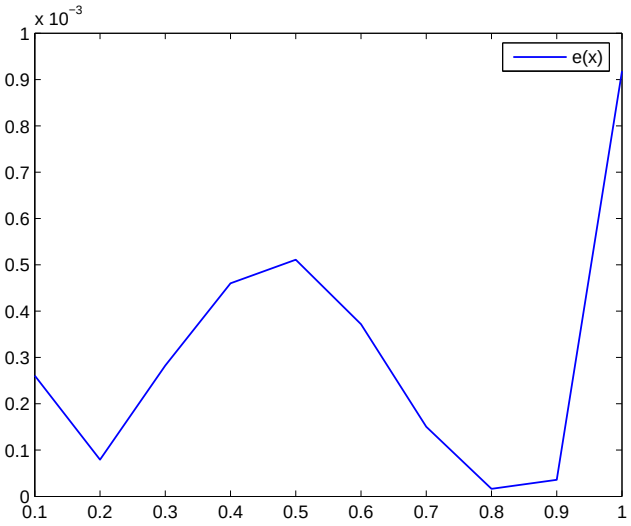


FIGURE 6. The error function of Example 4.

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TABLE 6. Comparison of $s(x)$ and $y(x)$, between present method and exact solution for Nonhomogeneous case in Example 5.

x_i	$s_i(h = 1/20)$	$s_i(h = 1/60)$	$s_i(h = 1/80)$	Exact Solution
0.000	0.003316	0.000210	-0.000112	0
0.025	-	-	0.000613	0.0006
0.050	0.003687	0.002938	0.002397	0.002500
0.075	-	-	0.005431	0.0056
0.100	0.009490	0.010115	0.009944	0.010000
0.200	0.038595	0.040223	0.039869	0.040000
0.300	0.091837	0.089873	0.090041	0.090000
0.400	0.162574	0.160074	0.159941	0.160000
0.500	0.247651	0.250276	0.249994	0.250000
0.600	0.360507	0.359981	0.359902	0.360000
0.700	0.485189	0.489921	0.489860	0.490000
0.800	0.639024	0.640047	0.639944	0.640000
0.900	0.804949	0.810226	0.809861	0.810000
1	1.002203	1.000145	0.999964	1.000000

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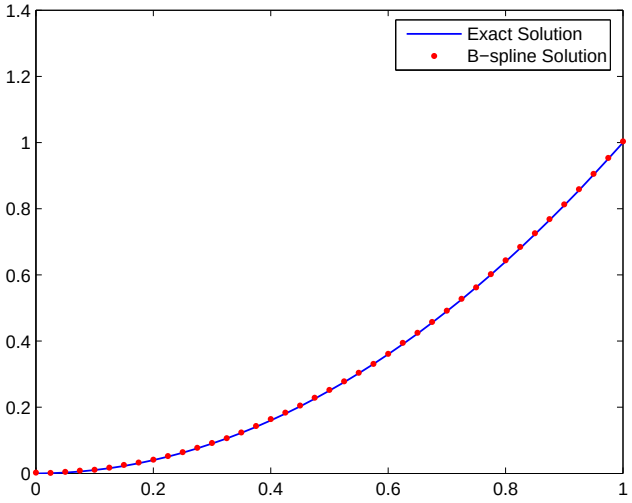


FIGURE 7. Graph of equation of Example 5 in comparing the presented method and the analytical solution.

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