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Some Equations in Semiprime Rings with Multiplicative Generalized Semiderivation

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Abstract. Let R be a semiprime ring. A mapping F on R is said to be a multiplicative generalized semiderivation of R if there exists a multiplicative semiderivation d associated with a map g on R such that (i) $F(xy) = F(x)y + g(x)\,d(y) = d(x)g(y) + xF(y)$ and (ii) $F(g(x)) = g(F(x))$, for all $x, y \in R$. The purpose of this paper is to study multiplicative generalized semiderivations satisfying certain differential identities on semiprime rings.

Keywords: Semiprime ring, Semiderivation, Generalized emiderivation, Multiplicative derivation.

2000 Mathematics subject classification: 16W25, 16U80.

1. Introduction

First, let's talk about some algebraic concepts used throughout this study. The first study on derivations of prime rings was conducted in 1957 by Posner [7]. In this paper, Posner describes the definition of derivation in any ring as follows: An additive mapping d on R is a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. The idea of multiplicative derivation was put forward in 1991 by Daif [3] as follows: A mapping d on R is said to be multiplicative derivation

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of R if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. These maps are not additive, so it is a generalization of the derivations in a sense.

In 1991, Bresar [2], the concept of derivation has been generalized as follows: Let d be a derivation of R, an additive mapping F on R is called generalized derivation of R associated with d if $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. In 1998, Hvala [6] studied the algebraic properties of generalized derivation of prime rings. Obviously, each derivation is generalized derivation of R, so the concept of generalized derivation covers the concept of derivation. In [4], Daif and Tammam El Sayiad have been introduced the notion of multiplicative generalized derivation of ring which extend the concept of generalized derivations of a ring as follows: A mapping F on R is called multiplicative generalized derivation of R associated with a derivation d on R if $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. Clearly, every generalized derivation is a multiplicative generalized derivation on R but the converse need not be true in general.

On the other hand, in 1983, J. Bergen [1] has been defined the concept of semiderivation of ring as follows: An additive mapping d on R is called a semiderivation of ring if there exist a map g on R such that (i) $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and (ii) $d(g(x)) = g(d(x))$, for all $x, y \in R$. In this definition if g is the identity map on R, the concept of semiderivation covers the concept of derivation.

The problems related to the commutativity of prime and semiprime rings admitting derivation and generalized derivation are still being investigated at the present time. Besides, in recent studies the algebraic properties of rings are investigated under the conditions which are formed by using the concept of semiderivation on prime and semiprime rings.

The principal aim in this study is to motivate the definition of semiderivation given by Bergen. Inspired by the definition multiplicative generalized derivation in [5], the authors defined the notion of multiplicative generalized semiderivation such as: A mapping F on R is said to be a multiplicative generalized semiderivation of R if there exists a multiplicative semiderivation d associated with a map g on R such that (i) $F(xy) = F(x)y + g(x)d(y) = d(x)g(y) + xF(y)$ and (ii) $F(g(x)) = g(F(x))$, for all $x, y \in R$. As multiplicative generalized semiderivation is an extended notion of semiderivation, derivation and generalized derivation. The main objective of this paper is to take care of this definition and investigate the some certain identities on a semiprime ring R admitting multiplicative generalized semiderivation.

2. Results

Throughout the paper, R will be a semiprime ring and F a multiplicative generalized semiderivation of R associated with a multiplicative semiderivation d of R and g is an epimorphism on R. For any $x, y \in R$, as usual $[x, y] =$

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$xy-yx$ and $xoy=xy+yx$ will denote the well-known Lie and Jordan product, respectively and make extensive use of basic commutator identities:

$$\begin{gathered}[x,yz]=y[x,z]+[x,y]z\\ [xy,z]=[x,z]y+x[y,z]\\ xo(yz)=(xoy)z=y[x,z]=y(xoz)+[x,y]z\\ (xy)oz=x(yoz)=[x,z]y=(xoz)y+x[y,z].\end{gathered}$$

Theorem 2.1. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F([x,y])=0$, for all $x,y\in R$, then $[F(x),x]=0$, for all $x\in R$.*

Proof. We are assuming that

$$F([x,y])=0,\ \text{for all } x,y\in R. \tag{2.1}$$

Substituting yx in place of y in (2.1), we get

$$F([x,y]x)=0,\ \text{for all } x,y\in R.$$

This implies that

$$F([x,y])x+g\left([x,y]\right)d\left(x\right)=0,\ \text{for all } x,y\in R.$$

Using the hypothesis, we have

$$g([x,y])d(x)=0,\ \text{for all } x,y\in R. \tag{2.2}$$

Replace y by yz in the last expression, we get

$$g([x,yz])d(x)=0,\ \text{for all } x,y\in R.$$

Since g is an epimorphism of R and using (2.2), we can write this equation such as

$$g([x,y])g(z)d(x)=0,\ \text{for all } x,y\in R. \tag{2.3}$$

Again replacing z by zx in (2.3), we get

$$g([x,y])g(z)g(x)d(x)=0,\quad \text{for all } x,y,z\in R. \tag{2.4}$$

Right multiply (2.3) by $g\left(x\right)$, we get

$$g([x,y])g(z)d(x)g\left(x\right)=0,\ \text{for all } x,y,z\in R. \tag{2.5}$$

Subtract (2.4) from (2.5), we arrive at

$$g([x,y])g(z)\left[g(x),d\left(x\right)\right]=0,\ \text{for all } x,y,z\in R.$$

Since g is an epimorphism of R, we have

$$\left[g\left(x\right),t\right]R\left[g(x),d\left(x\right)\right]=(0),\ \text{for all } x,t\in R. \tag{2.6}$$

Writing $d\left(x\right)$ instead of t in the last equation, we get

$$\left[g\left(x\right),d\left(x\right)\right]R\left[g(x),d\left(x\right)\right]=(0),\ \text{for all } x\in R.$$

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By the semiprimeness of R, we arrive at

$$[g(x), d(x)] = 0, \text{ for all } x \in R. \tag{2.7}$$

On the other hand, using F is a multiplicative generalized semiderivation of R

$$F(x^2) = F(x)x + g(x)\, d(x) = d(x)g(x) + xF(x)$$

for all $x \in R$. Simplifying last equation, we find that

$$[F(x), x] = 0, \text{ for all } x \in R.$$

This completes the proof. □

Theorem 2.2. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F(xoy) = 0$, for all $x, y \in R$, then $[F(x), x] = 0$, for all $x \in R$.*

Proof. Assume that

$$F(xoy) = \pm(xoy), \text{ for all } x, y \in R. \tag{2.8}$$

Writing yx instead of y in (2.8), we have

$$F((xoy)\, x) = \pm\,(xoy)\, x, \text{ for all } x, y \in R.$$

Since F is a multiplicative generalized semiderivation of R, we get

$$F(xoy)x + g\,(xoy)\, d\,(x) = 0, \text{ for all } x, y \in R.$$

Using (2.8), we obtain

$$g\,(xoy)\, d\,(x) = 0, \text{ for all } x, y \in R. \tag{2.9}$$

Put yz for y in last equation, we have

$$g\,(xoyz)\, d\,(x) = 0, \text{ for all } x, y \in R.$$

Expanding this equation and using the (2.9), we arrive at

$$g\,([x, y])\, g\,(z)\, d\,(x) = 0, \text{ for all } x, y \in R.$$

Further, the proof follows from Teorem 1, after equation (2.3). The same technique can be followed, we obtain $[F(x), x] = 0$, for all $x \in R$. □

Theorem 2.3. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F([x, y]) \pm xy = 0$, for all $x, y \in R$, then $[F(x), x] = 0$, for all $x \in R$.*

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Proof. Assume that

$$F([x,y]) \pm xy = 0, \text{ for all } x, y \in R. \tag{2.10}$$

Replacing y by yx in (2.10), we get

$$F([x,y]x) \pm xyx = 0$$

which can be expanded as

$$F([x,y])x \pm g([x,y])\, d(x) \pm xyx = 0, \text{ for all } x, y \in R.$$

This can be written as

$$(F([x,y]) \pm xy)\, x \pm g([x,y])\, d(x) = 0, \text{ for all } x, y \in R.$$

Using the hypothesis, this equation reduces to

$$g([x,y])\, d(x) = 0, \text{ for all } x, y \in R.$$

This equation is same as (2.2) in the proof of Theorem 1. Applying the same arguments in the proof of Theorem 1, we get the required result. □

Theorem 2.4. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F([x,y]) \pm yx = 0$, for all $x, y \in R$, then $[F(x), x] = 0$, for all $x \in R$.*

Proof. Assume that

$$F([x,y]) \pm yx = 0, \text{ for all } x, y \in R. \tag{2.11}$$

Replacing y by yx in the above equation and using it, we can easily get

$$g([x,y])\, d(x) = 0, \text{ for all } x, y \in R.$$

Applying the same techniques as used after the equation (2.2) in the proof of Theorem 1, we get the required result. □

Theorem 2.5. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F(xoy) \pm xy = 0$, for all $x, y \in R$, then $[F(x), x] = 0$, for all $x \in R$.*

Proof. By the hypothesis, we have

$$F(xoy) \pm xy = 0, \text{ for all } x, y \in R. \tag{2.12}$$

Replacing y by yx in (2.12) and using this, we obtain that

$$g(xoy)d(x) = 0, \text{ for all } x, y \in R.$$

This equation is same as (2.9) in the proof of Theorem 2. Applying the same arguments in the proof of Theorem 2, we get the required result. □

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Theorem 2.6. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F(xoy) \pm yx = 0$, for all $x, y \in R$, then $[F(x), x] = 0$, for all $x \in R$.*

Proof. Assume that

$$F(xoy) \pm yx = 0, \text{ for all } x, y \in R. \tag{2.13}$$

Replacing y by yx in (2.13) and using this equation, we arrive that

$$g(xoy)d(x) = 0, \text{ for all } x, y \in R.$$

Further, proceed as Theorem 2, we obtain that $[F(x), x] = 0$, for all $x \in R$. □

Theorem 2.7. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F([x, y]) = \pm [x, y]$, for all $x, y \in R$, then $[F(x), x] = 0$, for all $x \in R$.*

Proof. We assume that

$$F([x, y]) = \pm [x, y], \text{ for all } x, y \in R. \tag{2.14}$$

Writing y by yx in (2.14), we get

$$F([x, y] x) = \pm [x, y] x, \text{ for all } x, y \in R.$$

Using the definition of F, we have

$$F([x, y])x + g([x, y]) d(x) = \pm [x, y] x$$

By the hypothesis, we arrive that

$$g([x, y]) d(x) = 0, \text{ for all } x, y \in R.$$

This equation is same as (2.2) in the proof of Theorem 1. The required result is obtained using the same arguments in Theorem 1. □

Theorem 2.8. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F([x, y]) = \pm (xoy)$, for all $x, y \in R$, then $[F(x), x] = 0$, for all $x \in R$.*

Proof. By our hypothesis, we get

$$F([x, y]) = \pm (xoy), \text{ for all } x, y \in R. \tag{2.15}$$

Replacing y by yx in (2.15) and using this equation, we arrive that

$$g([x, y])d(x) = 0, \text{ for all } x, y \in R.$$

This equation is same as (2.2) in the proof of Theorem 1. Using the same arguments after the equation (2.2) in the proof of Theorem 1, we get the required result. □

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Theorem 2.9. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F(xoy) = \pm(xoy)$, for all $x, y \in R$, then $[F(x), x] = 0$, for all $x \in R$.*

Proof. By our hypothesis, we get

$$F(xoy) = \pm(xoy), \text{ for all } x, y \in R. \tag{2.16}$$

Replacing y by yx in (2.15), we get

$$F((xoy)x) = \pm(xoy)x, \text{ for all } x, y \in R.$$

Since F is a multiplicative generalized semiderivation of R, we get

$$F(xoy)x + g(xoy)d(x) = \pm(xoy)x$$

By the hypothesis, we find that

$$g(xoy)d(x) = 0, \text{ for all } x, y \in R.$$

This equation is same as (2.9) in the proof of Theorem 2. Using the same arguments in the proof of Theorem 2, we get the required result. □

Theorem 2.10. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F(xoy) = \pm[x, y]$, for all $x, y \in R$, then $[F(x), x] = 0$, for all $x \in R$.*

Proof. By our hypothesis, we get

$$F(xoy) = \pm[x, y], \text{ for all } x, y \in R. \tag{2.17}$$

Replacing y by yx in (2.17) and using (2.17), we obtain

$$g(xoy)d(x) = 0, \quad \text{for all } x, y \in R.$$

Using the same arguments after the equation (2.9) in the proof of Theorem 2, we get the required result. □

Theorem 2.11. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F([x, y]) = \pm x^m[x, y]x^n$ for all $x, y \in R, m, n \in \mathbb{N}$, then $[F(x), x] = 0$, for all $x \in R$.*

Proof. By our hypothesis

$$F([x, y]) = \pm x^m[x, y]x^n, \text{ for all } x, y \in R. \tag{2.18}$$

Taking y by yx in (2.18), we get

$$F([x, yx]) = \pm x^m[x, yx]x^n$$

and so

$$F([x, y]x) = \pm x^m[x, y]x^{n+1}$$

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which can be expanded as

$$F([x,y])x + g([x,y])\,d(x) = \pm x^m[x,y]x^{n+1}, \text{ for all } x,y \in R.$$

Using (2.18) , we arrive that

$$g([x,y])d(x) = 0, \text{ for all } x,y \in R. \tag{2.19}$$

Applying the same arguments after the equation (2.2) in the proof of Theorem 1, we get the required result. □

Theorem 2.12. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F([x,y]) = \pm x^m(xoy)x^n$ for all $x,y \in R, m,n \in \mathbb{N}$, then $[F(x),x] = 0$, for all $x \in R$.*

Proof. Let assume that

$$F([x,y]) = \pm x^m(xoy)x^n, \text{ for all } x,y \in R. \tag{2.20}$$

Writing yx instead of y in (2.20) and using (2.20), we obtain that

$$g([x,y])d(x) = 0, \text{ for all } x,y \in R.$$

which is same equation (2.2) in the proof of Theorem 1. Applying the same arguments after the equation (2.2) in the proof of Theorem 1, we get the required conclusion. □

Theorem 2.13. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F(xoy) = \pm x^m(xoy)x^n$ for all $x,y \in R, m,n \in \mathbb{N}$, then $[F(x),x] = 0$, for all $x \in R$.*

Proof. We assume that

$$F(xoy) = \pm x^m(xoy)x^n, \text{ for all } x,y \in R. \tag{2.21}$$

Taking y by yx in (2.21) and using (2.21), we find that

$$g(xoy)d(x) = 0, \text{ for all } x,y \in R. \tag{2.22}$$

Applying the same arguments after the equation (2.9) in the proof of Theorem 2, we get the required result. □

Theorem 2.14. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F(xoy) = \pm x^m[x,y]x^n$ for all $x,y \in R, m,n \in \mathbb{N}$, then $[F(x),x] = 0$, for all $x \in R$.*

Proof. Let assume that

$$F(xoy) = \pm x^m[x,y]x^n, \text{ for all } x,y \in R. \tag{2.23}$$

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Replacing y by yx in (2.23) and using (2.23), we arrive that

$$g(xoy)d(x) = 0, \text{ for all } x, y \in R$$

This equation is same as (2.9) in the proof of Theorem 2. Using the same arguments after the equation (2.9) in the proof of Theorem 2, we get the required result. □

Theorem 2.15. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F(xy) \pm xy = 0$, for all $x, y \in R$, then $[F(x), x] = 0$, for all $x \in R$.*

Proof. Assume that $F = 0$. By the hypothesis, $xy = 0$, for all $x, y \in R$. We can easily get $xRx = 0$, for all $x, y \in R$. By the semiprimeness of R, we arrive at $x = 0$, for all $x \in R$, a contradiction.
Now we assume $F \neq 0$. By our hypothesis, we get

$$F(xy) \pm xy = 0, \text{ for all } x, y \in R. \tag{2.24}$$

Taking yz by y in (2.24), we have

$$F(xyz) \pm xyz = 0, \text{ for all } x, y, z \in R.$$

Since F is a multiplicative generalized semiderivation of R, we get

$$F(xy)z + g(xy)\, d(z) \pm xyz = 0$$

which can be written as

$$(F(xy) \pm xy)\, z + g(xy)\, d(z) = 0, \text{ for all } x, y, z \in R.$$

Using the hypothesis, we arrive at

$$g(xy)\, d(z) = 0, \text{ for all } x, y, z \in R.$$

Since g is an epimorphism of R, we have

$$g(x)g(y)\, d(z) = 0, \text{ for all } x, y, z \in R. \tag{2.25}$$

and so

$$g(y)\, d(z)\, Rg(y)\, d(z) = (0), \text{ for all } y, z \in R.$$

By the semiprimeness of R, we arrive at

$$g(y)\, d(z) = 0, \text{ for all } y, z \in R.$$

Hence, we can write $F(xy) = F(x)\, y + g(x)\, d(y) = F(x)\, y$, for all $x, y \in R$ by the definition F. Using last expression in (2.24), we obtain that

$$(F(x) \pm x)\, y = 0, \text{ for all } x, y \in R. \tag{2.26}$$

and so

$$x\,(F(x) \pm x)\, Rx\,(F(x) \pm x) = (0), \text{ for all } x \in R.$$

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Semiprimeness of R forces to

$$x\left(F(x)\pm x\right)=0,\ \text{for all } x\in R. \tag{2.27}$$

Taking x by y in (2.26) and subtracting the relations (2.26) and (2.27), we obtain that

$$\left[\left(F(x)\pm x\right),x\right]=0,\ \text{for all } x\in R$$

and so

$$[F(x),x]=0,\ \text{for all } x\in R.$$

This completes the proof. □

Theorem 2.16. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F(xy)\pm yx=0$, for all $x,y\in R$, then $[F(x),x]=0$, for all $x\in R$.*

Proof. Let $F=0$. By the hypothesis, $yx=0$, for all $x,y\in R$. We can easily get $xRx=0$, for all $x,y\in R$. By the semiprimeness of R, we arrive at $x=0$, for all $x\in R$, a contradiction.
Now we assume that $F\neq 0$. By our hypothesis, we get

$$F\left(x\left(yz\right)\right)\pm\left(yz\right)x=0$$

and

$$F(\left(xy\right)z)\pm z\left(xy\right)=0,\ \text{for all } x,y,z\in R.$$

Subtracting these two equations, we have

$$yzx\mp zxy=0$$

and so

$$[y,zx]=0,\ \text{for all } x,y,z\in R.$$

Writing z by y in this equation and using this, we arrive at

$$z\left[x,z\right]=0,\ \text{for all } x,z\in R. \tag{2.28}$$

Replace x by yx in (2.28) and using (2.28), we get

$$zy\left[x,z\right]=0,\ \text{for all } x,y,z\in R.$$

and so

$$zyw\left[x,z\right]=0,\ \text{for all } x,y,z,w\in R. \tag{2.29}$$

Similary, (2.28) gives that

$$yzw\left[x,z\right]=0,\ \text{for all } x,y,z,w\in R. \tag{2.30}$$

Subtracting (2.29) from (2.30), we arrive at

$$\left[y,z\right]w\left[x,z\right]=0,\ \text{for all } x,y,z,w\in R.$$

and so

$$\left[x,z\right]R\left[x,z\right]=\left(0\right),\ \text{for all } x,y,z\in R.$$

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By the semiprimeness of R, we get

$$[x, z] = 0, \text{ for all } x, z \in R.$$

Replacing $F(x)z$ by z in this equation and using this, we get

$$[x, F(x)]z = 0, \text{ for all } x, z \in R.$$

and so

$$[x, F(x)]R[x, F(x)] = (0), \text{ for all } x \in R.$$

Again using semiprimeness of R, we get the required result. □

Theorem 2.17. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F(x)F(y) \pm xy = 0$, for all $x, y \in R$, then $[F(x), x] = 0$, for all $x \in R$.*

Proof. Assume that $F = 0$. By the hypothesis, $xy = 0$, for all $x, y \in R$. We can easily get $xRx = 0$, for all $x, y \in R$. By the semiprimeness of R, we arrive at $x = 0$, for all $x \in R$, a contradiction.
Now we assume $F \neq 0$. By our hypothesis, we have

$$F(x)F(y) \pm xy = 0, \text{ for all } x, y \in R. \tag{2.31}$$

Replacing y by yz in (2.31), we get

$$F(x)F(yz) \pm xyz = 0, \text{ for all } x, y, z \in R.$$

which can be expanded as

$$F(x)F(y)z + F(x)g(y)d(z) \pm xyz = 0, \text{ for all } x, y, z \in R.$$

and so

$$(F(x)F(y) \pm xy)z \pm F(x)g(y)d(z) = 0, \text{ for all } x, y, z \in R.$$

Using (2.31), above equation reduces to

$$F(x)g(y)d(z) = 0, \text{ for all } x, y, z \in R. \tag{2.32}$$

Taking xr instead of x in (2.32) and using the definition multiplicative generalized derivation, we obtain that

$$d(x)g(r)g(y)d(z) + xF(r)g(y)d(z) = 0, \text{ for all } x, y, z, r \in R.$$

Using (2.32) in last equation, we have

$$d(x)g(r)g(y)d(z) = 0, \text{ for all } x, y, z, r \in R.$$

Since g is surjective, we obtain that

$$d(x)Rg(y)d(z) = (0), \text{ for all } x, y, z \in R. \tag{2.33}$$

Left multiply (2.33) by $g(y)$, we get

$$g(y)d(x)Rg(y)d(z) = (0), \text{ for all } x, y, z \in R.$$

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In particular, we have

$$g(y)\,d(x)\,Rg(y)\,d(x)=(0),\ \text{for all } x,y\in R.$$

Since R is a semiprime ring, we obtain

$$g(y)\,d(x)=0,\ \text{for all } x,y\in R. \tag{2.34}$$

Using this in the following equation, we get

$$F(xy)=F(x)\,y+g(x)\,d(y)=F(x)\,y,\ \text{for all } x,y\in R. \tag{2.35}$$

Writing xy instead of x in (2.31) and using the last equation, we obtain

$$F(x)\,yF(y)\pm xyy=0,\ \text{for all } x,y\in R. \tag{2.36}$$

On the other hand, right multiply (2.31) by y, we get

$$F(x)F(y)\,y\pm xyy=0,\ \text{for all } x,y\in R. \tag{2.37}$$

Comparing last two expressions, one can obtain

$$F(x)\,[F(y),y]=0,\ \text{for all } x,y\in R.$$

Replacing x by xr, we get

$$F(xr)\,[F(y),y]=0,\ \text{for all } x,y,r\in R.$$

Using (2.35), we obtain that

$$F(x)\,r\,[F(y),y]=0,\ \text{for all } x,y,r\in R. \tag{2.38}$$

Taking xr by r in last equation, we have

$$F(x)\,xr\,[F(y),y]=0,\ \text{for all } x,y,r\in R. \tag{2.39}$$

Left multiply (2.38) by x, we get

$$xF(x)\,r\,[F(y),y]=0,\ \text{for all } x,y,r\in R. \tag{2.40}$$

Comparing (2.39) and (2.40), we obtain

$$[F(x),x]\,r\,[F(y),y]=0$$

and so

$$[F(x),x]\,R\,[F(x),x]=(0),\ \text{for all } x\in R.$$

Semiprimeness of R forces to $[F(x),x]=0$, for all $x\in R$. □

Theorem 2.18. *Let R be a semiprime ring admitting a multiplicative generalized semiderivation F associated with a multiplicative semiderivation d and g is an epimorphism on R. If $F(R)\subseteq Z$, then $[d(x),x]=0$, for all $x\in R$.*

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Proof. Let $F(xy) \in Z$, for $x, y \in R$. Hence, we have

$$\begin{aligned} F(xy)z &= zF(xy) \\ (F(x)y + g(x)d(y))z &= z(F(x)y + g(x)d(y)) \\ F(x)yz + g(x)d(y)z &= zF(x)y + zg(x)d(y) \end{aligned}$$

By the hypothesis, we obtain

$$F(x)[y,z] = zg(x)d(y) - g(x)d(y)z, \text{ for all } x, y, z \in R. \tag{2.41}$$

Replacing y by z in (2.41), we get

$$g(x)d(y)y = yg(x)d(y), \text{ for all } x, y \in R. \tag{2.42}$$

Writting xr for x in (2.42) and using g is an epimorphism of R, we have

$$g(x)g(r)d(y)y = yg(x)g(r)d(y), \text{ for all } x, y, r \in R.$$

By (2.42), we obtain

$$[g(x), y]g(r)d(y) = 0, \text{ for all } x, y, r \in R.$$

Since g is surjective, we have

$$[z, y]td(y) = 0, \text{ for all } y, z, t \in R. \tag{2.43}$$

Replacing z by $d(y)$ in (2.43), we get

$$[d(y), y]td(y) = 0, \text{ for all } y, t \in R. \tag{2.44}$$

Right multiplying this equation with y, we have

$$[d(y), y]td(y)y = 0, \text{ for all } y, t \in R. \tag{2.45}$$

Substituting ty for t in (2.44), we have

$$[d(y), y]tyd(y) = 0, \text{ for all } y, t \in R. \tag{2.46}$$

Subtracting (2.45) and (2.46), we arrive

$$[d(y), y]t[d(y), y] = 0, \text{ for all } y, t \in R.$$

That is

$$[d(y), y]R[d(y), y] = (0), \text{ for all } y \in R.$$

By the semiprimeness of R, we conclude that $[d(y), y] = 0$, for all $y \in R$. This completes the proof. □

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