Iranian Journal of Mathematical Sciences and Informatics
Vol. 10, No. 2 (2015), pp 23-27
DOI: 10.7508/ijmsi.2015.02.003

A Generalized Singular Value Inequality for Heinz Means

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Abstract. In this paper we will generalize a singular value inequality that was proved before. In particular we obtain an inequality for numerical radius as follows:

$$2\sqrt{t(1-t)}\omega(tA^{\nu}B^{1-\nu}+(1-t)A^{1-\nu}B^{\nu}) \le \omega(tA+(1-t)B),$$

where, A and B are positive semidefinite matrices, $0 \le t \le 1$ and $0 \le \nu \le \frac{3}{2}$.

Keywords: Matrix monotone functions, Numerical radius, Singular values, Unitarily invariant norms.

2000 Mathematics subject classification: 15A42, 15A60, 47A30.

1. Introduction

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices. A norm $|||.|||$ on $\mathbb{M}_n$ is said to be unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathbb{M}_n$ and all unitary $U, V \in \mathbb{M}_n$. Special examples of such norms are the "Ky Fan norms"

$$\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A), \qquad 1 \le k \le n.$$

Note that the operator norm, in this notation, is $\|A\| = \|A\|_{(1)} = s_1(A)$; see [4] and [9] for more information.

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Received 21 August 2013; Accepted 08 September 2014

If $\|A\|_{(k)} \leq \|B\|_{(k)}$ for $1 \leq k \leq n$, then $|||A||| \leq |||B|||$ for all unitary invariant norms. This is called the "Fan dominance theorem." If A is a Hermitian element of $\mathbb{M}_n$, then we arrange its eigenvalues in decreasing order as $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. If A is arbitrary, then its singular values are enumerated as $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$. These are the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{1/2}$. If A and B are Hermitian matrices, and $A - B$ is positive semidefinite, then we say that $B \leq A$.
Weyl's monotonocity theorem [4, p. 63] says that $B \leq A$ implies $\lambda_j(A) \leq \lambda_j(B)$, for all $j = 1, \ldots, n$. Let f be a real valued function on an interval I. Then f is said to be matrix monotone if $A, B \in \mathbb{M}_n$ are Hermitian matrices with all their eigenvalues in I and $A \geq B$, then $f(A) \geq f(B)$ and also, f is said to be matrix convex if

$$f(tA + (1-t)B) \leq tf(A) + (1-t)f(B),\ 0 \leq t \leq 1$$

and matrix concave if

$$f(tA + (1-t)B) \geq tf(A) + (1-t)f(B),\ 0 \leq t \leq 1.$$

In response to a conjecture by Zhan [13], Audenaert [2] has proved that if $A, B \in \mathbb{M}_n$ are positive semidefinite, then the inequality

$$s_j(A^\nu B^{1-\nu} + A^{1-\nu}B^\nu) \leq s_j(A + B), \quad 1 \leq j \leq n$$

holds, for all $0 \leq \nu \leq 1$. In this paper we generalize this inequality as follows: If $A, B \in \mathbb{M}_n$ are positive semidefinite matrices, then for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{3}{2}$

$$2\sqrt{t(1-t)}s_j(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu}B^\nu) \leq s_j(tA + (1-t)B).$$

For more details about inequalities and their generalizations with their history of origin, the reader may refer to [1, 5, 6, 11, 12, 13].

2. Main Results

Lemma 2.1. [14] *If* $X = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ *is positive, then* $2s_j(C) \leq s_j(X)$ *for all* $1 \leq j \leq n$.

Theorem 2.2. *Let f be a matrix monotone function on $[0, \infty)$ and A and B be positive semidefinite matrices. Then*

$$tAf(A)+(1-t)Bf(B) \geq (tA+(1-t)B)^{1/2}(tf(A)+(1-t)f(B))(tA+(1-t)B)^{1/2} \tag{2.1}$$

for all $0 \leq t \leq 1$.

Proof. The function f is also matrix concave, and $g(x) = xf(x)$ is matrix convex. (See [4]). The matrix convexity of g implies the inequality

$$(tA + (1-t)B)f(tA + (1-t)B) \leq tAf(A) + (1-t)Bf(B), \quad 0 \leq t \leq 1. \tag{2.2}$$

Since the matrix $tA+(1-t)B$ is positive semidefinite, in view of the spectral decomposition theorem, it is easy to see that for all $0 \leqslant t \leqslant 1$,

$$(tA+(1-t)B)f(tA+(1-t)B) = (tA+(1-t)B)^{1/2}f(tA+(1-t)B)(tA+(1-t)B)^{1/2}. \tag{2.3}$$

Also, the matrix concavity of f implies that

$$tf(A)+(1-t)f(B) \leq f(tA+(1-t)B), \quad 0 \leq t \leq 1. \tag{2.4}$$

Combining the relations (2.2), (2.3) and (2.4), we get (2.1). □

Theorem 2.3. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{3}{2}$*

$$2\sqrt{t(1-t)}s_j(tA^\nu B^{1-\nu}+(1-t)A^{1-\nu}B^\nu) \leq s_j(tA+(1-t)B). \tag{2.5}$$

Proof. The proof depends on the fact that the matrices XY and YX have the same eigenvalues. Let $f(x)=x^r, 0 \leq r \leq 1$. This function is matrix monotone on $[0,\infty)$. Hence from (2.1) and Weyl's monotonocity theorem we have

$$\lambda_j(tA^{r+1}+(1-t)B^{r+1}) \geq \lambda_j\left((tA+(1-t)B)(tA^r+(1-t)B^r)\right). \tag{2.6}$$

Except for trivial zeroes the eigenvalues of $(tA+(1-t)B)(tA^r+(1-t)B^r)$ are the same as those of the matrix

$$\begin{bmatrix} tA+(1-t)B & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{t}A^{r/2} & \sqrt{1-t}B^{r/2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{t}A^{r/2} & 0 \\ \sqrt{1-t}B^{r/2} & 0 \end{bmatrix}$$

and in turn, these are the same as the eigenvalues of

$$\begin{bmatrix} \sqrt{t}A^{r/2} & 0 \\ \sqrt{1-t}B^{r/2} & 0 \end{bmatrix} \begin{bmatrix} tA+(1-t)B & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{t}A^{r/2} & \sqrt{1-t}B^{r/2} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} tA^{r/2}(tA+(1-t)B)A^{r/2} & \sqrt{t(1-t)}A^{r/2}(tA+(1-t)B)B^{r/2} \\ \sqrt{t(1-t)}B^{r/2}(tA+(1-t)B)A^{r/2} & (1-t)B^{r/2}(tA+(1-t)B)B^{r/2} \end{bmatrix}.$$

So, Lemma 2.1 and inequality (2.6) together give

$$\lambda_j(tA^{r+1}+(1-t)B^{r+1}) \geq 2\sqrt{t(1-t)}s_j(A^{r/2}(tA+(1-t)B)B^{r/2})$$

$$= 2\sqrt{t(1-t)}s_j(tA^{1+\frac{r}{2}}B^{r/2}+(1-t)A^{r/2}B^{1+\frac{r}{2}}).$$

Replacing A and B by $A^{1/r+1}$ and $B^{1/r+1}$, respectively, we get from this

$$s_j(tA+(1-t)B) \geq 2\sqrt{t(1-t)}s_j(tA^{\frac{r+2}{2r+2}}B^{\frac{r}{2r+2}}+(1-t)A^{\frac{r}{2r+2}}B^{\frac{2+r}{2r+2}}),\ 0 \leq r,t \leq 1.$$

Now, if we put $\nu = \dfrac{r+2}{2r+2}$, then trivially, we get

$$s_j(tA+(1-t)B) \geq 2\sqrt{t(1-t)}s_j(tA^\nu B^{1-\nu}+(1-t)A^{1-\nu}B^\nu),$$

[DOI: 10.7508/ijmsi.2015.02.003]

for all $0 \le t \le 1$ and $\frac{1}{2} \le \nu \le \frac{3}{2}$ and we have proved (2.5) for this special range.
Symmetry, if we put $\nu = \dfrac{r}{2r+2}$, then it is easy to see that the inequality (2.5) holds for all for all $0 \le t \le 1$ and $0 \le \nu \le \frac{1}{2}$. Hence the proof is complete. □

If in Theorem 2.3, we put $t = \frac{1}{2}$, then we have the following corollary, which obtained by Audenaert in [2] and by Bhatia and Kittaneh in [6].

Corollary 2.4. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all* $0 \le \nu \le 1$

$$s_j(A^\nu B^{1-\nu} + A^{1-\nu}B^\nu) \le s_j(A + B).$$

Corollary 2.5. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all* $0 \le t \le 1$ *and* $0 \le \nu \le \frac{3}{2}$

$$2\sqrt{t(1-t)}\, \left|\left|\left| tA^\nu B^{1-\nu} + (1-t)A^{1-\nu}B^\nu \right|\right|\right| \le |||tA + (1-t)B|||\, .$$

For $A \in \mathbb{M}_n$, the numerical radius of A is defined and denoted by

$$\omega(A) = \max\{|x^*Ax| : x \in \mathbb{C}^n, x^*x = 1\}.$$

The quantity $\omega(A)$ is useful in studying perturbations, convergence, stability, approximation problems, iterative method, etc. For more information see [3, 7]. It is known that $\omega(.)$ is a vector norm on $\mathbb{M}_n$, but is not unitarily invariant. We recall the following results about the numerical radius of matrices which can be found in [8] (see also [10, Chapter 1]).

Lemma 2.6. *Let $A \in \mathbb{M}_n$ and $\omega(.)$ be the numerical radius. Then the following assertions are true:*
(i) $\omega(U^*AU) = \omega(A)$, where U is unitary;
(ii) $\frac{1}{2}\|A\| \le \omega(A) \le \|A\|$;
(iii) $\omega(A) = \|A\|$ if (but not only if) A is normal.

Utilizing Lemma 2.6 (parts (ii) and (iii)) and by Corollary 2.5 we obtain the following corollary.

Corollary 2.7. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all* $0 \le t \le 1$ *and* $0 \le \nu \le \frac{3}{2}$

$$2\sqrt{t(1-t)}\omega(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu}B^\nu) \le \omega(tA + (1-t)B).$$

Acknowledgment

The author would like to thank the anonymous referee for careful reading and the helpful comments improving this paper.

[DOI: 10.7508/ijmsi.2015.02.003]

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[DOI: 10.7508/ijmsi.2015.02.003]