

## Lie Ideals and Generalized Derivations in Semiprime Rings

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ABSTRACT. Let  $R$  be a 2-torsion free ring and  $L$  a Lie ideal of  $R$ . An additive mapping  $F : R \rightarrow R$  is called a generalized derivation on  $R$  if there exists a derivation  $d : R \rightarrow R$  such that  $F(xy) = F(x)y + xd(y)$  holds for all  $x, y \in R$ . In the present paper we describe the action of generalized derivations satisfying several conditions on Lie ideals of semiprime rings.

**Keywords:** Derivations, Generalized derivations, Semiprime rings, Lie ideals.

**2000 Mathematics subject classification:** 16W25, 16U80, 16N60.

### 1. INTRODUCTION

Let  $R$  be an associative ring with center  $Z(R)$ . A ring  $R$  is said to be  $n$ -torsion free if  $nx = 0$  implies  $x = 0$  for all  $x \in R$ . For any  $x, y \in R$ , the symbol  $[x, y]$  will represent the commutator  $xy - yx$ . Recall that a ring  $R$  is prime if  $aRb = 0$  implies  $a = 0$  or  $b = 0$  and  $R$  is semiprime if  $aRa = 0$  yields  $a = 0$ . An additive mapping  $d : R \rightarrow R$  is said to be a derivation of  $R$  if

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$d(xy) = d(x)y + xd(y)$  for all  $x, y \in R$ . In particular, for a fixed  $a \in R$  the mapping  $I_a : R \rightarrow R$  given by  $I_a(x) = [x, a]$  is a derivation which is called an inner derivation determined by  $a$ . In 1991 Bresar [5] introduced the concept of generalized derivation: more precisely an additive mapping  $F : R \rightarrow R$  is said to be a generalized derivation with associated derivation  $d$  if  $F(xy) = F(x)y + xd(y)$  for all  $x, y \in R$ . For the sake of convenience, we shall denote by  $(F, d)$  a generalized derivation  $F$  with associated derivation  $d$ . A mapping  $f : R \rightarrow R$  is known to be centralizing on  $R$  if  $[f(x), x] \in Z(R)$  for all  $x \in R$ . In particular, if  $[f(x), x] = 0$  for all  $x \in R$ , then  $f$  is said to be commuting on  $R$ . We recall that an additive group  $L$  of  $R$  is said to be a Lie ideal of  $R$  if  $[L, R] \subseteq L$ .

A well known result of Posner [18] states that a prime ring admitting a nonzero centralizing derivation must be commutative. This theorem indicates that the global structure of a ring  $R$  is often tightly connected to the behaviour of additive mappings defined on  $R$ . Following this line of investigation, several authors studied derivations and generalized derivations acting on appropriate subsets of the ring.

For instance in [19] Quadri et al. prove that if  $R$  is a prime ring with a non-zero ideal  $I$  and  $F$  is a generalized derivation of  $R$  such that  $F([x, y]) = [x, y]$ , for all  $x, y \in I$ , then  $R$  is commutative (Theorem 2.1). Later in [7] Dhara extends all results contained in [19] to semiprime rings.

Further in [10] Gölbaşı and Koç investigate the properties of a prime ring  $R$  with a generalized derivation  $(F, d)$  acting on a Lie ideal  $L$  of  $R$ . They prove that if  $[F(u), u] \in Z(R)$ , for all  $u \in L$ , then either  $d = 0$  or  $L \subseteq Z(R)$  (Theorem 3.3). Moreover if  $F([u, v]) = [u, v]$ , for all  $u, v \in L$ , then either  $d = 0$  or  $L \subseteq Z(R)$  (Theorem 3.6).

In this note we will consider a similar situation and extend the cited results to the case of semiprime rings with a generalized derivation  $(F, d)$  acting on a Lie ideal. More precisely we prove the following:

**Theorem 1.** *Let  $R$  be a 2-torsion free semiprime ring and  $L$  be a non-central Lie ideal of  $R$ . Suppose that  $(F, d)$  is a generalized derivation of  $R$  such that  $F[x, y] \in Z(R)$ , for all  $x, y \in L$ . If  $d(L) \neq (0)$ , then all the following hold simultaneously:*

- (1)  $d(R)[L, R] = (0)$  and  $[d(R), L] = (0)$ ;
- (2)  $a[L, R] = (0)$  and  $[a, L] = (0)$ ;
- (3)  $aI = (0)$  and  $d(I) = (0)$  (that is  $F(I) = (0)$ ), where  $I$  denotes the ideal of  $R$  generated by  $[L, L]$ .

**Theorem 2.** *Let  $R$  be a 2-torsion free semiprime ring and  $L$  be a non-central Lie ideal of  $R$ . Suppose  $R$  admits a generalized derivation  $(F, d)$ , defined as  $F(x) = ax + d(x)$ , for all  $x \in R$  and fixed element  $a \in R$ . If  $[F(x), x] \in Z(R)$  for all  $x \in L$  and  $d(L) \neq (0)$ , then all the following hold simultaneously:*

- (1)  $d(R)[L, R] = (0)$  and  $[d(R), L] = (0)$ ;
- (2)  $[a, L] = a[L, R] = (0)$ ;
- (3)  $aI = d(I) = (0)$  (that is  $F(I) = (0)$ ), where  $I$  denotes the ideal of  $R$  generated by  $[L, L]$ .

**Theorem 3.** Let  $R$  be a 2-torsion free semiprime ring and  $L$  be a non-central Lie ideal of  $R$ . Suppose  $R$  admits two generalized derivations  $(F, d)$  and  $(G, g)$ . Write  $F(x) = ax + d(x)$  and  $G(x) = bx + g(x)$ , for some  $a, b \in U$ . If  $F([x, y]) = [y, G(x)]$  for all  $x, y \in L$ , then either

- (1)  $g(L) = (0)$ ;
- (2)  $d(R)[L, R] = (0)$  and  $[d(R), L] = (0)$ ;
- (3)  $(a + b)[L, R] = (0)$ ,  $[b, L] = (0)$  and  $[a, L] = (0)$ ;
- (4)  $(a + b)I = (0)$  and  $d(I) = (0)$ , where  $I$  denotes the ideal of  $R$  generated by  $[L, L]$ .

or

- (1)  $d(L) = (0)$ ;
- (2)  $g(R)[L, R] = (0)$  and  $[g(R), L] = (0)$ ;
- (3)  $[b, L] = (0)$  and  $a[L, L] = (0)$ ;
- (4)  $aI = (0)$  and  $g(I) = (0)$ , where  $I$  denotes the ideal of  $R$  generated by  $[L, L]$ .

or

- (1)  $d(R)[L, R] = (0)$  and  $[d(R), L] = (0)$ ;
- (2)  $g(R)[L, R] = (0)$  and  $[g(R), L] = (0)$ ;
- (3)  $[a, L] = (0)$ ,  $[b, L] = (0)$ ,  $b[L, R] = a[L, R] = (0)$ ;
- (4)  $d(I) = g(I) = (0)$  and  $aI = bI = (0)$  (that is  $F(I) = G(I) = (0)$ ), where  $I$  denotes the ideal of  $R$  generated by  $[L, L]$ .

In all that follows let  $R$  be a non-commutative semiprime ring,  $L$  a non-central Lie ideal of  $R$ ,  $U$  the right Utumi quotient ring of  $R$ . We refer the reader to [3] for the definition and the related properties of  $U$ .

We begin with the following:

**Fact 1.1.** Let  $R$  be a semiprime ring. Then every generalized derivation  $F$  of  $R$  is uniquely extended to its right Utumi quotient ring  $U$  and assumes the form  $F(x) = ax + d(x)$ , where  $a \in U$  and  $d$  is the derivation of  $U$  associated with  $F$  (see Theorem 4 in [17])

**Lemma 1.2.** Let  $R$  be a prime ring of characteristic different from 2 and  $L$  be a Lie ideal of  $R$ . Suppose  $R$  admits a nonzero generalized derivation  $(F, d)$  such that  $F(x)[x, y] = 0$  (or  $[x, y]F(x) = 0$ ) for all  $x, y \in L$ , then  $L \subseteq Z(R)$ .

*Proof.* Suppose by contradiction that  $L$  is not central in  $R$ . By [11] (pages 4-5) there exists a non-central ideal  $I$  of  $R$  such that  $0 \neq [I, R] \subseteq L$ . By our

assumption it follows that  $F(x)[x, y] = 0$  (or  $[x, y]F(x) = 0$ ) for all  $x, y \in [I, R]$ . Since  $I$  and  $R$  satisfy the same differential identities (see the main result in [16]), we also have that  $F(x)[x, y] = 0$  (or  $[x, y]F(x) = 0$ ) for all  $x, y \in [R, R]$ . Let  $y_0 \in [R, R]$  be such that  $y_0 \notin Z(R)$  and denote by  $\delta : R \rightarrow R$  the non-zero inner derivation of  $R$  induced by the element  $y_0$ . Therefore  $F(x)\delta(x) = 0$  (or  $\delta(x)F(x) = 0$ ) for all  $x \in [R, R]$ . In light of [6], since  $\delta \neq 0$  and  $[R, R]$  is not central in  $R$ , one has the contradiction that  $F = 0$ .  $\square$

**Lemma 1.3.** *Let  $R$  be a 2-torsion free semiprime ring and  $L$  be a non-central Lie ideal of  $R$ . Suppose  $R$  admits a nonzero generalized derivation  $(F, d)$ , defined as  $F(x) = ax + d(x)$ , for all  $x \in R$  and fixed element  $a \in R$ . If  $F(x)[x, y] = 0$  (or  $[x, y]F(x) = 0$ ) for all  $x, y \in L$ , then all the following hold simultaneously:*

- (1)  $d(R)[L, R] = (0)$  and  $[d(R), L] = (0)$ ;
- (2)  $a[L, R] = (0)$  and  $[a, L] = (0)$ ;
- (3)  $aI = (0)$  and  $d(I) = (0)$  (that is  $F(I) = (0)$ ), where  $I$  denotes the ideal of  $R$  generated by  $[L, L]$ .

*Proof.* Let  $P$  be a prime ideal of  $R$  such that  $[L, L] \not\subseteq P$ .

Assume first that  $d(P) \subseteq P$ . Then  $F$  induces a canonical generalized derivation  $\bar{F}$  on  $\bar{R} = \frac{R}{P}$ . Therefore  $\bar{F}(\bar{x})[\bar{x}, \bar{y}] = 0$  for all  $\bar{x}, \bar{y} \in \bar{L}$ . Moreover  $\bar{L}$  is a Lie ideal of  $\bar{R}$ , such that  $[\bar{L}, \bar{L}] \neq 0$  since  $[L, L] \not\subseteq P$ . By Lemma 1.2 it follows that  $\bar{F}(\bar{R}) = \bar{0}$  that is  $aR \subseteq P$ ,  $d(R) \subseteq P$  and  $F(R) \subseteq P$ .

Assume now that  $d(P) \not\subseteq P$ , then  $\overline{d(P)} \neq \bar{0}$  and  $\overline{d(P)R} \neq \bar{0}$ . Moreover note that, for any  $p \in P$  and  $r, s \in R$ ,  $d(pr)s = d(p)rs + pd(r)s$  implies that  $d(P)R \subseteq d(PR)R + P$ , in particular  $\overline{d(P)R}$  is a non-zero right ideal of  $\bar{R}$ .

Starting from our main assumption and linearizing we have that  $F(x)[z, y] + F(z)[x, y] = 0$ , for all  $x, y, z \in L$ . For any  $p \in P, r \in R, u \in L$ , replace  $x$  by  $[pr, u]$ . By computation it follows  $[\bar{v}, \bar{u}][\bar{z}, \bar{y}] = \bar{0}$ , for all  $\bar{v} \in \overline{d(P)R}$  and  $\bar{u}, \bar{z}, \bar{y} \in \bar{L}$ . By using the same argument of Lemma 1.2, since  $\bar{L}$  is not central in  $\bar{R}$ , one has that  $\overline{d(P)R}$  is a central right ideal of  $\bar{R}$ , which implies that  $\bar{R}$  is commutative, a contradiction.

Therefore, for any prime ideal  $P$  of  $R$ , either  $aR \subseteq P$ ,  $d(R) \subseteq P$  and  $F(R) \subseteq P$  or  $[L, L] \subseteq P$ . In this last case, by applying Theorem 3 in [15] in the prime ring  $\bar{R}$ , since  $\text{char}(\bar{R}) \neq 2$  and  $[\bar{L}, \bar{L}] = \bar{0}$ , we have that  $\bar{L}$  is central in  $\bar{R}$ , which means  $[L, R] \subseteq P$ .

Hence in any case it follows that  $d(R)[L, R] = (0)$ ,  $a[R, L] = (0)$  and  $[d(R), L] = (0)$ .

By  $a[R, L] = (0)$  we get  $aR[R, L] = (0)$  and so both  $aLR[a, L] = (0)$  and  $LaR[a, L] = (0)$ , that is  $[a, L]R[a, L] = (0)$ . By the semiprimeness of  $R$  it follows  $[a, L] = (0)$ .

Moreover, if  $I = R[L, L]R$  denotes the ideal of  $R$  generated by  $[L, L]$ , it follows that  $aI = (0)$  and  $d(I) = (0)$ , that is  $F(I) = (0)$ .  $\square$

**Corollary 1.4.** *Let  $R$  be a 2-torsion free semiprime ring and  $L$  be a non-central Lie ideal of  $R$ . Suppose  $a \in R$  be such that  $ax[x, y] = 0$  for all  $x, y \in L$ , then  $a[L, R] = (0)$ ,  $[a, L] = (0)$  and  $aI = (0)$ , where  $I$  denotes the ideal of  $R$  generated by  $[L, L]$ .*

**Theorem 1.5.** *Let  $R$  be a 2-torsion free semiprime ring and  $L$  be a non-central Lie ideal of  $R$ . Suppose that  $(F, d)$  is a generalized derivation of  $R$  such that  $F[x, y] \in Z(R)$ , for all  $x, y \in L$ . If  $d(L) \neq (0)$ , then all the following hold simultaneously:*

- (1)  $d(R)[L, R] = (0)$  and  $[d(R), L] = (0)$ ;
- (2)  $a[L, R] = (0)$  and  $[a, L] = (0)$ ;
- (3)  $aI = (0)$  and  $d(I) = (0)$  (that is  $F(I) = (0)$ ), where  $I$  denotes the ideal of  $R$  generated by  $[L, L]$ .

*Proof.* Assume first that  $R$  is prime and denote  $V = [L, L]$ . Hence we have  $F(V) \subseteq Z(R)$ . As a consequence of Lemma 2 in [9] we conclude that either  $F = 0$  or  $V \subseteq Z(R)$ . In the first case we have the contradiction  $d = 0$ , and in the other case one has  $L \subseteq Z(R)$  (see Lemma 2 in [12]), a contradiction again. Let now  $P$  be a prime ideal of  $R$  such that  $[L, L] \not\subseteq P$ .

Assume first that  $d(P) \subseteq P$ . Then  $F$  induces a canonical generalized derivation  $\bar{F}$  on  $\bar{R} = \frac{R}{P}$ . Therefore  $\bar{F}([\bar{x}, \bar{y}]) \in Z(\bar{R})$  for all  $\bar{x}, \bar{y} \in \bar{L}$ . Moreover  $\bar{L}$  is a Lie ideal of  $\bar{R}$ , such that  $[\bar{L}, \bar{L}] \neq 0$  since  $[L, L] \not\subseteq P$ . By previous argument it follows that  $\bar{F}(\bar{R}) = \bar{0}$  that  $d(R) \subseteq P$  and  $F(R) \subseteq P$ .

Assume now that  $d(P) \not\subseteq P$ , then  $\overline{d(P)} \neq \bar{0}$  and  $\overline{d(P)R} \neq \bar{0}$ . We remark again that  $\overline{d(P)R}$  is a non-zero right ideal of  $\bar{R}$ .

Starting from our main assumption and linearizing we have that

$$F(x)y + F(x)z + xd(y) + xd(z) - F(y)x - F(z)x - yd(x) - zd(x) \in Z(R), \quad \forall x, y, z \in L.$$

For any  $p, p', p'' \in P, r, s \in R, u, v \in L$ , replace  $y$  by  $[pr, u]$  and  $z$  by  $[[p's, v], p'']$ . By computation it follows

$$\bar{x}[\bar{t}, \bar{u}] - [\bar{t}, \bar{u}]\bar{x} \in Z(\bar{R})$$

that is

$$\left[ \bar{x}, [\bar{t}, \bar{u}] \right] \in Z(\bar{R})$$

for all  $\bar{t} \in \overline{d(P)R}$  and  $\bar{u}, \bar{x} \in \bar{L}$ . As above denote  $\bar{V} = [\bar{L}, \bar{L}]$ , which is a Lie ideal for  $\bar{R}$ , and  $\delta$  is the derivation of  $\bar{R}$  induced by  $\bar{t}$ . Hence we have  $\delta(\bar{V}) \subseteq Z(\bar{R})$ . Again as a consequence of Lemma 2 in [9] it follows that either  $\delta = 0$  or  $\bar{V} \subseteq Z(\bar{R})$ . Since  $\bar{R}$  is not commutative, then there exists some  $\bar{t} \in \bar{R}$  which is not central. Thus  $\bar{V} \subseteq Z(\bar{R})$ , and  $\bar{L} \subseteq Z(\bar{R})$  follows from Lemma 2 in [12].

Therefore, for any prime ideal  $P$  of  $R$ , either  $d(R) \subseteq P$  and  $F(R) \subseteq P$  or  $[L, L] \subseteq P$ . In this last case, by applying Theorem 3 in [15] in the prime ring  $\bar{R}$ , since  $\text{char}(\bar{R}) \neq 2$  and  $[\bar{L}, \bar{L}] = \bar{0}$ , we conclude that  $\bar{L}$  is central in  $\bar{R}$ , which

means  $[L, R] \subseteq P$ .

Hence in any case it follows that  $d(R)[L, R] = (0)$ ,  $a[R, L] = (0)$  and  $[d(R), L] = (0)$ . Finally we obtain the required conclusions by following the same argument as in Lemma 1.3.  $\square$

In the sequel we will use the following known result:

**Lemma 1.6.** *Let  $R$  be a 2-torsion free semiprime ring,  $L$  a Lie ideal of  $R$  such that  $L \not\subseteq Z(R)$ . Let  $a \in L$  be such that  $aLa = 0$ , then  $a = 0$ .*

*Remark 1.7.* If  $R$  is a prime ring of characteristic different from 2,  $a \in R$  and  $L$  is a non-central Lie ideal of  $R$  such that  $[a, L] \subseteq Z(R)$ , then  $a \in Z(R)$ .

*Proof.* Denote by  $\delta : R \rightarrow R$  the inner derivation of  $R$  induced by the element  $a \in R$ . Since  $[[a, x], r] = 0$  for all  $x \in L$  and  $r \in R$ , a fortiori we have  $[a, x]_2 = 0$ , that is  $[\delta(x), x] = 0$ , for all  $x \in L$ . Thus, by [14] it follows  $\delta = 0$ , that is  $a \in Z(R)$ .  $\square$

**Theorem 1.8.** *Let  $R$  be a 2-torsion free semiprime ring and  $L$  be a non-central Lie ideal of  $R$ . Suppose  $R$  admits a generalized derivation  $(F, d)$ , defined as  $F(x) = ax + d(x)$ , for all  $x \in R$  and fixed element  $a \in R$ . If*

$$[F(x), x] \in Z(R) \text{ for all } x \in L. \quad (1.1)$$

and  $d(L) \neq (0)$ , then all the following hold simultaneously:

- (1)  $d(R)[L, R] = (0)$  and  $[d(R), L] = (0)$ ;
- (2)  $[a, L] = a[L, R] = (0)$ ;
- (3)  $aI = d(I) = (0)$  (that is  $F(I) = (0)$ ), where  $I$  denotes the ideal of  $R$  generated by  $[L, L]$ .

*Proof.* Let  $P$  be a prime ideal of  $R$  such that  $[L, L] \not\subseteq P$ .

Assume first that  $d(P) \subseteq P$ . Then  $F$  induces a canonical generalized derivation  $\bar{F}$  on  $\bar{R} = \frac{R}{P}$ . Therefore  $[\bar{F}(\bar{x}), \bar{x}] \in Z(\bar{R})$  for all  $\bar{x} \in \bar{L}$ . Moreover  $\bar{L}$  is a Lie ideal of  $\bar{R}$ , such that  $[\bar{L}, \bar{L}] \neq 0$  since  $[L, L] \not\subseteq P$ . Since  $[L, L] \not\subseteq P$ , a fortiori we get  $\bar{L}$  is not central in  $\bar{R}$ . Therefore, by Theorem 3.3 in [10], it follows that  $\bar{d}(\bar{R}) = \bar{0}$  that is  $d(R) \subseteq P$ .

Assume now that  $d(P) \not\subseteq P$ , then  $\overline{d(P)} \neq \bar{0}$  and  $\overline{d(P)R} \neq \bar{0}$ . By using similar argument as in Lemma 1.3,  $\overline{Rd(P)}$  is a non-zero right ideal of  $\bar{R}$ .

Linearizing (1.1) and using (1.1), we obtain

$$[F(x), y] + [F(y), x] \in Z(R) \text{ for all } x, y \in L. \quad (1.2)$$

Now, replace  $y$  by  $[rp, u]$ , for  $r \in R$ ,  $p \in P$  and  $u \in L$  and use (1.2) to get

$$[\bar{F}(\overline{[rp, u]}), \bar{x}] \in Z(\bar{R}). \quad (1.3)$$

Moreover, since  $F(r) = ar + d(r)$ , for all  $r \in R$ , by (1.3) it follows

$$[\bar{d}(\overline{[rp, u]}), \bar{L}] \subseteq Z(\bar{R}). \quad (1.4)$$

By the primeness of  $\overline{R}$  and Remark 1.7, one has that  $\overline{d}(\overline{[rp, u]}) \in Z(\overline{R})$ . On the other hand, an easy computation shows that  $\overline{d}(\overline{[rp, u]}) = \overline{[rd(p), \overline{u}]}$ , which implies  $\overline{[Rd(P), \overline{L}]} \subseteq Z(\overline{R})$ . Once again by Remark 1.7, we have  $Rd(P) \subseteq Z(\overline{R})$ . Since  $\overline{Rd(P)}$  is a non-zero right ideal of  $\overline{R}$ , it follows  $[\overline{R}, \overline{R}] = (0)$ , which contradicts with  $[\overline{L}, \overline{L}] \neq (0)$ .

The previous argument shows that, for any prime ideal  $P$  of  $R$ , either  $[L, L] \subseteq P$  or  $d(R) \subseteq P$ . Thus  $d(R)[L, L] \subseteq \cap P_i = (0)$ . Hence, by Lemma 1.3 and since  $L \not\subseteq Z(R)$ , we finally get the required conclusions:

- (1)  $d(R)[L, R] = (0)$  and  $[d(R), L] = (0)$ ;
- (2)  $a[L, R] = (0)$  and  $[a, L] = (0)$ ;
- (3)  $aI = d(I) = (0)$ , where  $I$  denotes the ideal of  $R$  generated by  $[L, L]$ .

□

**Theorem 1.9.** *Let  $R$  be a 2-torsion free semiprime ring and  $L$  be a non-central Lie ideal of  $R$ . Suppose  $R$  admits two generalized derivations  $(F, d)$  and  $(G, g)$ . Write  $F(x) = ax + d(x)$  and  $G(x) = bx + g(x)$ , for some  $a, b \in U$ . If  $F([x, y]) = [y, G(x)]$  for all  $x, y \in L$ , then either*

- (1)  $g(L) = (0)$ ;
- (2)  $d(R)[L, R] = (0)$  and  $[d(R), L] = (0)$ ;
- (3)  $(a + b)[L, R] = (0)$ ,  $[b, L] = (0)$  and  $[a, L] = (0)$ ;
- (4)  $(a + b)I = (0)$  and  $d(I) = (0)$ , where  $I$  denotes the ideal of  $R$  generated by  $[L, L]$ .

or

- (1)  $d(L) = (0)$ ;
- (2)  $g(R)[L, R] = (0)$  and  $[g(R), L] = (0)$ ;
- (3)  $[b, L] = (0)$  and  $a[L, L] = (0)$ ;
- (4)  $aI = (0)$  and  $g(I) = (0)$ , where  $I$  denotes the ideal of  $R$  generated by  $[L, L]$ .

or

- (1)  $d(R)[L, R] = (0)$  and  $[d(R), L] = (0)$ ;
- (2)  $g(R)[L, R] = (0)$  and  $[g(R), L] = (0)$ ;
- (3)  $[a, L] = (0)$ ,  $[b, L] = (0)$ ,  $b[L, R] = a[L, R] = (0)$ ;
- (4)  $d(I) = g(I) = (0)$  and  $aI = bI = (0)$  (that is  $F(I) = G(I) = (0)$ ), where  $I$  denotes the ideal of  $R$  generated by  $[L, L]$ .

*Proof.* Assume first  $g(L) = (0)$ , then  $F([x, y]) = [y, bx]$  for all  $x, y \in L$ . Thus

$$a[x, y] + d([x, y]) = b[y, x] \quad (1.5)$$

for all  $x, y \in L$ , that is  $(a + b)[x, y] + d([x, y]) = 0$  for all  $x, y \in L$ . Therefore, applying Theorem 1.5, one has

- (1)  $d(R)[L, R] = (0)$  and  $[d(R), L] = (0)$ ;
- (2)  $(a + b)[L, R] = (0)$  and  $[a + b, L] = (0)$ ;

- (3)  $(a+b)I = (0)$  and  $d(I) = (0)$ , where  $I$  denotes the ideal of  $R$  generated by  $[L, L]$ .

In particular  $d([L, L]) = (0)$  and  $a[x, y] = -b[x, y]$  for all  $x, y \in L$ , so that (1.5) reduces to  $(by - yb)x = 0$ , for all  $x, y \in L$ , that is  $[b, L]L = (0)$ . Hence by Lemma 1.6, we have  $[b, L] = (0)$  and so also  $[a, L] = (0)$

Let now  $d(L) = (0)$ , then  $a[x, y] = [y, G(x)]$  for all  $x, y \in L$ . In this case, for  $x = y$ , we have  $[G(y), y] = 0$  and by Theorem 1.8 the following hold:

- (1)  $g(R)[L, R] = (0)$  and  $[g(R), L] = (0)$ ;
- (2)  $[b, L] = (0)$ ,  $b[L, R] = (0)$  and  $a[L, L] = (0)$ ;
- (3)  $bI = (0)$  and  $g(I) = (0)$ , where  $I$  denotes the ideal of  $R$  generated by  $[L, L]$ .

Moreover, since  $[[L, L], R] \subseteq [L, L]$ , we also have  $0 = a[[L, L], R] = aR[L, L]$ , which implies  $aI = (0)$ .

Assume finally that both  $g(L) \neq (0)$  and  $d(L) \neq (0)$ . Once again for  $x = y \in L$  we have  $[G(x), x] = 0$  for any  $x \in L$ . Thus by Theorem 1.8, we have that all the following hold:

- (1)  $g(R)[L, R] = (0)$  and  $[g(R), L] = (0)$ ;
- (2)  $[b, L] = (0)$  and  $b[L, R] = (0)$ ;
- (3)  $bI = (0)$  and  $g(I) = (0)$ , where  $I$  denotes the ideal of  $R$  generated by  $[L, L]$ .

Hence by the main assumption it follows that  $(a+b)[x, y] + d([x, y]) = 0$ , for all  $x, y \in L$ . Denote  $H(x) = (a-b)x + d(x)$ , then  $H(u) = 0$  for all  $u \in [L, L]$ . Finally, by applying Theorem 1.5, one has

- (1)  $d(R)[L, R] = (0)$  and  $[d(R), L] = (0)$ ;
- (2)  $(a+b)[L, R] = (0)$  and  $[a, L] = (0)$ ;
- (3)  $(a+b)I = (0)$  and  $d(I) = (0)$ , where  $I$  denotes the ideal of  $R$  generated by  $[L, L]$ .

Note that, since both  $bI = (0)$  and  $(a+b)I = (0)$ , we are done.  $\square$

We conclude our paper with some applications to generalized derivations acting on ideals of semiprime rings:

**Theorem 1.10.** *Let  $R$  be a 2-torsion free semiprime ring and  $I$  be a non-central ideal of  $R$ . Suppose  $R$  admits a generalized derivation  $(F, d)$ , defined as  $F(x) = ax + d(x)$ , for all  $x \in R$  and fixed element  $a \in R$ . If  $[F(x), x] = 0$  for all  $x \in I$ , then either  $d(I) = 0$  or  $R$  contains a non-zero central ideal.*

*Proof.* By Theorem 1.8, we have that if  $d(I) \neq 0$  then  $[d(R), I] = (0)$ . Hence, by applying Main Theorem in [13], it follows that  $R$  must contain a non-zero central ideal.  $\square$

**Corollary 1.11.** *Let  $R$  be a 2-torsion free semiprime ring  $F$  a generalized derivation of  $R$ . If  $[F(x), x] = 0$  for all  $x \in R$ , then either  $R$  contains a*

non-zero central ideal or there exists  $\lambda \in Z(R)$  such that  $F(x) = \lambda x$ , for all  $x \in R$ .

**Theorem 1.12.** *Let  $R$  be a 2-torsion free semiprime ring and  $I$  be a non-central ideal of  $R$ . Suppose  $R$  admits two generalized derivations  $(F, d)$  and  $(G, g)$ . Write  $F(x) = ax + d(x)$  and  $G(x) = bx + g(x)$ , for some  $a, b \in U$ . If  $F([x, y]) = [y, G(x)]$  for all  $x, y \in L$ , then either  $d(I) = g(I) = (0)$  or  $R$  contains a non-zero central ideal.*

*Proof.* Assume either  $d(I) \neq 0$  or  $g(I) \neq 0$ . Thus, by Theorem 1.9 respectively we have that either  $[d(R), I] = (0)$  or  $[g(R), I] = (0)$ . In any case, again by [13],  $R$  must contain some non-zero central ideals.  $\square$

**Corollary 1.13.** *Let  $R$  be a 2-torsion free semiprime ring and  $F, G$  two generalized derivations of  $R$ . If  $F([x, y]) = [y, G(x)]$  for all  $x, y \in R$ , then either  $R$  contains a non-zero central ideal or there exist  $\lambda \in Z(R)$  such that  $F(x) = G(x) = \lambda x$ , for all  $x \in R$ .*

#### ACKNOWLEDGMENTS

The authors wishes to thank the referees for their valuable comments, suggestions.

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