

Finite Groups with Given σ -conditionally Permutable Subgroups

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ABSTRACT. Let $\sigma = \{\sigma_i | i \in I\}$ be a partition of the set of all primes $\mathbb{P}$ and G a finite group. A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ -set* of G if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i -subgroup of G for some $i \in I$ and $\mathcal{H}$ contains exactly one Hall σ_i -subgroup of G for every i such that $\sigma_i \cap \pi(G) \neq \emptyset$. In this paper, we study the structure of G based on the notion of σ -conditionally permutable subgroups.

Keywords: Finite groups, σ -conditionally permutable subgroups, Supersoluble groups, σ -nilpotent subgroups, Complete Hall σ -set.

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1. INTRODUCTION

Throughout this paper, all groups are finite and G always denotes a finite group. Let n be an integer. Then $\pi(n)$ denotes the set of all primes dividing n ; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G .

In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of all primes $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. We write $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$.

Following [1, 2, 3, 4], G is said to be σ -primary if $|\sigma(G)| \leq 1$; σ -soluble if every chief factor of G is σ -primary. A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ -set* of G if every non-identity member of $\mathcal{H}$ is a Hall σ_i -subgroup of G for some σ_i , and $\mathcal{H}$ contains exactly one Hall σ_i -subgroup

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for every $\sigma_i \in \sigma(G)$. G is said to be σ -full if G possesses a complete Hall σ -set; σ -nilpotent if G has a complete Hall σ -set $\mathcal{H} = \{H_1, \dots, H_t\}$ such that $G = H_1 \times \dots \times H_t$. Clearly, a σ -nilpotent group is σ -soluble. G is said to be a σ -full group of Sylow type if every subgroup of G is a D_{σ_i} -group for all $\sigma_i \in \sigma(G)$. A complete Hall σ -set $\mathcal{H} = \{H_1, \dots, H_t\}$ of subgroups of G is said to be a σ -basis of G if $H_i H_j = H_j H_i$ for all i, j . A subgroup H of G is said to be σ -subnormal in G if there exists a subgroup chain $H = H_0 \leq H_1 \leq \dots \leq H_n = G$ such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i}$ is σ -primary for all $i = 1, 2, \dots, n$.

The permutability of subgroups possess a series of interesting properties. For example, a subgroup H of G is said to be: *permute* with a subgroup K if $HK = KH$; *permutable* in G if $HK = KH$ for any subgroup K of G ; *conditionally permutable* in G if for any subgroup K of G there exists an element $x \in G$ such that $HK^x = K^x H$ (see [5]); *s-permutable* in G if $HP = PH$ for any Sylow subgroup P of G (see [6]). A set $\mathcal{S}$ of Sylow subgroups of G is said to be a *complete set of Sylow subgroups* of G if $\mathcal{S}$ contains exactly one Sylow p -subgroup of G for every prime $p \in \pi(G)$. Let $\mathfrak{S}$ be a complete set of Sylow subgroups of G . A subgroup H of G is said to be: $\mathfrak{S}$ -*permutable* if $HA = AH$ for all $A \in \mathfrak{S}$ (see [7]); *s-conditionally permutable* in G if for any Sylow subgroup P of G there exists an element $x \in G$ such that $HP^x = P^x H$ (see [8]). A subgroup H of G is said to be: σ -*permutable* in G if G possesses a complete Hall σ -set $\mathcal{H}$ such that $HA^x = A^x H$ for all $A \in \mathcal{H}$ and all $x \in G$ (see [3]); $\mathcal{H}$ -*permutable* if $HA = AH$ for all $A \in \mathcal{H}$, where $\mathcal{H}$ is a complete Hall σ -set of G (see [2]). Many people have studied the structure of finite groups based on above subgroups and a lot of research has been given; see for example [2, 3, 5, 6, 7, 8, 9]. Recently, Mao et al. introduced the following concept (see [10]):

Definition 1.1. A subgroup H of G is said to be σ -*conditionally permutable* in G if G possesses a complete Hall σ -set $\mathcal{H}$ and, for any subgroup $A \in \mathcal{H}$, there exists an element $x \in G$ such that $HA^x = A^x H$.

It is clear that every normal subgroup, every permutable subgroup, every conditionally permutable subgroup, every σ -permutable subgroup and every $\mathcal{H}$ -permutable subgroup of G are all σ -conditionally permutable in G . In the case when $\sigma = \{\{2\}, \{3\}, \dots\}$, every s -permutable subgroup, every $\mathfrak{S}$ -permutable subgroup and every s -conditionally permutable subgroup of G are also σ -conditionally permutable in G . However, the converse for s -conditionally permutable subgroup, σ -permutable subgroup and $\mathcal{H}$ -permutable subgroup is not true (see [10, Example 1.2]).

In this paper, we continue the research of σ -conditionally permutable subgroups, and obtain the results in section 3.

2. PRELIMINARIES

Lemma 2.1. [10, Lemma 2.4] *Let H and K be subgroups of a σ -full group G . If K is normal in G and $K \leq H$, then H is σ -conditionally permutable in G if and only if H/K is σ -Conditionally permutable in G/K .*

Lemma 2.2. [11, Lemma 5] *Let H , K and N be pairwise permutable subgroups of G , and suppose that H is a Hall subgroup of G . Then $N \cap HK = (N \cap H)(N \cap K)$.*

We use $\mathfrak{S}_\sigma$ to denote the class of all σ -soluble groups.

Lemma 2.3. [3, Lemma 2.1] *The class $\mathfrak{S}_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, any extension of a σ -soluble group by a σ -soluble group is a σ -soluble group as well.*

Lemma 2.4. [10, Theorem 1.3] *Let G be a σ -soluble group and $\mathcal{H} = \{H_1, \dots, H_t\}$ be a complete Hall σ -set of G such that H_i is supersoluble for $i \in \{1, \dots, t\}$. If every maximal subgroup of any non-cyclic H_i is σ -conditionally permutable in G , then G is supersoluble.*

We use $\mathfrak{N}_\sigma$ to denote the class of all σ -nilpotent groups.

Lemma 2.5. [3, Corollary 2.4 and Lemma 2.5]

- (1) *The class $\mathfrak{N}_\sigma$ is closed under taking products of normal subgroups, homomorphic images and subgroups.*
- (2) *If G/N and G/R are σ -nilpotent, then $G/N \cap R$ is σ -nilpotent.*
- (3) *If E is a normal subgroup of G and $E/E \cap \Phi(G)$ is σ -nilpotent, then E is σ -nilpotent.*

3. RESULTS

Theorem 3.1. *Let G be a σ -full group of Sylow type, $\mathcal{H}$ a complete Hall σ -set of G such that every member of $\mathcal{H}$ is supersoluble. Suppose that G is soluble and for some $\sigma_j \in \sigma(G)$, H_j is nilpotent and $(|G|, p-1) = 1$ for any prime $p \in \sigma_j$. If every maximal subgroup of H_j is σ -conditionally permutable in G , then G is σ_j -nilpotent (that is, G is p -nilpotent for any prime $p \in \sigma_j$).*

Proof. Let $\mathcal{H} = \{H_1, \dots, H_t\}$ be a complete Hall σ -set of G . We can assume without loss of generality that H_i is a supersoluble σ_i -group for all $i = 1, \dots, t$, and $j = 1$. Clearly $t > 1$. Assume the contrary and let (G, H_1) be a counterexample with $|G| + |H_1|$ minimal. Then

- (1) $O_{\sigma'_1}(G) = 1$.

Assume that this is false. Let $K = O_{\sigma'_1}(G)$. Clearly, $\sigma_1 \in \sigma(G/K)$ and H_1K/K is a nilpotent Hall σ_1 -subgroup of G/K . Since $(|G|, p-1) = 1$ for any $p \in \sigma_1$, we have that $(|G/K|, p-1) = 1$. Let U/K be any maximal subgroup

of H_1K/K , then U is a maximal subgroup of H_1K . Since $U = U \cap H_1K$ and H_1K/K is supersoluble, so

$$\begin{aligned} |H_1K/K : U/K| &= |H_1K : U| \\ &= |H_1K : (U \cap H_1)K| \\ &= |H_1||K| |(U \cap H_1) \cap K| : |U \cap H_1||K||H_1 \cap K| \\ &= |H_1 : U \cap H_1| \end{aligned}$$

is a prime, and so $U \cap H_1$ is a maximal subgroup of H_1 . Then by the hypothesis and Lemma 2.1, U/K is σ -conditionally permutable in G/K . This shows that $(G/K, H_1K/K)$ satisfies the hypothesis, and so G/K is σ_1 -nilpotent by the choice of (G, H_1) . It follows that G is σ_1 -nilpotent, a contradiction. Hence we have claim (1).

(2) Let R be a minimal normal subgroup of G , then R is an elementary abelian p -group for some prime $p \in \sigma_1$ and G/R is σ_1 -nilpotent.

Since G is soluble and by claim (1), we have that R is an elementary abelian p -group for some prime $p \in \sigma_1$, and so $R \leq H_1$. If $R = H_1$, then G/R is a σ'_1 -group and hence G/R is σ_1 -nilpotent. Now we consider that $R \neq H_1$. Then $\sigma_1 \in \sigma(G/R)$ and H_1/R is a nilpotent Hall σ_1 -subgroup of G/R . Since $(|G|, p-1) = 1$ for any $p \in \sigma_1$, we have that $(|G/R|, p-1) = 1$. Let W/R be a maximal subgroup of H_1/R . Then W is a maximal subgroup of H_1 . Hence by the hypothesis and Lemma 2.1, W/R is σ -conditionally permutable in G/R . This shows that $(G/R, H_1/R)$ satisfies the hypothesis. Therefore G/R is σ_1 -nilpotent by the choice of (G, H_1) .

(3) R is the unique minimal normal subgroup of G , $\Phi(G) = 1$ and $R = O_p(G) = F(G) = C_G(R)$.

This directly follows from G is soluble, claim (2) and [12, Chapter A, Theorem 15.2].

(4) R is non-cyclic.

Assume that R is cyclic. Then R is a cyclic group of order p by claim (2). It follows that $G/R = N_G(R)/C_G(R) \lesssim \text{Aut}(R)$ is a cyclic group of order $p-1$. But as $(|G|, p-1) = 1$, $G = R$ is a cyclic group of order p , a contradiction. Hence we have claim (4).

(5) H_1 is a p -group and so $\sigma_1 \cap \pi(G) = \{p\}$.

It directly follows from $C_G(R) = R$ for H_1 is nilpotent.

(6) Final contradiction.

Since $\Phi(G) = 1$, $R \not\leq \Phi(H_1)$ by [13, Chapter III, Lemma 3.3]. Hence there exists a maximal subgroup M of H_1 such that $H_1 = RM$. Let $E = R \cap M$. Then by claim (2), we have that $E \leq H_1$. Since H_1 is supersoluble, $|R : E| = |RM : M| = |H_1 : M|$ is a prime. Hence E is a maximal subgroup of R , and so $E \neq 1$ by claim (4). By the hypothesis, G possesses a complete Hall σ -set $\mathcal{H} = \{H_1, \dots, H_t\}$ and for any subgroup $H_i \in \mathcal{H}$, there exists an

element $x_i \in G$ such that $MH_i^{x_i} = H_i^{x_i}M$. Assume that H_i is not σ_1 -group. Then $R \cap MH_i^{x_i} = (R \cap M)(R \cap H_i^{x_i}) = (R \cap M) = E$ by Lemma 2.2. Hence $H_i^{x_i} \leq N_G(E)$. Moreover, as $E \trianglelefteq H_1$, we obtain that $E \trianglelefteq G$, and therefore $E = 1$. This follows that R is cyclic, contrary to claim (4). This final contradiction completes the proof. $\square$

Recall that $G^{\mathfrak{N}_\sigma}$ denotes the σ -nilpotent residual of G , that is, the intersection of all normal subgroups N of G with σ -nilpotent quotient G/N .

Theorem 3.2. *Let G be a σ -full group of Sylow type, $\mathcal{H}$ a complete Hall σ -set of G such that every member of $\mathcal{H}$ is supersoluble. If every maximal subgroup of any noncyclic H_i is σ -conditionally permutable in G , then the derived subgroup G' of G is σ -nilpotent.*

Proof. Let $\mathcal{H} = \{H_1, \dots, H_t\}$ be a complete Hall σ -set of G . We can assume without loss of generality that H_i is a supersoluble σ_i -group for all $i = 1, \dots, t$. Suppose that this assertion is false and let G be a counterexample of minimal order. Then G is not σ -nilpotent, so $|\sigma(G)| > 1$. Let $D = G^{\mathfrak{N}_\sigma} \neq 1$ be the σ -nilpotent residual of G , and so G is σ -soluble by Lemma 2.3.

(1) G is soluble and so G is not simple group.

Let q be the smallest prime dividing the order of G . Without loss of generality, we may assume that $q \in \pi(H_1)$. If H_1 is cyclic, then sylow q -subgroup of G is cyclic. Hence G is q -nilpotent by [13, Chapter IV, Theorem 2.8] and so G is soluble. Now assume that H_1 is noncyclic, then by Lemma 2.4 G is soluble.

(2) Let R be a minimal normal subgroup of G . Then R is a elementary abelian for some $p \in \sigma_i$ and $(G/R)'$ is σ -nilpotent.

It is clear that $\overline{\mathcal{H}} = \{H_1R/R, H_2R/R, \dots, H_tR/R\}$ is a complete Hall σ -set of G/R and $H_iR/R \cong H_i/H_i \cap R$ is supersoluble. By claim (1), R is an elementary abelian p -group for some prime p . Without loss of generality, we can assume that $R \leq H_1$. If H_1/R is non-cyclic. Then H_1 is non-cyclic. For every maximal subgroup V/R of H_1/R , we have that V is a maximal subgroup of H_1 . Then by the hypothesis and Lemma 2.1, V/R is σ -conditionally permutable in G/R . Now assume that H_iR/R is non-cyclic for $i \neq 1$, and let V_i/R be a maximal subgroup of H_iR/R . Then $V_i = (H_i \cap V_i)R$ is a maximal subgroup of H_iR . If $V_i \cap H_i$ is not a maximal subgroup of H_i , then there exists a subgroup M of H_i such that $V_i \cap H_i < M < H_i$. Since $(|H_i|, |R|) = 1$, $V_i < MR < H_iR$. This contradiction shows that $H_i \cap V_i$ is a maximal subgroup of H_i . By the hypothesis and Lemma 2.1, V/K is σ -conditionally permutable in G/R . This shows that G/R satisfies the hypothesis. Hence $(G/R)' = G'R/R = G'/G' \cap R$ is σ -nilpotent by the choice of G .

(3) R is the unique minimal normal subgroup of G , $R \not\leq \Phi(G)$ and $C_G(R) \leq R$.

By claim (2) $R \leq G'$ and $R \not\leq \Phi(G)$ by Lemma 2.5(3). Moreover, if G has a minimal normal subgroup $N \neq R$, then $N \leq G'$ and $G' \cong G'/1 = G'/R \cap N$

is σ -nilpotent by Lemma 2.5(2), contrary to the choice of G . Therefore R is the unique minimal normal subgroup of G and $C_G(R) \leq R$ by [12, Chapter A, Theorem 15.2].

(4) $|R| > p$.

If $|R| = p$ for some prime p . But then $C_G(R) = R$ and $G/R = G/C_G(R) \lesssim \text{Aut}(R)$ is a cyclic. Hence G' is nilpotent. This contradiction shows that $|R| > p$.

(5) *Final contradiction.*

Without loss of genrality, we may assume that $R \leq H_1$. Then $|R| > p$ by claim (4), and $R \not\leq \Phi(H)$ by claim (3) and [13, Chapter III, Lemma 3.3]. Therefore there exists a maximal subgroup V of H_1 such that $H_1 = RV$. It follows from R is elementary abelian that $R_1 = R \cap V \trianglelefteq H_1$. It is clear that R_1 is a maximal subgroup of R . Since R is noncyclic by claim (4), and $R \leq H_1$, H_1 is noncyclic. Then by the hypothesis, there exists an element $x_i \in G$ such that $VH_i^{x_i} = H_i^{x_i}V$ for all $i \in \{2, \dots, t\}$. Then $R \cap VH_i^{x_i} = (R \cap V)(R \cap H_i^{x_i}) = (R \cap V) = R_1$ by Lemma 2.2. This implies that $H_i^{x_i} \leq N_G(R_1)$. Hence $R_1 \trianglelefteq G$, and so $R_1 = 1$. This implies that $|R| = p$, a contradiction to claim (3). This completes the proof. $\square$

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