

## Generalized Douglas-Weyl Finsler Metrics

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ABSTRACT. In this paper, we study generalized Douglas-Weyl Finsler metrics. We find some conditions under which the class of generalized Douglas-Weyl  $(\alpha, \beta)$ -metric with vanishing S-curvature reduce to the class of Berwald metrics.

**Keywords:** Generalized Douglas-Weyl metrics, S-curvature.

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### 1. INTRODUCTION

Let  $(M, F)$  be a Finsler manifold. In local coordinates, a curve  $c(t)$  is a geodesic if and only if its coordinates  $(c^i(t))$  satisfy  $\ddot{c}^i + 2G^i(\dot{c}) = 0$ , where the local functions  $G^i = G^i(x, y)$  are called the spray coefficients [10].  $F$  is called a Berwald metric, if  $G^i$  are quadratic in  $y \in T_x M$  for any  $x \in M$  or equivalently  $G^i = \frac{1}{2}\Gamma_{jk}^i(x)y^j y^k$ . As a generalization of Berwald curvature, Bácsó-Matsumoto introduced the notion of Douglas metrics which are projective invariants in Finsler geometry [2].  $F$  is called a Douglas metric if  $G^i = \frac{1}{2}\Gamma_{jk}^i(x)y^j y^k + P(x, y)y^i$ .

A Finsler metric  $F$  is called generalized Douglas-Weyl metric (briefly, GDW-metric) if  $D_{jkl|m}^i y^m = T_{jkl} y^i$  holds for some tensor  $T_{jkl}$ , where  $D_{jkl|m}^i$  denotes the horizontal covariant derivatives of  $D_{jkl}^i$  with respect to the Berwald

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connection of  $F$  [8][18]. For a manifold  $M$ , let  $\mathcal{GDW}(M)$  denotes the class of all Finsler metrics satisfying in above relation for some tensor  $T_{jkl}$ . In [3], Bácsó-Papp showed that  $\mathcal{GDW}(M)$  is closed under projective changes. Then, Najafi-Shen-Tayebi characterized generalized Douglas-Weyl Randers metrics [8]. In [18], it is proved that all generalized Douglas-Weyl spaces with vanishing Landsberg curvature have vanishing the quantity  $\mathbf{H}$ . For other works, see [12] and [13].

The notion of S-curvature is originally introduced by Shen for the volume comparison theorem [9]. The Finsler metric  $F$  is said to be of isotropic S-curvature if  $\mathbf{S} = (n+1)cF$ , where  $c = c(x)$  is a scalar function on  $M$ . In [14], it is shown that every isotropic Berwald metric has isotropic S-curvature. In [4], Cheng-Shen show that every  $(\alpha, \beta)$ -metric with constant Killing 1-form has vanishing S-curvature. Then, Bácsó-Cheng-Shen proved that a Finsler metric  $F = \alpha \pm \beta^2/\alpha + \epsilon\beta$  has vanishing S-curvature if and only if  $\beta$  is a constant Killing 1-form [1]. Therefore, the Finsler metrics with vanishing S-curvature are of some important geometric structures which deserve to be studied deeply.

An  $(\alpha, \beta)$ -metric is a Finsler metric on  $M$  defined by  $F := \alpha\phi(s)$ ,  $s = \beta/\alpha$ , where  $\phi = \phi(s)$  is a  $C^\infty$  function on the  $(-b_0, b_0)$  with certain regularity,  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  is a Riemannian metric and  $\beta(y) = b_i(x)y^i$  is a 1-form on  $M$  [6]. In this paper, we are going to study generalized Douglas-Weyl  $(\alpha, \beta)$ -metrics with vanishing S-curvature.

**Theorem 1.1.** *Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be an  $(\alpha, \beta)$ -metric on a manifold  $M$  of dimension  $n \geq 3$ . Suppose that*

$$F \neq c_3\alpha \left(\frac{\beta}{\alpha}\right)^{\frac{c_2}{1+c_2}} \left(c_1\frac{\beta}{\alpha} + c_2 + 1\right)^{\frac{1}{1+c_2}} \quad \text{and} \quad F \neq d_1\sqrt{\alpha^2 + d_2\beta^2} + d_3\beta.$$

where  $c_1, c_2, c_3, d_1, d_2$  and  $d_3$  are real constants. Let  $F$  has vanishing S-curvature. Then  $F$  is a GDW-metric if and only if it is a Berwald metric.

## 2. PRELIMINARY

Given a Finsler manifold  $(M, F)$ , then a global vector field  $\mathbf{G}$  is induced by  $F$  on  $TM_0$ , which in a standard coordinate  $(x^i, y^i)$  for  $TM_0$  is given by  $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$ , where

$$G^i := \frac{1}{4}g^{il} \left\{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \right\}, \quad y \in T_x M.$$

The  $\mathbf{G}$  is called the spray associated to  $F$ .

Define  $\mathbf{B}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$  and  $\mathbf{E}_y : T_x M \otimes T_x M \rightarrow \mathbb{R}$  by  $\mathbf{B}_y(u, v, w) := B^i_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$  and  $\mathbf{E}_y(u, v) := E_{jk}(y)u^j v^k$  where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} := \frac{1}{2}B^m_{jkm}.$$

$\mathbf{B}$  and  $\mathbf{E}$  are called the Berwald curvature and mean Berwald curvature, respectively.  $F$  is called a Berwald and weakly Berwald if  $\mathbf{B} = \mathbf{0}$  and  $\mathbf{E} = 0$ , respectively [5][7].

Let

$$D_{j\ kl}^i := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left( G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right).$$

It is easy to verify that  $\mathcal{D} := D_{j\ kl}^i dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$  is a well-defined tensor on slit tangent bundle  $TM_0$ . We call  $\mathcal{D}$  the Douglas tensor. A Finsler metric with  $\mathcal{D} = 0$  is called a Douglas metric. The notion of Douglas metrics was proposed by Bácsó-Matsumoto as a generalization of Berwald metrics [2]. The Douglas tensor  $\mathcal{D}$  is a non-Riemannian projective invariant, namely, if two Finsler metrics  $F$  and  $\bar{F}$  are projectively equivalent,  $G^i = \bar{G}^i + Py^i$ , where  $P = P(x, y)$  is positively  $y$ -homogeneous of degree one, then the Douglas tensor of  $F$  is same as that of  $\bar{F}$ . Finsler metrics with vanishing Douglas tensor are called Douglas metrics [11].

For a Finsler metric  $F$  on an  $n$ -dimensional manifold  $M$ , the Busemann-Hausdorff volume form  $dV_F = \sigma_F(x) dx^1 \cdots dx^n$  is defined by

$$\sigma_F(x) := \frac{\text{Vol}(\mathbb{B}^n(1))}{\text{Vol}\left[\left(y^i \in \mathbb{R}^n \mid F\left(y^i \frac{\partial}{\partial x^i} \Big|_x\right) < 1\right)\right]}.$$

Let  $G^i$  denote the geodesic coefficients of  $F$  in the same local coordinate system. The S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i} \left[ \ln \sigma_F(x) \right],$$

where  $\mathbf{y} = y^i \frac{\partial}{\partial x^i} \Big|_x \in T_x M$ .  $\mathbf{S}$  is said to be isotropic if there is a scalar functions  $c = c(x)$  on  $M$  such that  $\mathbf{S} = (n+1)cF$ .

For an  $(\alpha, \beta)$ -metric  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , put

$$\Phi := -(q - sq')[n\Delta + 1 + sq] - (b^2 - s^2)(1 + sq)q'',$$

where

$$q := \frac{\phi'}{\phi - s\phi'}, \quad \Delta := 1 + sq + (b^2 - s^2)q'.$$

In [4], Cheng-Shen characterize  $(\alpha, \beta)$ -metrics with isotropic S-curvature.

**Lemma 2.1.** ([4]) Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be an non-Riemannian  $(\alpha, \beta)$ -metric on a manifold  $M$  of dimension  $n \geq 3$ . Suppose that  $\phi \neq c_1\sqrt{1 + c_2s^2} + c_3s$  for any constant  $c_1 > 0$ ,  $c_2$  and  $c_3$ . Then  $F$  is of isotropic S-curvature  $\mathbf{S} = (n+1)cF$  if and only if one of the following holds

(a)  $\beta$  satisfies

$$r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0, \quad (2.1)$$

where  $\varepsilon = \varepsilon(x)$  is a scalar function,  $b := \|\beta_x\|_\alpha$  and  $\phi = \phi(s)$  satisfies

$$\Phi = -2(n+1)k \frac{\phi \Delta^2}{b^2 - s^2}, \quad (2.2)$$

where  $k$  is a constant. In this case,  $\mathbf{S} = (n+1)cF$  with  $c = k\varepsilon$ .

(b)  $\beta$  satisfies

$$r_{ij} = 0, \quad s_j = 0 \quad (2.3)$$

In this case,  $\mathbf{S} = 0$ .

The characterization of Finsler metrics with isotropic S-curvature in Cheng-Shen's paper is not complete [4]. Their result is correct for dimension  $n \geq 3$ . For the case  $\text{dimension}(M) = 2$ , see [16].

### 3. PROOF OF MAIN RESULTS

Let  $F := \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be an  $(\alpha, \beta)$ -metric on a manifold  $M$ , where  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  and  $\beta(y) = b_i(x)y^i$ . Define  $b_{i|j}$  by  $b_{i|j}\theta^j := db_i - b_j\theta_i^j$ , where  $\theta^i := dx^i$  and  $\theta_i^j := \tilde{\Gamma}_{ik}^j dx^k$  denote the Levi-Civita connection forms of  $\alpha$ . Let

$$\begin{aligned} r_{ij} &:= \frac{1}{2} [b_{i|j} + b_{j|i}], & s_{ij} &:= \frac{1}{2} [b_{i|j} - b_{j|i}], \\ r_{i0} &:= r_{ij}y^j, & r_{00} &:= r_{ij}y^i y^j, & r_j &:= b^i r_{ij}, & t_j^i &:= s^i_m s_j^m \\ s_{i0} &:= s_{ij}y^j, & s_j &:= b^i s_{ij}, & r_0 &:= r_j y^j, & s_0 &:= s_j y^j. \end{aligned}$$

Then  $\beta = b_i(x)y^i$  is a constant Killing one-form on  $M$  if  $r_{ij} = s_j = 0$  hold. By definition, we have

$$b_{i|j} = s_{ij} + r_{ij}.$$

Since  $y^i|_s = 0$ , then for a constant Killing 1-form  $\beta$  we have

$$r_{00} = 0, \quad r_i + s_i = 0.$$

For an  $(\alpha, \beta)$ -metric  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , the following hold.

**Proposition 3.1.** *Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be an  $(\alpha, \beta)$ -metric on an  $n$ -dimensional manifold  $M$  of dimension  $n \geq 3$ , where  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  is a Riemannian metric and  $\beta = b_i(x)y^i$  is a one-form on  $M$ . Suppose that  $F$  is of vanishing S-curvature. Then  $F$  is a GDW-metric if and only if the following holds*

$$\begin{aligned} C_1 s_{j0|0}y^i - (C_2 y_j + C_3 b_j)y^i t_{00} &= C_4 y_j s^i_{0|0} + C_5 (b_j s^i_{0|0} + s_{j0} s^i_0) \\ &+ C_6 s^i_{j|0} + C_7 (y_j t^i_0 + s_{j0} s^i_0) + C_8 b_j t^i_0, \end{aligned} \quad (3.1)$$

where

$$\begin{aligned}
C_1 &:= -\left[(n+1)Q_\alpha + 2\beta Q_{\alpha\beta}\right]\alpha^{-3} - \left[Q_{\alpha\alpha} + b^2 Q_{\beta\beta}\right]\alpha^{-2}, \\
C_2 &:= (n+1)\left[Q_\alpha^2 + QQ_{\alpha\alpha} - \alpha^{-1}QQ_\alpha\right]\alpha^{-4} - 2\left[Q_\alpha Q_\beta + QQ_{\alpha\beta}\right]\beta\alpha^{-5} \\
&\quad + 2\left[2Q_\alpha Q_{\alpha\beta} + Q_{\alpha\alpha}Q_\beta + QQ_{\alpha\alpha\beta}\right]\beta\alpha^{-4} + b^2\left[2Q_{\alpha\beta}Q_\beta + Q_\alpha Q_{\beta\beta}\right]\alpha^{-3} \\
&\quad + \left[b^2 QQ_{\alpha\beta\beta} + 3Q_\alpha Q_{\alpha\alpha} + QQ_{\alpha\alpha\alpha}\right]\alpha^{-3}, \\
C_3 &:= (n+3)\left[Q_\alpha Q_\beta + QQ_{\alpha\beta}\right]\alpha^{-3} + 2\left[Q_\alpha Q_{\beta\beta} + QQ_{\alpha\beta\beta}\right]\beta\alpha^{-3} \\
&\quad + \left[2Q_\alpha Q_{\alpha\beta} + Q_\beta Q_{\alpha\alpha} + QQ_{\alpha\alpha\beta} + 4\beta\alpha^{-1}Q_\beta Q_{\alpha\beta}\right]\alpha^{-2} \\
&\quad + b^2\left[3Q_\beta Q_{\beta\beta} + QQ_{\beta\beta\beta}\right]\alpha^{-2}, \\
C_4 &:= -\left[(n+1)Q_\alpha + 2\beta Q_{\alpha\beta}\right]\alpha^{-3} + 2\left[\beta Q_{\alpha\alpha\beta} + Q_{\alpha\alpha}\right]\alpha^{-2} \\
&\quad + \left[b^2 Q_{\alpha\beta\beta} + Q_{\alpha\alpha\alpha}\right]\alpha^{-1}, \\
C_5 &:= (n+3)\alpha^{-1}Q_{\alpha\beta} + Q_{\alpha\alpha\beta} + 2\beta\alpha^{-1}Q_{\alpha\beta\beta} + b^2 Q_{\beta\beta\beta}, \\
C_6 &:= (n+1)\alpha^{-1}Q_\alpha + Q_{\alpha\alpha} + 2\beta\alpha^{-1}Q_{\alpha\beta} + b^2 Q_{\beta\beta}, \\
C_7 &:= (n+1)\alpha^{-3}QQ_\alpha - (n+1)\alpha^{-2}(Q_\alpha^2 + QQ_{\alpha\alpha}) - 2\beta\alpha^{-2}QQ_{\alpha\alpha\beta} \\
&\quad + 2\left[QQ_{\alpha\beta} + Q_\alpha Q_\beta\right]\beta\alpha^{-3} - b^2\left[QQ_{\alpha\beta\beta} + 2Q_{\alpha\beta}Q_\beta\right]\alpha^{-1} \\
&\quad - 2\left[2Q_\alpha Q_{\alpha\beta} + Q_\beta Q_{\alpha\alpha}\right]\beta\alpha^{-2} \\
&\quad - b^2\alpha^{-1}Q_\alpha Q_{\beta\beta} - 3\alpha^{-1}Q_\alpha Q_{\alpha\alpha} - 2\alpha^{-1}QQ_{\alpha\alpha\alpha}, \\
C_8 &:= -(n+3)\left[QQ_{\alpha\beta} + Q_\alpha Q_\beta\right]\alpha^{-1} - 2\left[2Q_\beta Q_{\alpha\beta} + QQ_{\alpha\beta\beta} + Q_\alpha Q_{\beta\beta}\right]\beta\alpha^{-1} \\
&\quad - b^2\left[QQ_{\beta\beta\beta} + 3Q_\beta Q_{\beta\beta}\right] - Q_\beta Q_{\alpha\alpha} - QQ_{\alpha\alpha\beta} - 2Q_\alpha Q_{\alpha\beta}.
\end{aligned}$$

*Proof.* Let  $G^i$  and  $G_\alpha^i$  denote the spray coefficients of  $F$  and  $\alpha$ , respectively, in the same coordinate system. Then, we have

$$G^i = G_\alpha^i + Py^i + Q^i, \quad (3.2)$$

where

$$\begin{aligned}
Q &:= \alpha q = \frac{\alpha\phi'}{\phi - s\phi'}, \\
P &:= \alpha^{-1}\Theta(r_{00} - 2Qs_0), \quad Q^i := Qs^i_0 + \Psi(r_{00} - 2Qs_0)b^i, \\
\Theta &= \frac{q - sq'}{2\Delta} = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi[(\phi - s\phi') + (b^2 - s^2)\phi'']} \\
\Psi &:= \frac{q'}{2\Delta} = \frac{1}{2} \frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.
\end{aligned}$$

By Lemma 2.1, we have  $r_{00} = s_0 = 0$ . Then (3.2) reduces to following

$$G^i = G_\alpha^i + Qs_0^i. \quad (3.3)$$

Let “ $\parallel$ ” and “ $|$ ” denote the covariant differentiations with respect to  $G^i$  and  $G_\alpha^i$  respectively. Then by (3.3), we have

$$\begin{aligned} D_{jkl\parallel m}^i y^m &= D_{jkl\parallel m}^i y^m - 2Qs_0^p \frac{\partial D_{jkl}^i}{\partial y^p} + D_{jkl}^p \tilde{N}_p^i - D_{pkl}^i \tilde{N}_j^p \\ &\quad - D_{jpl}^i \tilde{N}_k^p - D_{jkp}^i \tilde{N}_l^p, \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} D_{jkl\parallel m}^i y^m &= \alpha^{-4}(Q_{\alpha\alpha} - \alpha^{-1}Q_\alpha)(A_{jk}y_l + A_{kl}y_j + A_{jl}y_k)s_{0|0}^i \\ &\quad + \alpha^{-3}Q_\alpha(A_{jk}s_{l|0}^i + A_{kl}s_{j|0}^i + A_{jl}s_{k|0}^i) \\ &\quad + \alpha^{-3}Q_{\alpha\beta}[(A_{jk}b_l + A_{kl}b_j + A_{jl}b_k)s_{0|0}^i \\ &\quad + (A_{jk}s_{l0} + A_{kl}s_{j0} + A_{jl}s_{k0})s_{0|0}^i] \\ &\quad + \alpha^{-2}Q_{\alpha\alpha\beta}[(y_j y_k b_l + y_k y_l b_j + y_j y_l b_k)s_{0|0}^i \\ &\quad + (y_j y_k s_{l0} + y_k y_l s_{j0} + y_j y_l s_{k0})s_{0|0}^i] \\ &\quad + \alpha^{-1}Q_{\alpha\beta\beta}[(y_j b_k b_l + y_k b_j b_l + y_l b_k b_j)s_{0|0}^i \\ &\quad + ((y_j b_l + y_l b_j)s_{k0} + (y_j b_k + y_k b_j)s_{l0} \\ &\quad + (y_k b_l + y_l b_k)s_{j0})s_{0|0}^i] + \alpha^{-2}Q_{\alpha\alpha}(y_j y_k s_{l|0}^i + y_k y_l s_{j|0}^i + y_j y_l s_{k|0}^i) \\ &\quad + Q_{\beta\beta\beta}(b_k b_l s_{j0} + b_j b_l s_{k0} + b_j b_k s_{l0})s_{0|0}^i + \alpha^{-3}Q_{\alpha\alpha\alpha}y_j y_k y_l s_{0|0}^i \\ &\quad + \alpha^{-1}Q_{\alpha\beta}[(y_j b_k + y_k b_j)s_{l|0}^i + (y_k b_l + y_l b_k)s_{j|0}^i + (y_l b_j + y_j b_l)s_{k|0}^i \\ &\quad + (y_j s_{k0} + y_k s_{j0})s_{l|0}^i + (y_k s_{l0} + y_l s_{k0})s_{j|0}^i + (y_l s_{j0} + y_j s_{l0})s_{k|0}^i] \\ &\quad + Q_{\beta\beta} [b_j b_k s_{l|0}^i + b_k b_l s_{j|0}^i + b_j b_l s_{k|0}^i + (s_{j0} b_k + b_j s_{k0})s_{l|0}^i \\ &\quad + (s_{k0} b_l + b_k s_{l0})s_{j|0}^i + (b_l s_{j0} + b_j s_{l0})s_{k|0}^i] + Q_{\beta\beta\beta} b_j b_k b_l s_{0|0}^i \end{aligned} \quad (3.5)$$

and

$$A_{ij} = \alpha^2 a_{ij} - y_i y_j, \quad (3.6)$$

$$\tilde{N}_p^i = Qs_p^i + [\alpha^{-1}Q_\alpha y_p + Q_\beta b_p]s_{0|0}^i, \quad (3.7)$$

$$\frac{\partial D_{jkl}^i}{\partial y^p} = Q_{jklp} s_{0|0}^i + Q_{jkl} s_p^i + Q_{jkp} s_l^i + Q_{jlp} s_k^i + Q_{klp} s_j^i. \quad (3.8)$$

Let  $F$  is a  $GDW$ -metric. Then there exists a tensor  $D_{jkl}$  such that

$$D_{jkl\parallel m}^i y^m = D_{jkl} y^i.$$

By (3.4), we have

$$D_{jkl}y^i = D_{jkl|m}y^m - 2Q \frac{\partial D_{jkl}^i}{\partial y^p} s^p_0 + D_{jkl}^p \tilde{N}_p^i - D_{pkl}^i \tilde{N}_j^p - D_{jpl}^i \tilde{N}_k^p - D_{jkp}^i \tilde{N}_l^p. \quad (3.9)$$

By contracting (3.9) with  $y_i$  and using (3.5), (3.7) and (3.8) we get the following

$$\begin{aligned} D_{jkl} = & D_1 [A_{jk}s_{l0|0} + A_{kl}s_{j0|0} + A_{jl}s_{k0|0}] \\ & + D_2 [y_j y_k s_{l0|0} + y_k y_l s_{j0|0} + y_j y_l s_{k0|0}] \\ & + D_3 [(y_j b_k + y_k b_j) s_{l0|0} + (y_k b_l + y_l b_k) s_{j0|0} + (y_j b_l + y_l b_j) s_{k0|0}] \\ & + D_4 [b_j b_k s_{l0|0} + b_k b_l s_{j0|0} + b_j b_l s_{k0|0}] \\ & + D_5 [A_{jk}y_l + A_{kl}y_j + A_{jl}y_k] t_{00} \\ & + D_6 [A_{jk}b_l + A_{kl}b_j + A_{jl}b_k] t_{00} \\ & + D_7 [y_j y_k b_l + y_k y_l b_j + y_j y_l b_k] t_{00} \\ & + D_8 [y_l b_j b_k + y_j b_k b_l + y_k b_j b_l] t_{00} \\ & + D_9 y_j y_k y_l t_{00} + D_{10} b_j b_k b_l t_{00} \\ & + D_{11} [y_l s_{j0} s_{k0} + y_j s_{k0} s_{l0} + y_k s_{j0} s_{l0}] \\ & + D_{12} [b_l s_{j0} s_{k0} + b_j s_{k0} s_{l0} + b_k s_{j0} s_{l0}], \end{aligned} \quad (3.10)$$

where

$$\begin{aligned} D_1 & := -\alpha^{-5} Q_\alpha, \\ D_2 & := -\alpha^{-4} Q_{\alpha\alpha}, \\ D_3 & := -\alpha^{-3} Q_{\alpha\beta}, \\ D_4 & := -\alpha^{-2} Q_{\beta\beta}, \\ D_5 & := -\alpha^{-6} Q_\alpha^2 - \alpha^{-6} Q Q_{\alpha\alpha} + \alpha^{-7} Q Q_\alpha, \\ D_6 & := -\alpha^{-5} Q_\alpha Q_\beta - \alpha^{-5} Q Q_{\alpha\beta}, \\ D_7 & := -\alpha^{-4} Q_{\alpha\alpha} Q_\beta - 2\alpha^{-4} Q_{\alpha\beta} Q_\alpha - \alpha^{-4} Q Q_{\alpha\alpha\beta}, \\ D_8 & := -\alpha^{-3} Q_{\beta\beta} Q_\alpha - 2\alpha^{-3} Q_{\alpha\beta} Q_\beta - \alpha^{-3} Q Q_{\alpha\beta\beta}, \\ D_9 & := -3\alpha^{-5} Q_{\alpha\alpha} Q_\alpha - \alpha^{-5} Q Q_{\alpha\alpha\alpha}, \\ D_{10} & := -3\alpha^{-2} Q_{\beta\beta} Q_\beta - \alpha^{-2} Q Q_{\beta\beta\beta}, \\ D_{11} & := -2\alpha^{-3} Q_{\alpha\beta} + 2\alpha^{-4} Q_\alpha^2 + 2\alpha^{-4} Q Q_{\alpha\alpha} - 2\alpha^{-5} Q Q_\alpha, \\ D_{12} & := -2\alpha^{-2} Q_{\beta\beta} + 2\alpha^{-3} Q Q_{\alpha\beta} + 2\alpha^{-3} Q_\alpha Q_\beta. \end{aligned}$$

Now, by plugging (3.10) into (3.9), and contracting the obtained result with  $a^{kl}$ , we get (3.1).  $\square$

**Proof of Theorem 1.1:** Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be an  $(\alpha, \beta)$ -metric on an  $n$ -dimensional manifold  $M$ . By multiplying (3.1) with  $y_i$  and  $y^j$ , we get

$$-\alpha Q Q_{\alpha\alpha\alpha} t_{00} = 0. \quad (3.11)$$

If  $Q_{\alpha\alpha\alpha} = 0$  then

$$Q = c_1\alpha + c_2\frac{\alpha^2}{\beta},$$

where  $c_1$  and  $c_2$  are real constants. Thus, we get

$$F = c_3\alpha\left(\frac{\beta}{\alpha}\right)^{\frac{c_2}{1+c_2}}\left(c_1\frac{\beta}{\alpha} + c_2 + 1\right)^{\frac{1}{1+c_2}},$$

where  $c_3$  is a real constant. This is a contradiction with our assumption. Then by (3.11), we get  $t_{00} = 0$  which results that  $s_{i0} = 0$ . This means that  $\beta$  is a closed one-form. By assumption,  $\beta$  is parallel one-form and then  $F$  reduces to a Berwald metric.  $\square$

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