Iranian Journal of Mathematical Sciences and Informatics
Vol. 19, No. 2 (2024), pp 189-194
DOI: 10.61186/ijmsi.19.2.189

A Necessary Condition for Zero Divisors in Complex Group Algebra of Torsion-Free Groups

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Abstract. It is proved that if $\sum_{g\in G} a_g g$ is a non-zero zero divisor element of the complex group algebra $\mathbb{C}G$ of a torsion-free group G then $2\sum_{g\in G}|a_g|^2 < \big(\sum_{g\in G}|a_g|\big)^2$.

Keywords: Hilbert space $\ell^2(G)$, Complex group algebras, Zero divisor conjecture, Torsion-free groups.

2020 Mathematics subject classification: 46C07, 46L10, 20C07, 16S34.

1. Introduction and Results

Let G be any group and $\mathbb{C}G$ be the complex group algebra of G, i.e. the set of finitely supported complex functions on G. We may represent an element α in $\mathbb{C}G$ as a formal sum $\sum_{g\in G} a_g g$, where $a_g \in \mathbb{C}$ is the value of α in g. The multiplication in $\mathbb{C}G$ is defined by

$$\alpha\beta = \sum_{g,h\in G} a_g b_h gh = \sum_{g\in G}\left(\sum_{x\in G} a_{gx^{-1}} b_x\right) x$$

[DOI: 10.61186/ijmsi.19.2.189]

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Received 2 January 2021; Accepted 15 February 2022

for $\alpha = \sum_{g\in G} a_g g$ and $\beta = \sum_{g\in G} b_g g$ in $\mathbb{C}G$. We shall say that α is a *zero divisor* if there exists $0 \neq \beta \in \mathbb{C}G$ such that $\alpha\beta = 0$. If there is a non-zero $\beta \in \ell^2(G)$ such that $\alpha\beta = 0$, then we may say that α is *analytical zero divisor*. If $\alpha\beta \neq 0$ for all $0 \neq \beta \in \mathbb{C}G$, then we say that α is *regular*. The following conjecture is called the *zero divisor conjecture*.

Conjecture 1.1. Let G be a torsion-free group. Then all elements in $\mathbb{C}G$ are regular.

Conjecture 1.1 is still open; it has been proven affirmative when G belongs to special classes of groups; ordered groups ([11] and [12]), supersolvable groups [6], polycyclic-by-finite groups ([1] and [5]) and uniqe product groups [2]. Delzant [3] deals with group rings of word-hyperbolic groups and proves the conjecture for certain word-hyperbolic groups. Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed unions and extensions with elementary amenable quotients. Let G be a torsion-free group in $\mathcal{C}$ then G satisfies Conjecture 1.1 [8].

The map $\langle\cdot,\cdot\rangle : \mathbb{C}G \times \mathbb{C}G \to \mathbb{R}$ defined by

$$\langle\alpha,\beta\rangle := \sum_{g\in G} a_g \bar{b}_g \quad (\alpha,\beta \in \mathbb{C}G)$$

is an inner product on $\mathbb{C}G$, so $\|\alpha\|_2 = \langle\alpha,\alpha\rangle^{\frac{1}{2}}$ becomes a norm, called 2-norm; the completion of $\mathbb{C}G$ w.r.t. 2-norm is the Hilbert space $\ell^2(G)$. Indeed, we have

$$\ell^2(G) = \left\{ \alpha : G \to \mathbb{C} : \sum_{g\in G} \|\alpha(g)\|^2 < \infty \right\}.$$

In [9], Linnell formulated an analytic version of the zero divisor conjecture.

Conjecture 1.2. Let G be a torsion-free group. If $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq \beta \in \ell^2(G)$, then $\alpha\beta \neq 0$.

In [7], it is shown that Since $\mathbb{C}G \subset \ell^2(G)$, the second conjecture implies the first one. In [4], it is proved that for finitely generated amenable groups, the two conjectures are actually equivalent. We prove this is true for all amenable torsion-free groups.

The so-called 1-norm is defined on $\mathbb{C}G$ by

$$\|\alpha\|_1 = \sum_{g\in G} |a_g|, \quad \text{for } \alpha = \sum_{g\in G} a_g g \text{ in } \mathbb{C}G.$$

The *adjoint* of an element $\alpha = \sum_{g\in G} a_g g$ in $\mathbb{C}G$, denoted by α^*, is $\alpha^* = \sum_{g\in G} \bar{a}_g g^{-1}$. We call an element $\alpha \in \mathbb{C}G$ *self-adjoint* if $\alpha^* = \alpha$, and use $(\mathbb{C}G)_{\mathbf{s}}$ to denote the set of self-adjoint elements of $\mathbb{C}G$. It is worthy of mention that

[DOI: 10.61186/ijmsi.19.2.189]

if $\alpha = \sum_{g\in G} a_g g$ is self-adjoint then a_1 should be a real number. For $\alpha \in \mathbb{C}G$, β and γ in $\ell^2(G)$, the following equalities hold:

$$\langle \alpha\beta, \gamma\rangle = \langle \beta, \alpha^*\gamma\rangle .$$

The goal of this paper is to give a criterion for an element in a complex group algebra to be regular:

Theorem 1.1. *Let G be a torsion free group. Then $\alpha \in \mathbb{C}G$ is regular if $2\|\alpha\|_2^2 \geq \|\alpha\|_1^2$.*

2. Preliminaries

In this section we provide some preliminaries needed in the following.

Let G be a group. The *support* of an element $\alpha = \sum_{g\in G} a_g g$ in $\mathbb{C}G$, supp (α), is the finite subset $\{g \in G : a_g \neq 0\}$ of G.

Let H be a subgroup of G, and T be a right transversal for H in G. Then every element $\alpha \in \mathbb{C}G$ (resp. $\alpha \in \ell^2(G)$) can be written uniquely as a finite sum of the form $\sum_{t\in T} \alpha_t t$ with $\alpha_t \in \mathbb{C}H$ (resp. $\alpha_t \in \ell^2(H)$).

For $S \subset G$, we denote by $\langle S\rangle$, the subgroup of G generated by S. We have the following key lemma:

Lemma 2.1. *Let G be a group, $\alpha \in \mathbb{C}G$ and $H = \langle supp\,(\alpha)\rangle$. Then α is regular in $\mathbb{C}G$ iff α is regular in $\mathbb{C}H$.*

Proof. Suppose that α is a zero divisor. Among elements $0 \neq \gamma$ in $\mathbb{C}G$ which satisfy $\alpha\gamma = 0$ consider an element β such that $1 \in$ supp (β) and $|$supp $(\beta)|$ is minimal, then one can easily show that $\beta \in \mathbb{C}H$, and this proves the result of the lemma. □

An immediate consequence of this lemma is:

Corollary 2.2. *A group G satisfies Conjecture 1.1 iff all its finitely generated subgroups satisfy the Conjecture 1.1.*

By Lemma 2.1 in hand, we can generalize the main theorem of [4]:

Theorem 2.3. *Let G be an amenable group. If $0 \neq \alpha \in \mathbb{C}G$, $0 \neq \beta \in \ell^2(G)$ and $\alpha\beta = 0$, then there exists $0 \neq \gamma \in \mathbb{C}G$ such that $\alpha\gamma = 0$.*

The above theorem along with results in [10] provides another proof for [7, Theorem 2].

For a normal subgroup N of a group G, we denote the natural quotient map by $q_N : G \to G/N$. We continue to show that:

Lemma 2.4. *Let N be a normal subgroup of a group G satisfying Conjecture 1.1. Consider a non-torsion element $q_N(t)$, $t \in G$, in the quotient group. Then $\alpha + \beta t$ is regular, for all $\alpha, \beta \in \mathbb{C}N \setminus \{0\}$.*

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Proof. Suppose that $\alpha + \beta t$ is a zero divisor for non zero elements $\alpha, \beta \in \mathbb{C}N$. Applying Lemma 2.1 and multiplying by a suitable power of t, we can assume that there are non zero elements γ_k, $k = 0, 1, \ldots, n$, such that

$$(\alpha + \beta t) \sum_{k=0}^{n} \gamma_k t^k = 0.$$

In particular, $0 = \beta t \gamma t^n = (\beta t \gamma_n t^{-1}) t^{n+1}$, whence $\beta t \gamma_n t^{-1} = 0$, a contradiction, because $t \gamma_n t^{-1}$ is a non zero element of $\mathbb{C}N$. □

Proposition 2.5. *Let N be an amenable normal subgroup of a group G satisfying Conjecture 1.1. Consider a non-torsion element $q_N(t)$, $t \in G$, in the quotient group. Then there is no $0 \neq \gamma \in \ell^2(G)$ such that $(\alpha + \beta t)\gamma = 0$. In particular, $a + bg$ is an analytical zero divisor, for all non-torsion element $g \in G$ and non zero complex numbers a, b.*

Proof. The group $\langle N, t \rangle$ is amenable. Hence Lemma 2.4 together with Theorem 2.3 yields the result. □

3. A cone of regular elements

The result of the Proposition 2.5 is true if we replace $\mathbb{C}$ by an arbitrary field $\mathbb{F}$. The field of complex numbers allows us to define inner product on the group algebra; with the help of inner product, we can construct new regular elements from the ones we have:

Proposition 3.1. *Let G be a group and $\mathcal{F}$ be a finite non-empty subset of $\mathbb{C}G$. If $\sum_{\alpha \in \mathcal{F}} \alpha^* \alpha$ is an analytical zero divisor then all elements of $\mathcal{F}$ are analytical zero divisors. In particular, $\alpha \in \mathbb{C}G$ is an analytical zero divisor if and only if $\alpha^* \alpha$ is an analytical zero divisor.*

Proof. Let $\tilde{\alpha} := \sum_{\alpha \in \mathcal{F}} \alpha^* \alpha$ and $\tilde{\alpha}\beta = 0$ for some $\beta \in \ell^2(G)$. Then

$$0 = \langle \tilde{\alpha}\beta, \beta \rangle = \sum_{\alpha \in \mathcal{F}} \langle \alpha^* \alpha \beta, \beta \rangle = \sum_{\alpha \in \mathcal{F}} \langle \alpha\beta, \alpha\beta \rangle = \sum_{\alpha \in \mathcal{F}} \|\alpha\beta\|_2^2,$$

whence $\alpha\beta = 0$ for all $\alpha \in \mathcal{F}$. This completes the proof. □

A *cone* in a vector space $\mathfrak{X}$ is a subset $\mathfrak{K}$ of $\mathfrak{X}$ such that $\mathfrak{K} + \mathfrak{K} \subset \mathfrak{K}$ and $\mathbb{R}_+ \mathfrak{K} \subset \mathfrak{K}$. We proceed by introducing a cone of regular elements in $\mathbb{C}G$. First a definition:

Definition 3.2. Let G be a group and $(\mathbb{C}G)_{\mathrm{s}}$ be the set of self adjoint elements $\alpha \in \mathbb{C}G$, we define a function $\Upsilon : (\mathbb{C}G)_{\mathrm{s}} \to \mathbb{R}$ by

$$\Upsilon(\alpha) := a_1 - \sum_{g \neq 1} |a_g|.$$

We call an element $\alpha \in (\mathbb{C}G)_{\mathrm{s}}$ **golden** if $\Upsilon(\alpha) \geq 0$. The set of all golden elements in $(\mathbb{C}G)_{\mathrm{s}}$ is denoted by $(\mathbb{C}G)_{\mathrm{gold}}$.

[DOI: 10.61186/ijmsi.19.2.189]

What is important about golden elements is:

Proposition 3.3. *For a torsion free group G, $(\mathbb{C}G)_{gold}$ is a cone of regular elements.*

Proof. It is obvious that if α is golden then so is $r\alpha$ for any $r > 0$. The triangle inequality for $\mathbb{C}$ shows that if α and γ are golden then so is $\alpha+\gamma$. For $\alpha \in (\mathbb{C}G)_{\mathrm{s}}$, we have

$$\begin{aligned}\alpha &= \frac{1}{2}(\alpha+\alpha^*)\\ &= a_1 + \frac{1}{2}\sum_{g\neq 1}\left(\bar{a}_g g^{-1} + a_g g\right)\\ &= \Upsilon(\alpha) + \frac{1}{2}\sum_{g\neq 1}\left(2|a_g| + \bar{a}_g g^{-1} + a_g g\right)\\ &= \Upsilon(\alpha) + \frac{1}{2}\sum_{g\neq 1}|a_g|\left(\frac{\bar{a}_g}{|a_g|}+g\right)^*\left(\frac{a_g}{|a_g|}+g\right)\end{aligned}$$

Hence, by Lemma 2.5 and Proposition 3.1, α is regular. □

Now, we are ready to prove our main result:

Proof of Theorem 1.1. For $\alpha = \sum_{g\in G} a_g g$ in $\mathbb{C}G$, $\alpha^*\alpha$ is self-adjoint, and one can easily show that

$$\Upsilon(\alpha^*\alpha) \geq 2\|\alpha\|_2^2 - \|\alpha\|_1^2.$$

Hence, by Proposition 3.3, the result of the Theorem is proved. □

Acknowledgments

The author wish to thank the referee for the helpful suggestions and comments.

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[DOI: 10.61186/ijmsi.19.2.189]