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Free Extended BCK-Module

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Abstract. In this paper, by considering the notion of extended BCK-module, we define the concepts of free extended BCK-module, free object in category of extended BCK-modules and we state and prove some related results. Specially, we define the notion of idempotent extended BCK-module and we get some important results in free extended BCK-modules. In particular, in category of idempotent extended BCK-modules, we give a method to make a free object on a nonempty set and in BCK-algebra of order 2, we give a method to make a basis for unitary extended BCK-modules. Finally, we define the notions of projective and productive modules and we investigate the relation between free modules and projective modules. In special case, we state the relation between free modules and productive modules.

Keywords: BCK-algebra, Extended BCK-module, Free extended BCK-module.

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1. Introduction

The notion of BCK-algebra was formulated first in 1966 by Imai and Iseki. This notion is originated from two different ways. One of the motivations

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is based on set theory. Another motivation is from classical and non-classical propositional calculus. As is well known, there is close relationship between the notion of the set difference in set theory and the implication functor in logical systems. Then the following problems arise from this relationship. What is the most essential and fundamental common properties? Can we establish a good theory of general algebra? To give answer to these problems, Y. Imai and K. Iseki introduced a notion of a new class of general algebras, which is called a BCK-algebra. This name is taken from BCK-system of C. A. Meredith. BCK-algebras have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. The notion of BCK-module was introduced in [3] as an action of a BCK-algebra over a commutative group by M. Aslam, A.B .Thaheem and H.A.S. Abujaabal. The idea was further explored by F. Kopa and C. Vance in [9]. The concept of BCK-module was extended by R. A. Borzooei, J. Shohani and M. Jafari in [6]. In following, this concept was extended in different way by R. A. Borzooei and S. Saidi Goraghani in [5]. In groups category and modules category, the study of free objects is important and interesting. In particular, free modules have numerous applications in mathematics. Now, since the notions of free module and projective module are fundamental notions in modules theory, then in this paper, we introduce and investigate them on BCK-modules. In studying of BCK-modules, founding a basis for a BCK-module is important. In general, founding a method to make a free object in category of BCK-modules can be interesting and important. So we start off this long way and we obtain some results as mentioned in the abstract.

2. Preliminaries

Definition 2.1. [10] A BCK-algebra is a structure $X = (X, *, 0)$ of type $(2, 0)$ such that:
$(BCK1)$ $((x * y) * (x * z)) * (z * y) = 0$,
$(BCK2)$ $(x * (x * y)) * y = 0$,
$(BCK3)$ $x * x = 0$,
$(BCK4)$ $0 * x = 0$,
$(BCK5)$ $x * y = y * x = 0$ implies that $x = y$, for all $x, y, z \in X$.
Let $(X, *, 0)$ be a BCK-algebra. The relation $x \leq y$, which is defined by $x * y = 0$, is a partial order with 0 as the least element. In BCK-algebra X, for any $x, y, z \in X$, we have
$(BCK6)$ $(x * y) * z = (x * z) * y$,
$(BCK7)$ $x * 0 = x$.
Moreover, $\emptyset \neq X_0 \subseteq X$ is called a *subalgebra* of X, if for any $x, y \in X_0$, $x * y \in X_0$, i.e., X_0 is closed under the binary operation "$*$" of X. X is called *bounded*, if there exists $1 \in X$ such that $x \leq 1$, for any $x \in X$ and in this case, we set $Nx = 1 * x$. X is said to be *commutative*, if $y * (y * x) = x * (x * y)$, for

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all $x, y \in X$. X is said to be *implicative*, if $x * (y * x) = x$, for all $x, y \in X$. In a BCK-algebra X, we let $x \wedge y = y * (y * x)$ and in a bounded BCK-algebra X, we let $x \vee y = N(Nx \wedge Ny)$, for all $x, y \in X$. In bounded commutative BCK-algebra X, $\vee$ is the least upper bound and $\wedge$ is the greatest lower bound of X and so $(X, \vee, \wedge)$ is a bounded lattice. $\emptyset \neq A \subseteq X$ is called an ideal of X, if $0 \in A$ and for any $x, y \in X$, $x * y \in A$ and $y \in A$ imply that $x \in A$. If X is commutative and A be a proper ideal of X, then A is called a *prime ideal* of X, if $a \wedge b \in A$ implies that $a \in A$ or $b \in A$, for any $a, b \in X$. Suppose A is an ideal of BCK-algebra X. Then we denote $x \sim y$ if and only if $x * y \in A$ and $y * x \in A$, for any $x, y \in X$. So $\sim$ is an equivalence relation on X. Denote the equivalence class containing x by C_x and $\frac{X}{A} = \{C_x : x \in X\}$. Moreover, $(\frac{X}{A}, \star, C_0)$ is a BCK-algebra, where $C_0 = A$ and $C_x \star C_y = C_{x*y}$, for all $x, y \in X$. The relation "$\leq$" which is defined by $C_x \leq C_y$ if and only if $x * y \in A$, is a partial order relation on $\frac{X}{A}$. If X is bounded and commutative, then $\frac{X}{A}$ is bounded and commutative, too. In addition C_1 is unit of $\frac{X}{A}$. Let $(X, *, 0)$ and $(Y, *', 0')$ be two BCK-algebras. A mapping $f : X \to Y$ is called a *homomorphism* if $f(0) = 0'$ and $f(x * y) = f(x) *' f(y)$, for any $x, y \in X$. If f is one to one (onto), then f is called monomorphism (epimorphism) and if f is onto and one to one, then f is called an isomorphism. Let $f : X \to Y$ be a BCK-epimorphism. Then $\frac{X}{Kerf} \cong Y$.

Lemma 2.2. [10] *Let X be a bounded implicative BCK-algebra . Then for all $x, y, z \in X$,*
(i) $x \wedge y = x * Ny$,
(ii) $x * (x \wedge y) = x * y$,
(iii) $x \wedge (y * z) = (x \wedge y) * (x \wedge z)$,
(iv) $(x * y) + (y * x) = x + y$, *where* $x + y = (x * y) \vee (y * x)$,
(v) $(x + y) \wedge z = (x \wedge z) + (y \wedge z)$,
(vi) $x + x = 0$ *and so* $x = -x$,
(vii) $x + 0 = 0 + x = x$.

Definition 2.3. [5] Let X be a BCK-algebra, M be an abelian group and operation $. : X \times M \longrightarrow M$ be defined by $(x, m) \mapsto x.m$, which satisfies the following axioms:
$(XM1)$ $(x \wedge y).m = x.(y.m)$,
$(XM2)$ $x.(m + n) = x.m + x.n$,
$(XM3)$ $0.m = 0$,
$(XM4)$ $(x * y).m = x.m - y.m$, where $x * y \neq 0$, for $x \neq y$,
for all $x, y \in X$ and $m, n \in M$. Then M is called an *extended BCK-module* or briefly X^E-module. If X is bounded and $1.m = m$, for any $m \in M$, then M is called a *unitary X^E-module.*

Proposition 2.4. [5] *Let X be a bounded implicative BCK-algebra such that "$\leq$" is totally ordered and operations "$+,.$": $X \times X \longrightarrow X$ be defined by*

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$x+y=(x*y)\vee(y*x)$, $x.y=x\wedge y$, for all $x,y\in X$. Then X is an X^E-module.

Proposition 2.5. [5] *Let X be a bounded commutative BCK-algebra such that X is an X^E-module and A be an ideal of X. Then $(\frac{X}{A},+')$ is an abelian group, where $C_x+'C_y=C_{x+y}$ and $x+y=x*y\vee y*x$, for any $x,y\in X$. Moreover, if operation $\bullet: X\times\frac{X}{A}\longrightarrow\frac{X}{A}$ is defined by $x\bullet C_y=C_{x.y}$, for any $x,y\in X$, then $\frac{X}{A}$ is an X^E-module.*

Definition 2.6. [5] A map $f:M\to N$, where M and N are X^E-modules, is called an X^E-*homomorphism*, if the following hold:
(i) $f(m+n)=f(m)+f(n)$,
(ii) $f(x.m)=x.f(m)$, for all $m,n\in M$ and $x\in X$.

Theorem 2.7. [5] *Let X be a bounded implicative BCK-algebra. Then $(\prod_{i\in I}X,+')$ is an abelian group, where $\{x_i\}_{i\in I}+'\{y_i\}_{i\in I}=\{x_i+y_i\}_{i\in I}$, for any $\{x_i\}_{i\in I},\{y_i\}_{i\in I}\in\prod_{i\in I}X$. Moreover, if operation $.: X\times\prod_{i\in I}X\longrightarrow\prod_{i\in I}X$ is defined by $x.\{x_i\}_{i\in I}=\{x\wedge x_i\}_{i\in I}$, for any $x,x_i\in X$, then $\prod_{i\in I}X$ is an X^E-module.*

Definition 2.8. [5] A subgroup N of X^E-module M is a *submodule* of M, if for any $x\in X$ and any $n\in N$, $x.n\in N$. N is called a *prime* submodule of M, if $N\neq M$ and for any $x\in X$, $x.m\in N$ implies that $m\in N$ or $x\in(N:M)$. Note that, for X^E-module M, $Y\subseteq X$ and submodule N of M, we consider

$$YM=Y.M=\{x.m: x\in Y, m\in M\},\ (N:M)=\{x\in X: x.M\subseteq N\}.$$

Proposition 2.9. [5] *Let M be an X^E-module and N be a submodule of M. Then $(N:M)$ is an ideal of X. Moreover, $\frac{M}{N}$ is an X^E-module.*

Lemma 2.10. [5] *Let X be a commutative BCK-algebra, M be an X^E-module, N be a submodule of M and A be an ideal of X. Then $AM+N=\{\sum_{i=1}^n t_i.m_i+n: t\in A, m\in M, n\in N\}$ is a submodule of M.*

Theorem 2.11. [5] *Let X be a bounded BCK-algebra, A be a proper ideal of X and M be an X^E-module. Then $\frac{M}{AM}$ is an $(\frac{X}{A})^E$-module.*

Theorem 2.12. [5] *Let M and M' be two X^E-modules and $\phi: M\longrightarrow M'$ be an X^E-homomorphism. Then $\frac{M}{Ker\phi}\simeq Img\phi$.*

Note. From now on, in this paper, M is an abelian group and X is a BCK-algebra.

3. Free Extended BCK-Module

Definition 3.1. Let M be an X^E-module, $\emptyset\neq T\subseteq M$ and $M=\{\sum_{i\in I}x_i.t_i: x_i\in X, t_i\in T\}$. Then we say M is generated by T and we set $M=\prec T\succ$. If $|T|<\infty$, then M is called finitely generated X^E-module.

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EXAMPLE 3.2. (i) Let $X = \{0, 1, 2\}$ and operation "$*$" is defined by

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

Then $(X, *, 0)$ is a BCK-algebra. Now, let operation $. : X \times \mathbb{Z} \longrightarrow \mathbb{Z}$ be defined by $2.n = n$, $1.n = 0.n = 0$, for any $n \in \mathbb{Z}$. Then $\mathbb{Z}$ is an X^E-module. For any $0 \neq n \in \mathbb{Z}$, $n = \underbrace{1 + \cdots + 1}_{n\ times} = 2.1 + \cdots + 2.1$ and so $\mathbb{Z} = \prec 1 \succ$.

(ii) Let $M = \{0, 1, 2, 3\}$, $X = \{0, a\}$ and the operations "$*_1$", "$*_2$" be defined by

$*_1$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

$*_2$	0	a
0	0	0
a	a	0

Then $(M, *_1, 0)$ is a bounded implicative BCK-algebra with unit 3 and $(X, *_2, 0)$ is a BCK-algebra, too. It is easy to show that $(M, +)$ is an abelian group, where $m + n = (m *_1 n) \vee (n *_1 m)$, for any $m, n \in M$. Let operation $. : X \times M \longrightarrow M$ be defined by $a.m = m$ and $0.m = 0$, for any $m \in M$. Then M is an X^E-module. Since $1 = a.1$, $2 = a.2$ and $3 = a.1 + a.2$, $M = \prec \{1, 2\} \succ$.

(iii) Let $D = \{0, \frac{1}{2}, 1\}$, $X_1 = \{a, b\}$ and $0, f, I$ be functions from X_1 to D such that $0(x) = 0$, $f(x) = \frac{1}{2}$ and $I(x) = 1$, for any $x \in X_1$. We define operation "$*$" by $(g * h)(x) = g(x) - min\{g(x), h(x)\}$, for any $g, h \in \{0, f, I\} = X$. Then it is easy to show that $(X, *, 0)$ is a BCK-algebra. Consider the abelian group $A = \{\frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}$. Let operation $. : X \times A \longrightarrow A$ be defined by $g.\frac{m}{2^n} = \frac{g(x)m}{2^n}$, for any $g \in X$, $\frac{m}{2^n} \in A$. Then A is an X^E-module. because, for any $\frac{m}{2^n}$, $\frac{m_1}{2^{n_1}}$, $\frac{m_2}{2^{n_2}} \in A$ and $x \in X_1$,

$(XM1)$: $(f \wedge I).\frac{m}{2^n} = min\{f, I\}.\frac{m}{2^n} = f.\frac{m}{2^n} = \frac{f(x)m}{2^n} = \frac{m}{2^{n+1}} = f.(I.\frac{m}{2^n})$. Similarly, $(g \wedge h).\frac{m}{2^n} = g.(h.\frac{f(x)m}{2^n})$, for any $g, h \in X$.

$(XM2)$: $g.(\frac{m_1}{2^{n_1}} + \frac{m_2}{2^{n_2}}) = g(x)(\frac{m_1}{2^{n_1}} + \frac{m_2}{2^{n_2}}) = \frac{g(x)m_1}{2^{n_1}} + \frac{g(x)m_2}{2^{n_2}}) = g.\frac{m_1}{2^{n_1}} + g.\frac{m_2}{2^{n_2}}$, for any $g \in X$.

$(XM3)$: It is clear.

$(XM4)$: We have $I * f \neq 0$. Then

$$\begin{aligned} (I * f).(\frac{m}{2^n}) &= f.\frac{m}{2^n} = \frac{f(x)m}{2^n} = \frac{m}{2^{n+1}} = \frac{m}{2^n} - \frac{m}{2^{n+1}} = \frac{I(x)m}{2^n} - \frac{f(x)m}{2^n} \\ &= I.\frac{m}{2^n} - f.\frac{m}{2^n}. \end{aligned}$$

Moreover, we claim that $T = \{1, \frac{1}{2}, \frac{1}{2^2}, \cdots, \frac{1}{2^n}, \cdots\}$ is a generator for A. Let $\frac{m}{2^n} \in A$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N} \cup \{0\}$. We have

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$\frac{m}{2^n} = \underbrace{\frac{1}{2^n} + \cdots + \frac{1}{2^n}}_{m\ time} = I(x).\frac{1}{2^n} + \cdots + I(x).\frac{1}{2^n}$. Then $A = \prec T \succ$.

(iv): Let X be the BCK-algebra which is defined in (iii). Consider the abelian group,

$$M = \{\frac{m}{2^n} + \mathbb{Z} : m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\},\ where\ \frac{m}{2^n} + \mathbb{Z}\ has\ uniquely\ represent\}.$$

If operation $. : X \times A \longrightarrow A$ is defined by $g.\frac{m}{2^n} + \mathbb{Z} = \frac{g(x)m}{2^n} + \mathbb{Z}$, then it is not difficult to prove that M is an X^E-module.
If $T = \{\mathbb{Z}, \frac{1}{2} + \mathbb{Z}, \frac{1}{2^2} + \mathbb{Z}, \cdots, \frac{1}{2^n} + \mathbb{Z}, \cdots\}$, then $M = \prec T \succ$.

Theorem 3.3. *Let X be bounded and commutative and A be an ideal of X. If X is an X^E-module such that $X = \prec T \succ$, where $T \subseteq X$, then $\frac{X}{A} = \prec \{C_t : t \in T\} \succ$ as an X^E-module.*

Proof. By Proposition 2.5, $(\frac{X}{A}, +')$ is an X^E-module. Let $T = \{t_i : i \in I\}$ such that $X = \prec T \succ$. Then for any $x \in X$, $x = \sum_{i \in I_0} x_i.t_i$, where $I_0 \subseteq I$ and so $C_x = C_{\sum_{i \in I_0} x_i.t_i} = \sum_{i \in I_0} C_{x_i.t_i} = \sum_{i \in I_0} x_i \bullet C_{t_i}$. Hence, $\frac{X}{A} = \prec \{C_{t_i} : i \in I\} \succ$. □

Definition 3.4. Let M be an X^E-module and $\emptyset \neq T \subseteq M$. We say that T is a *basis* for M if
(i) $M = \prec T \succ$,
(ii) If $\sum_{i \in I} x_i.t_i = 0$, for any $x_i \in X$ and $t_i \in T$, then $x_i = 0$, for any $i \in I$. (In this case, we say that T is a linearly independent set).

Definition 3.5. Let M be an X^E-module. Then M is called a *free* X^E-module, if M has a nonempty basis. Specially, if $M = \prec m \succ$, where $m \in M$, then M is a called a cyclic X^E-module.

Example 3.6. (i) Let $X = \{0, x\}$ and operation "$*$" on X be defined by

$*$	0	x
0	0	0
x	x	0

Then $(X, *, 0)$ is a BCK-algebra. Now, let operation $. : X \times \mathbb{Z} \longrightarrow \mathbb{Z}$ is defined by $x.n = n$ and $0.n = 0$, for any $n \in \mathbb{Z}$. It is easy to show that $\mathbb{Z}$ is an X^E-module. Now, we show that $\mathbb{Z}$ is a free X^E-module. For any $n \in \mathbb{Z}$, $n = 1 + \cdots + 1 = x.1 + \cdots + x.1$. So $\mathbb{Z} = \prec 1 \succ$ on X. Moreover, if $t.1 = 0$, then $t = 0$, for any $t \in X$. Therefore, $\mathbb{Z}$ is a free X^E-module.
(ii) In Example 3.2 (ii), if $x.1 + y.2 = 0$, for any $x, y \in X$, then $x = y = 0$. Hence, $M = \prec \{1, 2\} \succ$ is a free X^E-module.
(iii) In Example 3.2 (iii), we have $A = \prec T \succ$. If $\sum g.\frac{1}{2^n} = 0$, for any $g \in X$ and $n \in \mathbb{N} \cup \{0\}$, then $g = 0$. Therefore, A is a free X^E-module.

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(iv) In Example 3.2 (iv), T is not a basis for M. Since

$$\begin{aligned} I.\mathbb{Z}+I.(\frac{1}{2}+\mathbb{Z})+\cdots+I.(\frac{1}{2^n}+\mathbb{Z})+\cdots &= \mathbb{Z}+\frac{1}{2}+\mathbb{Z}+\cdots+\frac{1}{2^n}+\mathbb{Z}+\cdots \\ &= (1+\frac{1}{2}+\cdots+\frac{1}{2^n}+\cdots)+\mathbb{Z} \\ &= 2+\mathbb{Z} \\ &= \mathbb{Z}, \end{aligned}$$

T is not a linearly independent set.

Proposition 3.7. *Let X be of order 2. Then every unitary X^E-module is a free X^E-module.*

Proof. Let $X=\{0,1\}$ be a BCK-algebra of order 2. Then X is a bounded BCK-algebra with unit 1. Let M be a unitary X^E-module and $K=\{T\subseteq M : T \textit{ is linear independent}\}$. Since M is a unitary X^E-module, $1.a=a\neq 0$, for any $0\neq a\in M$. So $\{a\}$ is linear independent. It means that $\{a\}\in K$ and so $K\neq\emptyset$. Let $Y=\{T_i : i\in I\}$ be a chain of elements in K. We claim that $U=\bigcup_{i\in I}T_i$ is an upper bound for Y, with respect to " $\subseteq$ ". Since we have a chain, there exists $T_j\in K$ such that $U\subseteq T_j$ and so $U\in K$. Hence, by Zorn Lemma, K has a maximal element T_1. We claim that $M=\prec T_1\succ$. Let $M\neq\prec T_1\succ$. Then $\prec T_1\succ\subsetneq M$ and so there exists $m\in M$ such that $m\notin\prec T_1\succ$. We show that $T_1\cup\{m\}$ is linear independent. Let $x.m+x_1.t_1+x_2.t_2+\cdots=0$, for any $x,x_i\in X$ and $i\in I$. If $x\neq 0$, then $x=1$. So $m=-(x_1.t_1+x_2.t_2+\cdots)$ and so $m\in\prec T_1\succ$, which is a contradiction. Hence, $x=0$ and so $T_1\cup\{m\}$ is a linear independent set. Therefore, M is a free X^E-module. □

Theorem 3.8. *Let X be of order 2 and M be a unitary X^E-module. Then every $W\subseteq M$ such that $M=\prec W\succ$, contains a basis for M.*

Proof. The proof is similar to the proof of Proposition 3.7. □

Lemma 3.9. *Let X be bounded and commutative, X be an X^E-module and A be an ideal of X. Then $\frac{X}{A}$ is an $(\frac{X}{A})^E$-module.*

Proof. By Proposition 2.5, $(\frac{X}{A},+')$ is an abelian group. Now, let operation $\bullet:\frac{X}{A}\times\frac{X}{A}\longrightarrow\frac{X}{A}$ be defined by $C_x\bullet C_y=C_{x.y}$, for any $x,y\in X$. Then it is easy to prove that $\frac{X}{A}$ is an $(\frac{X}{A})^E$-module. □

Theorem 3.10. *Let X be bounded and commutative, P be a prime ideal in X, $t\in X-P$ and $X=\prec t\succ$ be a free X^E-module, where $x.y=x\wedge y$, for any $x,y\in X$. Then $\frac{X}{P}$ is a free $(\frac{X}{P})^E$-module.*

Proof. By Lemma 3.9, $\frac{X}{P}$ is an $(\frac{X}{P})^E$-module. Let $C_y\in\frac{X}{P}$, for any $y\in X$. Then there exists $x\in X$ such that $C_y=C_{x.t}=C_x\bullet C_t$ and so $\frac{X}{P}=\prec C_t\succ$. Now, let $C_x\bullet C_t=C_0$, for any $x\in X$. Then $C_0=C_x\bullet C_t=C_{x.t}=C_{x\wedge t}$

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and so by $(BCK7)$, $x \wedge t = (x \wedge t) * 0 \in P$. Since $t \notin P$, then $x \in P$ and so $C_x = C_0$. Therefore, $\{C_t\}$ is a basis for $\frac{X}{P}$. $\square$

Definition 3.11. Let M be an X^E-module such that $2m = m + m = 0$, for any $m \in M$. Then M is called an *idempotent* X^E-module.

EXAMPLE 3.12. (i) In Example 3.2 (ii), M is an idempotent X^E-module.
(ii) If bounded implicative BCK-algebra X be totally ordered, then by Proposition 2.4, X is an idempotent X^E-module.
(iii) Let $X = \{0, 1, 2, 3, 4\}$ and the operation " $*$ " is defined by

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	3	2	0

Then $(X, *, 0)$ is a bounded BCK-algebra with unit 4. Let $Y = \{0, 1, 4\}$ and $M = \{0, 2, 3, 4\}$. It is clear that Y is a subalgebra of X and so it is a BCK-algebra. It is easy to show that $(M, +)$ is an abelian group, where $x + y = (x * y) \vee (y * x)$, for any $x, y \in M$. Now, we define the operation $. : Y \times M \to M$ by $y.m = y \wedge m$, for any $y \in Y$ and $m \in M$. Then M is an idempotent Y^E-module.

Theorem 3.13. *Let X be bounded implicative and totally ordered, operations "$+, .$" $: X \times X \longrightarrow X$ be defined by $x + y = (x * y) \vee (y * x)$, $x.y = x \wedge y$, for any $x, y \in X$ and M be an idempotent X^E-module. Then M is a free X^E-module if and only if $M \simeq \prod_{i \in I} X$, where I is a nonempty set.*

Proof. $(\Rightarrow)$ Let $M = \prec T \succ$ be a free idempotent X^E-module, where $T = \{t_i : i \in I\}$. By Theorem 2.7, $(\prod_{i \in I} X, +')$ is an X^E-module, where $\{x_i\}_{i \in I} +' \{y_i\}_{i \in I} = \{x_i + y_i\}_{i \in I}$ and $x.\{x_i\}_{i \in I} = \{x \wedge x_i\}_{i \in I}$, for any $\{x_i\}_{i \in I}, \{y_i\}_{i \in I} \in \prod_{i \in I} X$ and $x \in X$. We define $\phi : \prod_{i \in I} X_i \longrightarrow M$, by $\phi(\{x_i\}_{i \in I}) = \sum_{i \in I} x_i.t_i$, for any $t_i \in T$ and $x_i \in X$. We show that ϕ is an X^E-homomorphism. It is clear that ϕ is well defined. Now, since M is idempotent, $x.t - y.t = x.t + y.t$, for any $x, y \in X$ and $t \in T$. On the other hand, $x_i * y_i = 0$ or

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$y_i * x_i = 0$, for any $i \in I$. Hence, by $(XM4)$, for any $\{x_i\}_{i\in I}, \{y_i\}_{i\in I} \in \prod_{i\in I} X$,

$$\begin{aligned}
\phi(\{x_i\}_{i\in I} +' \{y_i\}_{i\in I}) &= \phi(\{x_i + y_i\}_{i\in I}) = \sum_{i\in I}(x_i + y_i).t_i \\
&= \sum_{i\in I}(x_i * y_i \vee y_i * x_i).t_i \\
&= \sum_{j\in J}(x_j * y_j).t_j + \sum_{k\in K}(y_k * x_k).t_k \\
&= \sum_{j\in J} x_j.t_j + \sum_{j\in J} y_j.t_j + \sum_{k\in K} y_k.t_k + \sum_{k\in K} x_k.t_k \\
&= \sum_{i\in I} x_i.t_i + \sum_{i\in I} y_i.t_i \\
&= \phi(\{x_i\}_{i\in I}) + \phi(\{y_i\}_{i\in I}), \text{ where, } J \cup K = I.
\end{aligned}$$

Moreover, for any $x \in X$, by $(XM1)$ and $(XM2)$,

$$\begin{aligned}
\phi(x.\{x_i\}_{i\in I}) &= \phi(\{x \wedge x_i\}_{i\in I}) = \sum_{i\in I}(x \wedge x_i).t_i = \sum_{i\in I} x.(x_i.t_i) = x.\sum_{i\in I} x_i.t_i \\
&= x.\phi(\{x_i\}).
\end{aligned}$$

Then ϕ is an X^E-homomorphism. It is clear that ϕ is an epimorphism. Now, let $\phi(\{x_i\}_{i\in I}) = \sum_{i\in I} x_i.t_i = 0$. Since T is linear independent, $x_i = 0$, for any $i \in I$ and so $Ker\phi = \{0\}$. On the other hand, by Theorem 2.12, $\frac{\prod_{i\in I} X}{Ker\phi} \simeq M$ and so $\prod_{i\in I} X \simeq M$.

($\Leftarrow$) Let $M \simeq \prod_{t\in T} X$, where T is a nonempty set. We construct a basis for $\prod_{t\in T} X$. Let $\theta_t = \{u_i\}_{i\in I}$ such that

$$u_i = \begin{cases} 0 & \text{if } i \neq t \\ 1 & \text{if } i = t \end{cases}$$

We show that $K = \{\theta_t : t \in T\}$ is a basis for $\prod_{t\in T} X$. Let $\{x_t\}_{t\in T} \in \prod_{t\in T} X$. We have

$$\begin{aligned}
\{x_i\}_{i\in I} &= \{0, \cdots, x_1, 0, \cdots\} +' \{0, \cdots, x_2, 0, \cdots\} +' \cdots \\
&= \{0, \cdots, x_1 \wedge 1, 0, \cdots\} +' \{0, \cdots, x_2 \wedge 1, 0, \cdots\} +' \cdots \\
&= x_1.\{0, \cdots, 1, \cdots\} +' x_2.\{0, \cdots, 1, 0, \cdots\} +' \cdots \\
&= x_1.\theta_{t_1} +' x_2.\theta_{t_2} +' \cdots
\end{aligned}$$

Then $\prod_{t\in T} X = \prec K \succ$. Now, let $\sum_{i\in I} x_i.\theta_{t_i} = 0$. Then

$$\begin{aligned}
0 = \sum_{i\in I} x_i.\theta_{t_i} &= x_1.\{0, 0, \cdots, 1, 0, \cdots\} +' x_2.\{0, 0, \cdots, 1, 0, \cdots\} +' \cdots \\
&= \{0, \cdots, x_1 \wedge 1, 0, \cdots\} +' \{0, \cdots, x_2 \wedge 1, 0, \cdots\} +' \cdots \\
&= \{0, \cdots x_1, 0, \cdots\} +' \{0, \cdots, x_2, 0, \cdots\} +' \cdots \\
&= \{x_i\}_{i\in I}
\end{aligned}$$

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Hence, $x_i = 0$, for any $i \in I$ and so K is a basis for $\prod_{t\in T} X$. Finally, let $\varphi : \prod_{t\in T} X \longrightarrow M$ be an X^E-isomorphism. Then $\varphi(K)$ is a basis for M. $\square$

Definition 3.14. Let Y be a nonempty set, F be an X^E-module and $i : Y \longrightarrow F$ be a map. If for any mapping $f : Y \longrightarrow A$, where A is an X^E-module, there exists a unique X^E-homomorphism $\bar{f} : F \longrightarrow A$ such that $\bar{f} \circ i = f$, then F is called a *free object* on Y.

Proposition 3.15. *Let F_1 and F_2 be two X^E-modules and F_1 and F_2 be two free objects on Y_1 and Y_2, respectively. If $|Y_1| = |Y_2|$, then $F_1 \backsimeq F_2$.*

Proof. The proof is straitforward. $\square$

Theorem 3.16. *By assumptions of Theorem 3.13, every free object in the category of idempotent X^E-modules is isomorphic with $\prod_{i\in I} X$, where I is a nonempty set. (In this category, objects are idempotent X^E-modules and morphisms are X^E-homomorphisms.)*

Proof. Let F be a free object on T, where T is a nonempty set. Similar to the proof of Theorem 3.13, $K = \{\theta_t : t \in T\}$ is a basis for $\prod_{t\in T} X$, as an X^E-module. We show that $\prod_{t\in T} X$ is a free object on K. Let G be an idempotent X^E-module and $i : K \longrightarrow \prod_{t\in T} X$, $f : K \longrightarrow G$ be two maps. We define $h : \prod_{t\in T} X \longrightarrow G$ by $h(\sum_{t\in T} y_t.\theta_t) = \sum_{t\in T} y_t.f(\theta_t)$, where $y_t \in X$, for any $t \in T$. Let $\sum_{t\in T} y_t.\theta_t = \sum_{t\in T} y'_t.\theta_t$, for any $y_t, y'_t \in X$. So $\{y_t\}_{t\in T} = \{y'_t\}_{t\in T}$ and so $\sum_{t\in T} y_t.f(\theta_t) = \sum_{t\in T} y'_t.f(\theta_t)$. It means that h is well defined. Since $(X, \leq)$ is totally ordered, $y_t * y'_t = 0$ or $y'_t * y_t = 0$, for any $y_t, y'_t \in X$. Also, since G is idempotent, for any $\sum_{t\in T} y_t.\theta_t, \sum_{t\in T} y'_t.\theta_t \in \prod_{t\in T} X$, by $(XM4)$,

$$
\begin{aligned}
h(\sum_{t\in T} y_t.\theta_t +' \sum_{t\in T} y'_t.\theta_t) &= h(\sum_{j\in J} (y_j * y'_j).\theta_j +' \sum_{k\in K} (y'_k * y_k).\theta_k) \\
&= \sum_{j\in J} (y_j * y'_j).f(\theta_j) +' \sum_{k\in K} (y'_k * y_k).f(\theta_k) \\
&= \sum_{j\in J} y_j.f(\theta_j) +' \sum_{j\in J} y'_j.f(\theta_j) \\
&+' \sum_{k\in K} y'_k.f(\theta_k) +' \sum_{k\in K} y_k.f(\theta_k) \\
&= \sum_{t\in T} y_t.f(\theta_t) +' \sum_{t\in T} y'_t.f(\theta_t) \\
&= h(\sum_{t\in T} y_t.\theta_t) +' h(\sum_{t\in T} y'_t.\theta_t),
\end{aligned}
$$

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where $T = J \cup K$. Now, for any $x \in X$, by $(XM1)$ and $(XM2)$,

$$\begin{aligned} h(x.\sum_{t\in T} y_t.\theta_t) &= h(\sum_{t\in T} x.(y_t.\theta_t)) = h(\sum_{t\in T}(x \wedge y_t).\theta_t) \\ &= \sum_{t\in T}(x \wedge y_t).f(\theta_t) = \sum_{t\in T} x.(y_t.f(\theta_t)) \\ &= x.\sum_{t\in T} y_t.f(\theta_t) = x.h(\sum_{t\in T} y_t.\theta_t). \end{aligned}$$

Then h is an X^E-homomorphism. On the other hand, by definition h, $h \circ i(\theta_t) = h(\theta_t) = f(\theta_t)$, for any $t \in T$. It is easy to show that h is a unique X^E-homomorphism. Hence, $\prod_{t\in T} X$ is a free object on K. Since $|K| = |T|$, then by Proposition 3.15, $\prod_{t\in T} X \simeq F$. $\square$

Notation: If X is totally ordered and bounded implicative, then by the proof of Theorem 3.13, we obtain a method to make a free object on a nonempty set in the category of idempotent X^E-modules. If A is a nonempty set, then $K = \{\theta_a : a \in A\}$ is a basis for $\prod_{a\in A} X$. By Theorem 3.16, $\prod_{a\in A} X$ is a free object on K.

Theorem 3.17. *By assumptions of Theorem 3.13, every X^E-module in the category of idempotent X^E-modules is homomorphic image of a free X^E-module.*

Proof. Let M be an idempotent X^E-module such that $M = \prec A \succ$, where A is a nonempty set. By the above notation, $\prod_{a\in A} X$ is a free object on $K = \{\theta_a : a \in A\}$. Then there exists a unique X^E-homomorphism $\phi : \prod_{a\in A} X \longrightarrow M$ such that $\phi \circ i = f$, where $i : K \longrightarrow \prod_{a\in A} X$ is an inclusion map and $f : K \longrightarrow M$ is defined by $f(\theta_a) = a$. Now, let $m \in M$. We have

$$\begin{aligned} m &= \sum_{i\in I} x_i.a_i = \sum_{i\in I} x_i.f(\theta_{a_i}) = \sum_{i\in I} x_i.\phi \circ i(\theta_{a_i}) = \sum_{i\in I} x_i.\phi(\theta_{a_i}) \\ &= \sum_{i\in I} \phi(x_i.\theta_{a_i}) = \phi(\sum_{i\in I} x_i.\theta_{a_i}), \end{aligned}$$

where $x_i \in X$, for any $i \in I$. Therefore, ϕ is an X^E-epimorphism. $\square$

Lemma 3.18. *Let M and N be two X^E-modules. Then $M \times N = \{(m, n) : m \in M, n \in N\}$ is an X^E-module.*

Proof. Let $\bullet : X \times (M, N) \longrightarrow (M, N)$ is defined by $x \bullet (m, n) = (x.m, x.n)$, for any $m \in M$, $n \in N$ and $x \in X$. It is easy to show that $M \times N$ is an X^E-module. $\square$

Theorem 3.19. *Let M and N be free X^E-modules. Then $M \times N$ is a free X^E-module.*

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Proof. Let $M =\prec T \succ$ and $N =\prec K \succ$, where $T = \{t_i : i \in I\}$ and $K = \{k_j : j \in J\}$ are basises of M, N, respectively. It is easy to show that $M \times N =\prec \{(t_i, 0) : i \in I\} \cup \{(0, k_j) : j \in J\} \succ$ is a free X^E-module. □

Theorem 3.20. *Let X be bounded and implicative, A be a proper ideal of X and M be a free X^E-module with basis Y. Then $\frac{M}{AM}$ is a free $(\frac{X}{A})^E$-module. Moreover, the cardinality of Y is equal to cardinality of the basis of $\frac{M}{AM}$.*

Proof. By Lemma 2.11, $\frac{M}{AM}$ is an $(\frac{X}{A})^E$-module. Let $\beta : M \to \frac{M}{AM}$ be canonical epimorphism. We show that $\frac{M}{AM}$ is a free $(\frac{X}{A})^E$-module by basis $\beta(Y)$. For any $m + AM \in \frac{M}{AM}$, there exists $x_1, \cdots, x_n \in X$ such that

$$\begin{aligned} m + AM &= \sum_{i=1}^{n} x_i.y_i + AM = (x_1.y_1 + AM) + \cdots + (x_n.y_n + AM) \\ &= C_{x_1} \bullet (y_1 + AM) + \cdots + C_{x_n} \bullet (y_n + AM) \\ &= C_{x_1} \bullet (\beta(y_1)) + \cdots + C_{x_n} \bullet (\beta(y_n)). \end{aligned}$$

Then $\frac{M}{AM} =\prec \beta(Y) \succ$. Now, let $\sum_{i=1}^{n} C_{x_i} \bullet (y_i + AM) = AM$. Hence, $\sum_{i=1}^{n} x_i.y_i + AM = AM$ and so $\sum_{i=1}^{n} x_i.y_i \in AM$. This means that $\sum_{i=1}^{n} x_i.y_i = \sum_{i=1}^{m} s_i.m_i$, where $s_i \in A, m_i \in M, m \in \mathbb{Z}$, $1 \leq i \leq m$. For any $m_i \in M$, we have $m_i = \sum_{j=1}^{l_i} t_{ij}.y_j$, where $t_{ij} \in X, 1 \leq j \leq n, l_i \in \mathbb{Z}$. Then

$$\begin{aligned} \sum_{i=1}^{n} x_i.y_i &= \sum_{i=1}^{m} s_i.\Sigma_{j=1}^{l_i} t_{ij}.y_j \\ &= s_1.\Sigma_{j=1}^{l_1} t_{1j}.y_j + \cdots + s_m.\Sigma_{j=1}^{l_m} t_{mj}.y_j \\ &= \sum_{j=1}^{l_1} s_1.(t_{1j}.y_j) + \cdots + \sum_{j=1}^{l_m} s_m.(t_{mj}.y_j) \\ &= \sum_{j=1}^{l_1} (s_1 \wedge t_{1j}).y_j + \cdots + \sum_{j=1}^{l_m} (s_m \wedge t_{mj}).y_j. \end{aligned}$$

Therefore, $\sum_{i=1}^{n} x_i.y_i - (\sum_{j=1}^{l_1} (s_1 \wedge t_{1j}).y_j + \cdots + \sum_{j=1}^{l_m} (s_m \wedge t_{mj}).y_j) = 0$. If y_1 be only in the first summation, then we have $x_1 = 0$ and so $C_{x_1} = C_0 = A$ and similarly for other x_i, where $1 \leq i \leq n$. If y_1 be in two summation, then $x_1.y_1 - (s_1 \wedge t_{11}).y_1 + \cdots = 0$. If $x_1 * (s_1 \wedge t_{11}) \neq 0$, where $x_1 \neq (s_1 \wedge t_{11})$, then by $(XM4)$, $(x_1 * (s_1 \wedge t_{11})).y_1 + \cdots = 0$. Since Y is a basis of M, $x_1 * (s_1 \wedge t_{11}) = 0$, which is a contradiction. Hence, $x_1 * (s_1 \wedge t_{11}) = 0$. By lemma 2.2 (i), $x_1 * (s_1 * Nt_{11}) = 0$. It results that $C_{x_1} \star (C_{s_1} \star C_{Nt_{11}}) = C_{x_1 * (s_1 * N_{t_{11}})} = C_0$. Since $C_{s_1} = C_0$, $C_0 = C_{x_1} \star (C_{s_1} \star C_{1 * t_{11}}) = C_{x_1} \star C_{0 * 1 * t_{11}} = C_{x_1 * 0} = C_{x_1}$ and so $C_{x_1} = C_0 = A$. Similarly, $C_{x_i} = A$, for any $1 \leq i \leq n$. Therefore, $\beta(Y)$ is a basis of $\frac{M}{AM}$. Now, we show that $|\beta(Y)| = |Y|$. We define $\phi : Y \to \beta(Y)$ by $\phi(y) = \beta(y)$. It is clear that ϕ is well defined and onto. Let $\beta(y) = \beta(z)$, for some $y, z \in Y$ and $y \neq z$. Hence, $C_1 \bullet \beta(y) = C_1 \bullet \beta(z)$ and so

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$C_1 \bullet \beta(y) - C_1 \bullet \beta(z) = AM$. Since $\beta(Y)$ is a basis of $\frac{M}{AM}$, $C_1 = C_0$, which is a contradiction. Therefore, ϕ is one to one and $|\phi(Y)| = |Y|$. □

Theorem 3.21. *Let $f : X \to Y$ be an epimorphism of bounded implicative BCK-algebras, $Y \neq \{0\}$ and every two basises of any Y^E-module have equal cardinality. Then every two basises of any X^E-module have equal cardinality, too.*

Proof. Let M be an arbitrary free X^E-module with basises K and U. We must show that $|K| = |U|$. Let $A = Kerf$. If $A = X$, then $f(X) = 0$. Since f is an epimorphism, $f(X) = \{0\} = Y$, which is a contradiction and so $A \neq X$. By Theorem 3.20, $\frac{M}{AM}$ is a free $(\frac{X}{A})^E$-module with basis of $\beta(K)$ such that $|\beta(K)| = |K|$ and $\frac{M}{AM}$ is a free $(\frac{X}{A})^E$-module with basis of $\beta(U)$ such that $|\beta(U)| = |U|$. Since $Y \simeq \frac{X}{A}$, $\frac{M}{AM}$ is a free Y^E-module. It results that $|\beta(K)| = |\beta(U)|$ and so $|K| = |U|$. □

4. Productive and Projective X^E-Module

Definition 4.1. Let M be an X^E-module. If for any submodule N of M, there exists an ideal A of X such that $N = AM$, then M is called a *productive* X^E-module.

Theorem 4.2. *Let X be commutative. Then every cyclic X^E-module is a productive X^E-module.*

Proof. Since M is a cyclic X^E-module, there exists $m \in M$ such that $M = \prec m \succ$. Let N be a submodule of M. By Proposition 2.9, $(N : M)$ is an ideal of X and by Lemma 2.10, $(N : M)M$ is a submodule of M. We show that $N = (N : M)M$. It is clear that $(N : M)M \subseteq N$. Now, let $n \in N$. Then there exists $x \in X$ such that $n = x.m$. Since

$$\begin{aligned} x.M &= x. \prec m \succ = \{x.(x_i.m) : x_i \in X\} = \{(x \wedge x_i).m : x_i \in X\} \\ &= \{x_i.(x.m) : x_i \in X\} = \{x_i.n : x_i \in X\} \subseteq N, \end{aligned}$$

$x \in (N : M)$ and so $n \in (N : M)M$. Hence, $N \subseteq (N : M)M$. Therefore, $N = (N : M)M$. □

Definition 4.3. Let A, B, P be three X^E-modules. Then P is called a projective X^E-module if for any X^E-homomorphism $g : P \to B$ and X^E-epimorphism $f : A \to B$, there exists X^E-homomorphism $h : P \to A$ such that $f \circ h = g$.

Theorem 4.4. *Let M be an idempotent X^E-module with basis $\emptyset \neq Y$ and for any $x, y \in X$ and $0 \neq x$, $x * y = 0$ implies that $x = y$. Then M is a free object on Y. Moreover, if M is free X^E-module, then M is a projective X^E-module.*

Proof. Let $i : Y \to M$ be a map. We will show that for any mapping $f : Y \to G$, where G is an idempotent X^E-module, there exists a unique X^E-homomorphism $h : M \to G$ such that $h \circ i = f$. We have

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$M = \{\sum_{i\in I} x_i.y_i : x_i \in X, y_i \in Y, i \in I\}$. We define
$h(\sum_{i\in I} x_i.y_i) = \sum_{i\in I} x_i.f(y_i)$. Let $\sum_{i\in I} x_i.y_i = \sum_{i\in I} x'_i.y_i$, where $x_i, x'_i \in X$. Hence, $x_1.y_1 - x'_1.y_1 + x_2.y_2 - x'_2.y_2 + \cdots = 0$. If $x_i * x'_i \neq 0$, where $x_i \neq x'_i$, then by $(XM4)$, we have $(x_1 * x'_1).y_1 + (x_2 * x'_2).y_2 + \cdots = 0$. Since Y is a basis of M, $x_i * x'_i = 0$, which is a contradiction. So $x_i * x'_i = 0$ and so $x_i = x'_i$, for any $i \in I$. It results that h is well defined. Now,

$$\begin{aligned} h(\sum_{i\in I} x_i.y_i + \sum_{i\in I} x'_i.y_i) &= x_1.f(y_1) + x'_1.f(y_1) + x_2.f(y_2) + x'_2.f(y_2) + \cdots \\ &= \sum_{i\in I} x_i.f(y_i) + \sum_{i\in I} x'_i.f(y_i) \\ &= h(\sum_{i\in I} x_i.y_i) + h(\sum_{i\in I} x'_i.y_i), \end{aligned}$$

where, $x_i, x'_i \in X$ and $i \in I$. On the other hand, for any $x \in X$, by $(XM1)$, $(XM2)$, we have

$$\begin{aligned} h(x.\sum_{i\in I} x_i.y_i) &= h(\sum_{i\in I} x.(x_i.y_i)) = h(\sum_{i\in I}(x \wedge x_i).y_i) = \sum_{i\in I}(x \wedge x_i).f(y_i) \\ &= \sum_{i\in I} x.(x_i.f(y_i)) = x.\sum_{i\in I} x_i.f(y_i) = x.h(\sum_{i\in I} x_i.y_i). \end{aligned}$$

Then h is an X^E-homomorphism. By definition h, $h \circ i(y) = h(y) = f(y)$ for any $y \in Y$ and so $h \circ i = f$. Finally, h is unique, because if there exists an X^E-homomorphism $h' : M \to G$ such that $h' \circ i = f$, then we have
$h'(y) = h' \circ i(y) = f(y) = h \circ i(y) = h(y)$, for any $y \in Y$. Therefore, M is a free object on Y.
Now, we prove that the second part of theorem. Let M be a free X^E-module with basis Y, $f : A \to B$ be an X^E-epimorphism and $g : M \to B$ be an X^E-homomorphism. Let $y \in Y$. Then $i(y) \in M$, where $i : Y \to M$ is inclusion map. It results that $g(i(y)) \in B$. Since f is an X^E-epimorphism, there exists $a_y \in A$ such that $f(a_y) = g(i(y))$. Since choosing θ is at the discretion of us, W. O. L. G, suppose that a_y is unique. Hence, we can define $\theta : Y \to A$ by $\theta(y) = a_y$. Since M is a free object on Y, there exists $h : M \to A$ such that $h \circ i = \theta$. It is easy to show that $f \circ h \circ i(y) = f(a_y) = g \circ i(y)$ and so $f \circ h \circ i = g \circ i : Y \to B$. Since M is a free object on Y, $f \circ h = g$. □

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