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Numerical Approach of Cattaneo Equation with Time Caputo-Fabrizio Fractional Derivative

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Abstract. In the paper, we consider a type of Cattaneo equation with time fractional derivative without singular kernel based on fourth-order compact finite difference (CFD) in the space directions. In case of two dimensional, two alternating direction implicit (ADI) methods are proposed to split the equation into two separate one dimensional equations. The time fractional derivation is described in the Caputo-Fabrizio's sense with scheme of order $O(\tau^2)$. The solvability, unconditional stability and H^1 norm convergence of the scheme are proved. Numerical results confirm the theoretical results and the effectiveness of the proposed scheme.

Keywords: Caputo-Fabrizio fractional derivative, Compact finite difference, Cattaneo equation, Alternating direction implicit method.

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1. Introduction

In recent years, fractional calculus has played an important role in many fields of physics, chemistry, mechanics, electricity, signal and processing, etc [14, 15, 21, 28, 29, 30, 31].

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In this paper, we consider the following time fractional Cattaneo equation:

$$\frac{\partial u(X,t)}{\partial t} +_0^{\mathrm{CF}} \mathcal{D}_t^{\alpha} u(X,t) = \Delta u(X,t) + f(X,t), X \in \Omega,\ 0 \le t \le T. \tag{1.1}$$

${}_0^{\mathrm{C\text{-}F}}\mathcal{D}_t^{\alpha}$ is the α-th Caputo-Fabrizio fractional derivative defined by

$${}_0^{\mathrm{CF}}\mathcal{D}_t^{\alpha} u(X,t) = \frac{M(\alpha)}{2-\alpha} \int_0^t u''(X,s) \exp\left[(1-\alpha\frac{t-s}{2-\alpha}\right]\mathrm{ds}, \quad 1 < \alpha < 2, \tag{1.2}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$.
In 2015, Caputo and Fabrizio suggested a new definition of fractional derivative based on the exponential kernel [7]. They considered two different representations for the temporal and spatial variables. It is important and interesting that this approach describes the behavior of classical viscoelastic materials, thermal media, electromagnetic systems, etc. Another interesting property of this definition is that it opens up new avenues in the mechanical phenomena, related with plasticity, fatigue, damage and with electromagnetic hysteresis [7]. In connection with the application of this new fractional derivative, interested reader may find details in [2, 3, 5].
Recently, studies of Caputo-Fabrizio fractional derivative have been carried out by some authors. Authors of [6] investigated the existence of solution for two high-order fractional integro-differential equations including the Caputo-Fabrizio derivative. Atangana and Alqahtani [4] considered a numerical approximation of the space and time Caputo-Fabrizio fractional derivative in connection with ground water pollution equation. In [16] authors presented a Crank-Nicolson finite difference scheme to solve fractional Cattaneo equation by a new fractional derivative. Furthermore, they analysed the stability and convergence order of the scheme. The main aim of [8] is to prove existence and uniqueness of the flow of water within a confined aquifer with Caputo-Fabrizio fractional diffusion for the temporal and spatial variables. A type of Fokker-Planck equation with Caputo-Fabizio fractional derivative based on the Ritz method with known basis functions is considered in [11]. In 2017, Mirza and Vieru proposed the fundamental solutions to advection-diffusion equation with time-fractional without singular kernel. They applied the Laplace transform and Fourier transform with respect to the temporal variable and space coordinates, respectively. Authors of [35] proposed a new fractional derivative without singular kernel. They investigated the potential application for modeling the steady heat-conduction problem and obtained the analytical solution of the fractional-order heat flow by means of the Laplace transform. In [17] authors constructed the shifted Legendre polynomials operational matrix in order to solve problems with left-sided Caputo-Fabrizio operator. A second-order scheme for the space fractional diffusion equation with Caputo-Fabrizio is provided in [27]. The main aim of [13] is to solve two problems in nonlocal

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quantum machanics wherein the nonlocal shrödinger equation has been transformed to an ordinary linear differential equation. Other interesting papers in the field of the Caputo-Fabrizio derivative are found in [1, 9, 18, 19, 32, 34].

The time fractional Cattaneo equation has been considered by some reserchers [10, 12, 22, 23, 24]. For instance, Ren and Gan [24] considered the new numerical methods for the solution of two-dimensional Cattaneo equation with time fractional derivative. Authors of [10] studied a family of generalized fractional Cattaneo's equation by using fractional substitutions in integer-order rational transfer functions. The solution of the space-time fractional Cattaneo diffusion model is considered in [23]. In such way the solutions of the Cattaneo equation are obtained under integral and series form in terms of the H-functions. In [12] authors developed two finite difference schemes based on the explicit predictor-corrector and totally implicit schemes for Cattaneo equation. The main of [22] is to present a Cattaneo type time fractional heat conduction equation for laser heating. They obtained the analytical solution for the temperature distribution by the Laplace transformation method.

Previous studies for fractional Cattaneo equation have been limited to the singular kernel. Therefore, it is interesting to discuss numerical schemes for the fractional derivatives with the non-singular kernel. It is clear from the truncation error estimate of the L1 method for singular kernel that the accuracy is dependent on the fractional order α and when the order of the Caputo fractional derivative $\alpha \approx 2$ $(1 < \alpha < 2)$, its accuracy decreases to the first order. Hence, It is important to improve numerical accuracy of the L1 approximation of fractional derivative. What distinguishes this paper from previous study is its two-dimensional problem type which is arised from one-dimensional case of [16].

In this work, we turn our attention to the time fractional derivative with non-singular kernel. First, we apply a second-order scheme for approximating the time fractional derivative by Caputo-Fabrizio derivative. Then, we use a fourth-order CFD scheme to solve one-dimensional fractional Cattaneo equation. For two-dimensional case, we design two ADI schemes in time stepping which resulting two-dimensional system is reduced to series of one-dimensional equations. In this paper, we introduce the $\|.\|_{\tilde{H}^1}$ norm, which is proved to be equivalent to the standard H^1 norm. Furthermore, we prove that present scheme is unconditionally stable and convergent in $\tilde{H}^1$ norm with the order $O(\tau^2 + h^4)$.

The reminder of the paper is organized as follows. In section 2, the Caputo fractional derivative is described. In section 3, we develop the CFD for the one and two-dimensional fractional Cattaneo equation. Section 4 is devoted to theoretical analysis of this scheme. In section 5, the stability and convergence analysis of the compact ADI scheme is discussed. In section 6, the numerical

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examples are carried out which confirm the accuracy of the scheme. Concluding remarks are given in section 7.

2. Proposed scheme

2.1. One-dimensional fractional Cattaneo equation. In this section, we study the Caputo-Fabrizio type Cattaneo equation where the fourth-order compact difference method is used to discretize the spatial derivative.

Consider the following time fractional Cattaneo equation given by:

$$\frac{\partial u(x,t)}{\partial t} + {}_0^{\mathtt{CF}}\mathcal{D}_t^{\alpha} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad (x,t) \in \Omega = [0,L]\times[0,T], \quad 1 < \alpha < 2, \tag{2.1}$$

with the initial conditions

$$u(x,0) = \phi(x), \quad \frac{\partial u(x,0)}{\partial t} = \psi(x), \quad 0 \le x \le L, \tag{2.2}$$

and the boundary conditions

$$u(0,t) = \varphi_1(t), \quad u(L,t) = \varphi_2(t), \quad 0 < t \le T, \tag{2.3}$$

here, $f(x,t)$ is the source function with sufficient smoothness, $\psi(x), \phi(x), \varphi_1(t)$ and $\varphi_2(t)$ are given continuous functions.

Consider Eq. (2.1). For spatial and temporal approximations, let $h = \frac{L}{M}$ and $\tau = \frac{T}{N}$ be the spatial and the temporal step sizes respectively, where M and N are some given positive integers, so we define

$$x_i = ih, \qquad i = 0,1,\dots,M,$$
$$t_n = n\tau, \qquad n = 0,1,\dots,N.$$

We introduce the following notations:

$$v_i^{n+\frac{1}{2}} = \frac{1}{2}\big(v_i^{n+1} + v_i^n\big), \qquad \delta_t v_i^{n+\frac{1}{2}} = \frac{1}{\tau}\big(v_i^{n+1} - v_i^n\big),$$
$$\delta_x v_{i-\frac{1}{2}}^n = \frac{1}{h}(v_i^n - v_{i-1}^n), \qquad \delta_x^2 v_i^n = \frac{1}{h}(\delta_x v_{i+\frac{1}{2}}^n - \delta_x v_{i-\frac{1}{2}}^n),$$

and

$$\mathcal{H} v_i = \begin{cases} \frac{1}{12}(v_{i+1} + 10 v_i + v_{i-1}), & 1 \le i \le M-1, \\ v_i, & i = 0 \text{ or } M. \end{cases}$$

It is obvious that

$$\mathcal{H} v_i = \left(I + \frac{h^2}{12}\delta_x^2\right) v_i, \quad 1 \le i \le M-1.$$

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Lemma 2.1. *(See [33]). Let function* $g(x) \in \mathcal{C}^6[x_{i-1}, x_{i+1}]$*, and* $\xi(s) = 5(1-s)^3 - 3(1-s)^5$*, then*

$$\frac{g''(x_{i+1}) + 10g''(x_i) + g''(x_{i-1})}{12} = \frac{g(x_{i+1}) - 2g(x_i) + g(x_{i-1})}{h^2} + \frac{h^4}{360}\int_0^1 \left[g^6(x_i - sh) + g^6(x_i + sh)\right]\xi(s)\mathrm{ds}.$$

Define the grid functions

$$U_i^n = u(x_i, t_n), \quad f_i^n = f(x_i, t_n), \quad 0 \le i \le M,\ 1 \le n \le N.$$

According to [16], we define

$$\begin{aligned}
{}_0^{\mathtt{CF}}\mathcal{D}_t^{\alpha} u(x_i, t_{n+\frac{1}{2}}) &= \frac{1}{\alpha - 1}\sum_{k=1}^{n}\left(\frac{u_i^{k+1} - 2u_i^k + u_i^{k-1}}{\tau^2}\right)M_{n-k} \\
&+ \frac{1}{\alpha - 1}\left(\frac{u_i^1 - u_i^0}{\tau^2} - \frac{\psi}{\tau}\right)M_n + R_i^{n+\frac{1}{2}} \\
&= \frac{1}{(\alpha-1)\tau}\Bigg[M_0\delta_t u_i^{n+\frac{1}{2}} - \sum_{k=1}^{n}\left(M_{n-k} - M_{n-k+1}\right)\delta_t u_i^{k-\frac{1}{2}} \\
&- M_n\psi_i\Bigg] + R_i^{n+\frac{1}{2}}, \qquad (2.4)
\end{aligned}$$

where

$$M_k = \exp\left(\frac{1-\alpha}{2-\alpha}\tau k\right) - \exp\left(\frac{1-\alpha}{2-\alpha}\tau(k+1)\right), \qquad (2.5)$$

and

$$\begin{aligned}
R_i^{n+\frac{1}{2}} &= \frac{1}{2-\alpha}\sum_{k=0}^{n}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\left(r^k + (s - t_k)u_t^{(3)}(x_i, c_k)\right) \\
&\exp\left((1-\alpha)\frac{t_{n+\frac{1}{2}} - s}{2-\alpha}\right)\mathrm{ds} \\
&- \frac{1}{2-\alpha}\int_{-t_{\frac{1}{2}}}^{0} O(\tau)\exp\left((1-\alpha)\frac{t_{\frac{1}{2}} - s}{2-\alpha}\right)\mathrm{ds}, \qquad (2.6)
\end{aligned}$$

where

$$r^k = \frac{1}{48}\tau^2\left[u_t^{(4)}(x_i, \eta_1) + u_t^{(4)}(x_i, \eta_2)\right] + O(\tau^2) = O(\tau^2),\ \eta_1 \in (t_k, t_{k+\frac{1}{2}}), \ \eta_2 \in (t_{k-\frac{1}{2}}, t_k). \qquad (2.7)$$

Lemma 2.2. *([16]). For the definition* M_k*, we have* $M_k > 0$ *and* $M_{k+1} < M_k$*,* $\forall k \le n$*.*

Lemma 2.3. *Let* $1 < \alpha < 2$ *and* $M_k = \exp\left(\frac{1-\alpha}{2-\alpha}\tau k\right) - \exp\left(\frac{1-\alpha}{2-\alpha}\tau(k+1)\right)$*, then*

$$\frac{\alpha-1}{2-\alpha}\tau\exp\left(\frac{1-\alpha}{2-\alpha}t_{k+1}\right) < M_k < \frac{\alpha-1}{2-\alpha}\tau\exp\left(\frac{1-\alpha}{2-\alpha}t_k\right).$$

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Proof. Noticing that $M_k = \frac{\alpha-1}{2-\alpha}\int_{t_k}^{t_{k+1}} \exp\big(\frac{1-\alpha}{2-\alpha}x\big)\mathrm{dx}$, and $\exp\big(\frac{1-\alpha}{2-\alpha}t_{k+1}\big) < \exp\big(\frac{1-\alpha}{2-\alpha}x\big) < \exp\big(\frac{1-\alpha}{2-\alpha}t_k\big)$, by properties of the integral, it is easy to verify that

$$\int_{t_k}^{t_{k+1}} \exp\big(\frac{1-\alpha}{2-\alpha}t_{k+1}\big)\mathrm{dx} \leq \int_{t_k}^{t_{k+1}} \exp\big(\frac{1-\alpha}{2-\alpha}x\big)\mathrm{dx} \leq \int_{t_k}^{t_{k+1}} \exp\big(\frac{1-\alpha}{2-\alpha}t_k\big)\mathrm{dx},$$

$$\begin{aligned}\tau \exp\big(\frac{1-\alpha}{2-\alpha}t_{k+1}\big) &\leq \frac{2-\alpha}{1-\alpha}\big(\exp\big(\frac{1-\alpha}{2-\alpha}t_{k+1}\big) - \exp\big(\frac{1-\alpha}{2-\alpha}t_k\big)\big)\\ &\leq \tau \exp\big(\frac{1-\alpha}{2-\alpha}t_k\big),\end{aligned}$$

multiplying the above equation by $\frac{\alpha-1}{2-\alpha}$, we complete the proof. □

Lemma 2.4. *([16]). Suppose $u(x,t) \in C_{x,t}^{4,6}([0,L]\times[0,T])$, then we have*

$$\big|R_i^{n+\frac{1}{2}}\big| \leq \frac{1}{\alpha-1}\max_{0\leq k\leq n}\big|u_t^{(3)}(x_i,c_k)\big|\exp\big(\frac{2\alpha-2}{2-\alpha}\big)\tau^2 + O(\tau^2). \tag{2.8}$$

Let $f_i^{n+\frac{1}{2}} = f(x_i,t_{n+\frac{1}{2}})$ and $t_{n+\frac{1}{2}} = \frac{t_{n+1}+t_n}{2}$. Then according to the new Caputo-Fabrizio formula (2.4), Eq. (2.1) can be written as follows:

$$\begin{aligned}&\delta_t U_i^{n+\frac{1}{2}} + \frac{1}{(\alpha-1)\tau}\Big[M_0\delta_t U_i^{n+\frac{1}{2}} - \sum_{k=1}^{n}\big(M_{n-k}-M_{n-k+1}\big)\delta_t U_i^k - M_n\psi_i\Big]\\ &\quad = \frac{\partial^2 u}{\partial x^2}(x_i,t_{n+\frac{1}{2}}) + f_i^{n+\frac{1}{2}} + (\mathrm{R}_t)_i^{n+\frac{1}{2}},\\ &\quad 1\leq i\leq M-1, \quad 0\leq n\leq N-1.\end{aligned}$$

Operating $\mathcal{H}$ on the above equality, we have:

$$\begin{aligned}&\mathcal{H}\delta_t U_i^{n+\frac{1}{2}} + \frac{1}{(\alpha-1)\tau}\mathcal{H}\Big[M_0\delta_t U_i^{n+\frac{1}{2}} - \sum_{k=1}^{n}\big(M_{n-k}-M_{n-k+1}\big)\delta_t U_i^{k-\frac{1}{2}} - M_n\psi_i\Big]\\ &\quad = \mathcal{H}\frac{\partial^2 u}{\partial x^2}(x_i,t_{n+\frac{1}{2}}) + \mathcal{H}f_i^{n+\frac{1}{2}} + \mathcal{H}(\mathrm{R}_t)_i^{n+\frac{1}{2}},\\ &\quad 1\leq i\leq M-1, \quad 0\leq n\leq N-1.\end{aligned} \tag{2.9}$$

Lemma (2.1) implies that

$$\mathcal{H}\frac{\partial^2 u}{\partial x^2}(x_i,t_{n+\frac{1}{2}}) = \delta_x^2 U_i^{n+\frac{1}{2}} + (\mathrm{R}_x)_i^{n+\frac{1}{2}}, \quad 1\leq i\leq M-1,\ 0\leq n\leq N-1, \tag{2.10}$$

where

$$(\mathrm{R}_x)_i^{n+\frac{1}{2}} = \frac{h^4}{360}\int_0^1\Big[\frac{\partial^6 u}{\partial x^6}(x_i-sh) + \frac{\partial^6 u}{\partial x^6}(x_i+sh)\Big]\xi(s)\mathrm{ds}. \tag{2.11}$$

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Substituting (2.10) into (2.9), we obtain

$$(\alpha-1)\tau\mathcal{H}\delta_t U_i^{n+\frac{1}{2}} + \mathcal{H}\Big[M_0\delta_t U_i^{n+\frac{1}{2}} - \sum_{k=1}^{n}\big(M_{n-k}-M_{n-k+1}\big)\delta_t U_i^{k-\frac{1}{2}} - M_n\psi_i\Big]$$
$$= (\alpha-1)\tau\delta_x^2 U_i^{n+\frac{1}{2}} + (\alpha-1)\tau\mathcal{H}f_i^{n+\frac{1}{2}} + \mathcal{H}(\mathrm{R}_t)_i^{n+\frac{1}{2}},$$
$$1\le i\le M-1,\quad 0\le n\le N-1, \tag{2.12}$$

where

$$\mathrm{R}_i^{n+\frac{1}{2}} = \mathcal{H}(\mathrm{R}_t)_i^{n+\frac{1}{2}} + (\mathrm{R}_x)_i^{n+\frac{1}{2}}. \tag{2.13}$$

Noticing that

$$\big|(R_t)_i^{n+\frac{1}{2}}\big| \le \frac{1}{\alpha-1}\max_{0\le k\le n}\big|u_t^{(3)}(x_i,c_k)\big|\exp\big(\frac{2\alpha-2}{2-\alpha}\big)\tau^2 + O(\tau^2).$$

Then it holds that

$$|\mathrm{R}_i^n| \le C_R(\tau^2+h^4),\quad 1\le i\le M-1,\ 0\le n\le N-1, \tag{2.14}$$

where C_R is a positive constant, that is independent of the time step size τ and grid spacing h. In addition, it follows from the initial and boundary value conditions that

$$U_0^n = \varphi_1(t_n),\ U_M^n = \varphi_2(t_n),\quad 0\le n\le N-1, \tag{2.15}$$

$$U_i^0 = \phi(x_i),\quad 0\le i\le M. \tag{2.16}$$

Omitting the small term $\mathrm{R}_i^{n+\frac{1}{2}}$ and replacing the function U_i^n with its numerical approximation u_i^n, we obtain the following difference scheme

$$(\alpha-1)\tau\mathcal{H}\delta_t u_i^{n+\frac{1}{2}} + \mathcal{H}\Big[M_0\delta_t u_i^{n+\frac{1}{2}} - \sum_{k=1}^{n}\big(M_{n-k}-M_{n-k+1}\big)\delta_t u_i^{k-\frac{1}{2}} - M_n\psi_i\Big]$$
$$= (\alpha-1)\tau\delta_x^2 u_i^{n+\frac{1}{2}} + (\alpha-1)\tau\mathcal{H}f_i^{n+\frac{1}{2}},$$
$$1\le i\le M-1,\quad 0\le n\le N-1, \tag{2.17}$$

$$u_0^n = \varphi_1(t_n),\ u_M^n = \varphi_2(t_n),\quad 1\le n\le N, \tag{2.18}$$

$$u_i^0 = \phi(x_i),\quad 0\le i\le M. \tag{2.19}$$

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3. Matrix form of the numerical scheme

If we write Eq. (2.17) at each mesh point and $u^n = [u_1^n, u_2^n, \dots, u_{M-1}^n]^T$, we can give the matrix form of the proposed method by

$$Au^{n+1} = Bu^n - \sum_{k=1}^{n} C_k^n (u^k - u^{k-1}) + F^{n+1}, \quad n = 0, \dots, N-1, \tag{3.1}$$

where the tridiagonal matrices in (3.1) are given by

$$A = \text{tri}\Big[a_1, a_2, a_3\Big]_{M-1\times M-1},$$
$$B = \text{tri}\Big[b_1, b_2, b_3\Big]_{M-1\times M-1},$$

where

$$a_1 = \frac{1}{12}\big[(\alpha-1)+M_0\big] - \frac{(\alpha-1)\tau^2}{h^2},$$
$$a_2 = \frac{10}{12}\big[(\alpha-1)+M_0\big] + 2\frac{(\alpha-1)\tau^2}{h^2},$$
$$a_3 = \frac{1}{12}\big[(\alpha-1)+M_0\big] - \frac{(\alpha-1)\tau^2}{h^2},$$
$$b_1 = \frac{1}{12}\big[(\alpha-1)+M_0\big] + \frac{(\alpha-1)\tau^2}{h^2},$$
$$b_2 = \frac{10}{12}\big[(\alpha-1)+M_0\big] - 2\frac{(\alpha-1)\tau^2}{h^2},$$
$$b_3 = \frac{1}{12}\big[(\alpha-1)+M_0\big] + \frac{(\alpha-1)\tau^2}{h^2},$$

and finally, the column vector in $R^{(M-1)}$ is given by

$$F^n = \begin{pmatrix} F_1^n \\ F_2^n \\ \vdots \\ \\ F_{M-2}^n \\ F_{M-1}^n \end{pmatrix},$$

where

$$\begin{aligned} F_1^n = &-\left(\frac{1}{12}\big[(\alpha-1)+M_0\big] - \frac{(\alpha-1)\tau^2}{h^2}\right)u_0^{n+1} \\ &-\left(\frac{1}{12}\big[(\alpha-1)+M_0\big] - \frac{(\alpha-1)\tau^2}{h^2}\right)u_0^{n} + \tau M_n \psi_1, \\ &+(\alpha-1)\tau^2 \mathcal{H} f_1^{n+\frac{1}{2}}, \end{aligned}$$

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$$F_2^n = \tau M_n \psi_2 + (\alpha-1)\tau^2 \mathcal{H} f_2^{n+\frac{1}{2}},$$

$$F_{M-2}^n = \tau M_n \psi_{M-2} + (\alpha-1)\tau^2 \mathcal{H} f_{M-2}^{n+\frac{1}{2}},$$

$$\begin{aligned} F_{M-1}^n = &-\left(\frac{1}{12}\big[(\alpha-1)+M_0\big] - \frac{(\alpha-1)\tau^2}{h^2}\right)u_M^{n+1} \\ &-\left(\frac{1}{12}\big[(\alpha-1)+M_0\big] - \frac{(\alpha-1)\tau^2}{h^2}\right)u_M^{n} \\ &+\tau M_n \psi_{M-1} + (\alpha-1)\tau^2 \mathcal{H} f_{M-1}^{n+\frac{1}{2}}, \end{aligned}$$

$$C_k^n = \big(M_{n-k} - M_{n-k+1}\big)\begin{pmatrix} \frac{10}{12} & \frac{1}{12} & & & \\ \frac{1}{12} & \frac{10}{12} & \frac{1}{12} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{12} & \frac{10}{12} & \frac{1}{12} \\ & & & \frac{1}{12} & \frac{10}{12} \end{pmatrix},$$

and M_k is defined in (2.5). Also $\text{tri}[a_1, a_2, a_3]_{(M-1)\times(M-1)}$ denotes a $(M-1)\times(M-1)$ tri-diagonal matrix. Each row of this matrix contains the values a_1, a_2 and a_3 on its sub-diagonal, diagonal, and super-diagonal, respectively.

4. Analysis of the proposed scheme

4.1. **Solvability.** For the solvability of the scheme we have:

Theorem 4.1. *The difference system (2.17) has a unique solution.*

Proof. For any possible values of τ, α and h, the coefficient matrix A is strictly diagonal dominant. Consequently, it is non-singular, thus it is invertible. Hence, in this point of view, we can conclude that the difference scheme (2.17)-(2.19) has a unique solution. □

4.2. **Stability.** Here, we first introduce the following space of grid functions which are basic in the whole theory.

$$S_h = \{v | v = (v_0, v_1, \ldots, v_M),\ v_0 = v_M = 0\}.$$

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For any grid function $u, v \in S_h$, define the inner products and L_2 norm, H^1 and $\|.\|_{\mathcal{H}}$ norm as follows:

$$\langle v, w\rangle_h = h\sum_{i=1}^{M-1} v_i.w_i, \qquad \|v\| = \sqrt{\langle v, v\rangle_h},$$
$$\|\delta_x v\| = \big(h\sum_{i=1}^{M}(\delta_x v_{i-\frac{1}{2}})^2\big)^{\frac{1}{2}}, \quad \|v\|_{H^1} = \big(\|v\|^2 + \|\delta_x v\|^2\big)^{\frac{1}{2}},$$
$$\langle u, v\rangle_{\mathcal{H}} = \langle \delta_x u, \delta_x v\rangle_h - \frac{h^2}{12}\langle \delta_x^2 u, \delta_x^2 v\rangle_h, \qquad \|v\|_{\mathcal{H}} = \sqrt{\langle u, u\rangle_{\mathcal{H}}},$$

and we denote $\|\delta_x^2 v\|$ similarly.

Lemma 4.2. *See([26]). For any grid function $v \in S_h$, it holds that $\|v\| \leq \frac{1}{\sqrt{6}}\|\delta_x v\|$.*

Lemma 4.3. *See([26]). For any grid function $v \in S_h$, we have*

$$\frac{2}{3}\|\delta_x v\|^2 \leq \|v\|^2_{\mathcal{H}} \leq \|\delta_x v\|^2. \tag{4.1}$$

From the Lemmas 4.2-4.3, one can result that the norm $\|.\|_{\mathcal{H}}$ is equivalent to the standard H^1 norm and is more convenient than the standard H^1 norm for stability and convergence analysis.

Lemma 4.4. *See ([25]). For any grid function $u \in S_h$, we have*

$$-\big\langle \delta_x^2 u^{n+\frac{1}{2}}, \mathcal{H}\delta_t u^{n+\frac{1}{2}}\big\rangle_h = \frac{1}{2\tau}\big(\|u^{n+1}\|^2_{\mathcal{H}} - \|u^n\|^2_{\mathcal{H}}\big). \tag{4.2}$$

In proceeding to determine the stability of the difference scheme (2.17)-(2.19) with respect to the initial values $\phi(x)$, $\psi(x)$ and the source term f, we now prove the following theorem.

Theorem 4.5. *Suppose $\{u_i^n | 0 \leq i \leq M, 1 \leq n \leq N\}$ is the solution of the difference scheme (2.17)-(2.19), where $u_0^n = u_M^n = 0$; then it holds that*

$$\Theta(u^m) \leq (\alpha-1)\|\phi\|^2_{\mathcal{H}} + \frac{1}{2}\big(1 - \exp(\frac{1-\alpha}{2-\alpha}T)\big)\|\mathcal{H}\psi\|^2 + \frac{1}{2}(\alpha-1)\tau\sum_{n=0}^{m-1}\|\mathcal{H}f^{n+\frac{1}{2}}\|^2,$$
$$0 \leq m \leq N.$$

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Proof. Multiplying the Eq. (2.17) by $h\mathcal{H}\delta_t u_i^{n+\frac{1}{2}}$, summing over i for $i = 1,2,\ldots,M-1$, we have:

$$\begin{aligned}&(\alpha-1)\tau\|\mathcal{H}\delta_t u^{n+\frac{1}{2}}\|^2 + M_0\|\mathcal{H}\delta_t u^{n+\frac{1}{2}}\|^2\\&-\sum_{k=1}^{n}(M_{n-k}-M_{n-k+1})\big\langle\mathcal{H}\delta_t u^{k-\frac{1}{2}},\mathcal{H}\delta_t u^{n+\frac{1}{2}}\big\rangle_h - M_n\big\langle\mathcal{H}\psi,\mathcal{H}\delta_t u^{n+\frac{1}{2}}\big\rangle_h\\&=(\alpha-1)\tau\big\langle\delta_x^2 u^{n+\frac{1}{2}},\mathcal{H}\delta_t u^{n+\frac{1}{2}}\big\rangle_h + (\alpha-1)\tau\big\langle\mathcal{H}f^{n+\frac{1}{2}},\mathcal{H}\delta_t u^{n+\frac{1}{2}}\big\rangle_h. \qquad (4.3)\end{aligned}$$

It follows from Lemma (4.4) that

$$(\alpha-1)\tau\langle\delta_x^2 u^{n+\frac{1}{2}},\mathcal{H}\delta_t u^{n+\frac{1}{2}}\rangle_h = -\frac{\alpha-1}{2}\big(\|u^{n+1}\|_{\mathcal{H}}^2 - \|u^n\|_{\mathcal{H}}^2\big). \qquad (4.4)$$

Substituting Eq. (4.4) into Eq. (4.3) and noticing that both M_n and $M_{n-k} - M_{n-k+1}$ are positive, we obtain

$$\begin{aligned}&(\alpha-1)\tau\|\mathcal{H}\delta_t u^{n+\frac{1}{2}}\|^2 + M_0\|\mathcal{H}\delta_t u^{n+\frac{1}{2}}\|^2 + \frac{\alpha-1}{2}\big(\|u^{n+1}\|_{\mathcal{H}}^2 - \|u^n\|_{\mathcal{H}}^2\big)\\&\le\sum\frac{1}{2}(M_{n-k}-M_{n-k+1})\|\mathcal{H}\delta_t u^{k-\frac{1}{2}}\|^2 + \frac{1}{2}(M_0-M_n)\|\mathcal{H}\delta_t u^{n+\frac{1}{2}}\|^2\\&+\frac{M_n}{2}\big(\|\psi\|^2 + \|\mathcal{H}\delta_t u^{n+\frac{1}{2}}\|^2\big) + \frac{1}{2}(\alpha-1)\tau\big(\|\mathcal{H}f^{n+\frac{1}{2}}\|^2 + \|\mathcal{H}\delta_t u^{n+\frac{1}{2}}\|^2\big). \qquad (4.5)\end{aligned}$$

Combining Eqs. (4.3) and (4.5), we have:

$$\begin{aligned}\frac{\alpha-1}{2}\|u^{n+1}\|_{\mathcal{H}}^2 + \frac{1}{2}\sum_{k=1}^{n+1}M_{n-k+1}\|\mathcal{H}\delta_t u^{k-\frac{1}{2}}\|^2 \le{}& \frac{\alpha-1}{2}\|u^n\|_{\mathcal{H}}^2\\&+\frac{1}{2}\sum_{k=1}^{n}M_{n-k}\|\mathcal{H}\delta_t u^{k-\frac{1}{2}}\|^2\\&+\frac{M_n}{2}\|\mathcal{H}\psi\|^2\\&+\frac{1}{2}(\alpha-1)\tau\|\mathcal{H}f^{n+\frac{1}{2}}\|^2. \qquad (4.6)\end{aligned}$$

Let $\Theta(u^0) = \|u^0\|_{\mathcal{H}}^2$ and $\Theta(u^n) = \frac{\alpha-1}{2}\|u^{n+1}\|_{\mathcal{H}}^2 + \frac{1}{2}\sum_{k=1}^{n}M_{n-k}\|\mathcal{H}\delta_t u^{k-\frac{1}{2}}\|^2$. Sum with respect to n from to N to obtain

$$\begin{aligned}\Theta(u^N) &\le \Theta(u^{n-1}) + \frac{1}{2}M_n\|\mathcal{H}\psi\|^2 + \frac{1}{2}(\alpha-1)\tau\|\mathcal{H}f^{n+\frac{1}{2}}\|^2\\&\le\Theta(u^0) + \frac{1}{2}\sum_{n=0}^{N-1}M_n\|\mathcal{H}\psi\|^2 + \frac{1}{2}(\alpha-1)\tau\sum_{n=0}^{N-1}\|\mathcal{H}f^{n+\frac{1}{2}}\|^2. \qquad (4.7)\end{aligned}$$

It is easy to verify by straightforward calculation that $\sum_{n=0}^{N-1}M_n = 1-\exp(\frac{1-\alpha}{2-\alpha}t_N)$, and thus

$$\frac{1}{2}\sum_{n=0}^{N-1}M_k\|\mathcal{H}\psi\|^2 = \frac{1}{2}\big(1-\exp(\frac{1-\alpha}{2-\alpha}T)\big)\|\mathcal{H}\psi\|^2. \qquad (4.8)$$

Substituting Eq. (4.8) into Eq. (4.7), we obtain the desired result. □

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4.3. **Optimal error estimate.** Now we can consider the error estimate of the difference scheme (2.17)-(2.19). Let

$$e_i^n = u(x_i,t_n) - u_i^n, \quad 0 \le i \le M, \quad 0 \le n \le N.$$

Subtracting Eqs. (2.17)-(2.19) from Eqs. (2.12), (2.15)-(2.16), we obtain the error equations

$$(\alpha-1)\tau\mathcal{H}\delta_t e_i^{n+\frac{1}{2}} + \mathcal{H}\Big[M_0\delta_t e_i^{n+\frac{1}{2}} - \sum_{k=1}^{n}\big(M_{n-k}-M_{n-k+1}\big)\delta_t e_i^{k-\frac{1}{2}}\Big] = (\alpha-1)\tau\delta_x^2 e_i^{n+\frac{1}{2}}, \quad 1 \le i \le M-1, \quad 0 \le n \le N-1, \tag{4.9}$$

$$e_0^n = e_M^n = 0, \quad 1 \le n \le N, \tag{4.10}$$

$$e_i^0 = 0, \quad 0 \le i \le M. \tag{4.11}$$

Multiplying the Eq. (2.17) by $h\mathcal{H}\delta_t e_i^{n+\frac{1}{2}}$, summing over i for $i = 1,2,\dots,M-1$, we have:

$$\begin{aligned}(\alpha-1)\tau\|\mathcal{H}\delta_t e^{n+\frac{1}{2}}\|^2 &+ M_0\|\mathcal{H}\delta_t e^{n+\frac{1}{2}}\|^2 \\ &- \sum_{k=1}^{n}(M_{n-k}-M_{n-k+1})\big\langle\mathcal{H}\delta_t e^{k-\frac{1}{2}}, \mathcal{H}\delta_t u^{n+\frac{1}{2}}\big\rangle_h \\ &- M_n\big\langle\mathcal{H}\psi, \mathcal{H}\delta_t e^{n+\frac{1}{2}}\big\rangle_h = (\alpha-1)\tau\big\langle\delta_x^2 e^{n+\frac{1}{2}}, \mathcal{H}\delta_t e^{n+\frac{1}{2}}\big\rangle_h \\ &+ (\alpha-1)\tau\big\langle\mathcal{H}f^{n+\frac{1}{2}}, \mathcal{H}\delta_t e^{n+\frac{1}{2}}\big\rangle_h. \end{aligned} \tag{4.12}$$

The following relation can now be obtianed in a similar way to the stability analysis as follows:

$$\begin{aligned}\frac{\alpha-1}{2}\|e^{n+1}\|_{\mathcal{H}}^2 + \frac{1}{2}\sum_{k=1}^{n+1}M_{n-k+1}\|\mathcal{H}\delta_t e^{k-\frac{1}{2}}\|^2 \le \frac{\alpha-1}{2}\|e^n\|_{\mathcal{H}}^2 &+ \frac{1}{2}\sum_{k=1}^{n}M_{n-k}\|\mathcal{H}\delta_t e^{k-\frac{1}{2}}\|^2 \\ &+ \frac{M_n}{2}\|\mathcal{H}\psi\|^2 \\ &+ \frac{1}{2}(\alpha-1)\tau\|\mathcal{H}f^{n+\frac{1}{2}}\|^2. \end{aligned} \tag{4.13}$$

By considering the definition of Θ in stability analysis, the above relation can be rewritten as follows:

$$\Theta(e^{n+1}) \le \Theta(e^n) + \frac{1}{2}(\alpha-1)\tau\|R^{n+\frac{1}{2}}\|^2, \tag{4.14}$$

applying Eq. (2.14) and sum with respect to $0 \le n \le N-1$ to obtain:

$$\Theta(e^N) \le \Theta(e^0) + C(\tau^2+h^4)^2. \tag{4.15}$$

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By Gronwall inequality, we received the following:

$$\frac{\alpha-1}{2}\|e^N\|_{\mathcal{H}}^2 \leq \Theta(e^N) \leq \Theta(e^0) + C(\tau^4 + h^8), \tag{4.16}$$

and

$$\frac{1}{2}\sum_{k=1}^{N} M_{N-k}\|\mathcal{H}\delta_t e^{k-\frac{1}{2}}\|^2 \leq \Theta(e^N) \leq \Theta(e^N) + C(\tau^4 + h^8). \tag{4.17}$$

Theorem 4.6. *Suppose that the equation (2.1) has smooth solution $u(x,t) \in C^{6,4}_{x,t}$, and let $\{u_i^n | 0 \leq i \leq M,\ 1 \leq n \leq N\}$ be the solution of the difference scheme (2.17)-(2.19), then it holds that*

$$\|e^n\|_{\mathcal{H}}^2 = O(\tau^2 + h^4).$$

4.4. **Two-dimensional fractional Cattaneo equation.** Consider the following problem involving the fractional Cattaneo equation in two-dimensional:

$$\frac{\partial u(x,y,t)}{\partial t} + {}_0^{\mathsf{CF}}\mathcal{D}_t^{\alpha} u(x,y,t) = \Delta u(x,y,t) + f(x,y,t), \quad (x,y) \in \Omega,\ 0 \leq t \leq T, \quad 1 < \alpha < 2, \tag{4.18}$$

with the initial conditions

$$u(x,y,0) = \phi(x,y), \quad \frac{\partial u(x,y,0)}{\partial t} = \psi(x,y), \quad (x,y) \in \bar{\Omega} = \Omega \cup \partial\Omega, \tag{4.19}$$

and the boundary conditions

$$u(x,y,t) = \varphi(x,y,t), \quad (x,y) \in \partial\Omega, \quad 0 \leq t \leq T, \tag{4.20}$$

here, $\Omega = (0, L_1) \times (0, L_2)$, $\partial\Omega$ is the boundary of Ω, $f(x,y,t)$ is the source function with sufficient smoothness, $\psi(x,y), \phi(x,y)$ and $\varphi(x,y,t)$ are given continuous functions.

Let $h_x = \frac{L_1}{M_1}$, $h_y = \frac{L_2}{M_2}$ and $\tau = \frac{T}{N}$ be the spatial and the temporal step sizes respectively, where M_1, M_2 and N are some given positive integers. So we define

$$x_i = ih_x, \qquad i = 0, 1, \ldots, M_1,$$
$$y_j = jh_y, \qquad j = 0, 1, \ldots, M_2,$$
$$t_n = n\tau, \qquad n = 0, 1, \ldots N.$$

$\overline{\Omega}_h = \{(x_i, y_j) | 0 \leq i \leq M_1,\ 0 \leq j \leq M_2\}$, $\Omega_h = \overline{\Omega}_h \cap \Omega$ and $\partial\Omega_h = \overline{\Omega}_h \cap \partial\Omega$. For any grid function $u = \{u_{ij}^n | 0 \leq i \leq M_1, 0 \leq j \leq M_2\}$, denote

$$\delta_x u_{i-\frac{1}{2},j} = \frac{1}{h_x}(u_{i,j} - u_{i-1,j}), \qquad \delta_x^2 u_{i,j} = \frac{1}{h_x}(\delta_x u_{i+\frac{1}{2},j} - \delta_x u_{i-\frac{1}{2},j}),$$

$$\delta_y\delta_x u_{i-\frac{1}{2},j-\frac{1}{2}} = \frac{1}{h_y}(\delta_x u_{i-\frac{1}{2},j} - \delta_x u_{i-\frac{1}{2},j-1}),$$

$$\delta_y\delta_x^2 u_{i,j-\frac{1}{2}} = \frac{1}{h_y}(\delta_x^2 u_{i,j} - \delta_x^2 u_{i,j-1}),$$

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$$\mathcal{H}_x u_{i,j} = \begin{cases} \frac{1}{12}(u_{i+1,j} + 10u_{i,j} + u_{i-1,j}), & 1 \le i \le M_1 - 1,\ 0 \le j \le M_2, \\ u_{i,j}, & i = 0 \text{ or } M_1, \end{cases}$$

$$\mathcal{H}_y u_{i,j} = \begin{cases} \frac{1}{12}(u_{i,j+1} + 10u_{i,j} + u_{i,j-1}), & 0 \le i \le M_1,\ 1 \le j \le M_2 - 1, \\ u_{i,j}, & j = 0 \text{ or } M_2. \end{cases}$$

Similarly, the notations $\delta_y u_{i,j-\frac{1}{2}}$, $\delta_y^2 u_{i,j}$, $\delta_x \delta_y^2 u_{i-\frac{1}{2},j}$ and $\delta_x^2 \delta_y^2 u_{i,j}$ can be defined. The discrete Laplace operator is denoted as $\Delta_h u_{ij} = \delta_x^2 u_{i,j} + \delta_y^2 u_{i,j}$.
It is obvious that

$$\mathcal{H}_x u_{i,j} = \left(I + \frac{h_x^2}{12}\delta_x^2\right) u_{i,j}, \qquad \mathcal{H}_y u_{i,j} = \left(I + \frac{h_y^2}{12}\delta_y^2\right) u_{i,j}, \quad (x_i, y_i) \in \Omega_h.$$

For simplicity of the formulas in our further consideration, we define:

$$\mathcal{H} u_{ij} = \mathcal{H}_x \mathcal{H}_y u_{i,j}, \qquad \Lambda_h u_{ij} = \left(\mathcal{H}_y \delta_x^2 + \mathcal{H}_x \delta_y^2\right), \quad (x_i, y_i) \in \Omega_h.$$

Let $f_{ij}^{n+\frac{1}{2}} = f(x_i, y_j, t_{n+\frac{1}{2}})$ and $t_{n+\frac{1}{2}} = \frac{t_{n+1}+t_n}{2}$. Then according to the new Caputo-Fabrizio formula (2.4), Eq. (4.18) at points $(x_i, y_j, t_{n+\frac{1}{2}})$ can be written as follows:

$$\begin{aligned} \delta_t U_{ij}^{n+\frac{1}{2}} &+ \frac{1}{(\alpha-1)\tau}\left[M_0 \delta_t U_{ij}^{n+\frac{1}{2}} - \sum_{k=1}^{n}\left(M_{n-k} - M_{n-k+1}\right)\delta_t U_{ji}^{k-\frac{1}{2}} - M_n \psi_{ij}\right] \\ &= \frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_{n+\frac{1}{2}}) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_{n+\frac{1}{2}}) \\ &+ f_{ij}^{n+\frac{1}{2}} + (\mathrm{R}_t)_{ij}^{n+\frac{1}{2}}, \quad (x_i, y_j) \in \Omega_h, \quad 0 \le n \le N-1. \end{aligned}$$

Operating $\mathcal{H} = \mathcal{H}_x \mathcal{H}_y$ on the above equality, it leads to:

$$\begin{aligned} \mathcal{H}\delta_t U_{ij}^{n+\frac{1}{2}} &+ \frac{1}{(\alpha-1)\tau}\mathcal{H}\left[M_0 \delta_t U_{ij}^{n+\frac{1}{2}} - \sum_{k=1}^{n}\left(M_{n-k} - M_{n-k+1}\right)\delta_t U_{ij}^{k-\frac{1}{2}} - M_n \psi_{ij}\right] \\ &= \mathcal{H}\frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_{n+\frac{1}{2}}) + \mathcal{H}\frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_{n+\frac{1}{2}}) \\ &+ \mathcal{H} f_{ij}^{n+\frac{1}{2}} + \mathcal{H}(\mathrm{R}_t)_{ij}^{n+\frac{1}{2}}, \quad (x_i, y_j) \in \Omega_h, \quad 0 \le n \le N-1. \end{aligned} \tag{4.21}$$

Lemma (2.1) implies that

$$\begin{aligned} \mathcal{H}_x \frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_n) &= \delta_x^2 U_{ij}^n + (\mathrm{R}_x)_{ij}^n,\ (x_i, y_j) \in \Omega_h,\ 0 \le n \le N, \\ \mathcal{H}_y \frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_n) &= \delta_y^2 U_{ij}^n + (\mathrm{R}_y)_{ij}^n,\ (x_i, y_j) \in \Omega_h,\ 0 \le n \le N, \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} (\mathrm{R}_x)_{ij}^n &= \frac{h_x^4}{360}\int_0^1 \left[\frac{\partial^6 u}{\partial x^6}(x_i - sh_x) + \frac{\partial^6 u}{\partial x^6}(x_i + sh_x)\right]\xi(s)\mathrm{d}s, \\ (\mathrm{R}_y)_{ij}^n &= \frac{h_y^4}{360}\int_0^1 \left[\frac{\partial^6 u}{\partial y^6}(y_j - sh_y) + \frac{\partial^6 u}{\partial y^6}(y_j + sh_y)\right]\xi(s)\mathrm{d}s. \end{aligned}$$

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Then

$$\mathcal{H}_y\mathcal{H}_x\frac{\partial^2 u}{\partial x^2}(x_i,y_j,t_{n+\frac{1}{2}})=\mathcal{H}_y\delta_x^2U_{ij}^{n+\frac{1}{2}}+\mathcal{H}_y(\mathrm{R}_x)_{ij}^{n+\frac{1}{2}},\ (x_i,y_j)\in\Omega_h,\ 0\le n\le N,$$
$$\mathcal{H}_x\mathcal{H}_y\frac{\partial^2 u}{\partial y^2}(x_i,y_j,t_{n+\frac{1}{2}})=\mathcal{H}_x\delta_y^2U_{ij}^{n+\frac{1}{2}}+\mathcal{H}_x(\mathrm{R}_y)_{ij}^{n+\frac{1}{2}},\ (x_i,y_j)\in\Omega_h,\ 0\le n\le N. \tag{4.23}$$

Substituting Eq. (4.23) into Eq. (4.21), we obtain:

$$(\alpha-1)\tau\mathcal{H}\delta_tU_{ij}^{n+\frac{1}{2}}+\mathcal{H}\Big[M_0\delta_tU_{ij}^{n+\frac{1}{2}}-\sum_{k=1}^{n}\big(M_{n-k}-M_{n-k+1}\big)\delta_tU_{ij}^{k-\frac{1}{2}}-M_n\psi_{ij}\Big]$$
$$=(\alpha-1)\tau\Lambda_hU_i^{n+\frac{1}{2}}+(\alpha-1)\tau\mathcal{H}f_{ij}^{n+\frac{1}{2}}+(\tilde{\mathrm{R}})_{ij}^{n+\frac{1}{2}},$$
$$(x_i,y_j)\in\Omega_h,\quad 0\le n\le N-1, \tag{4.24}$$

where

$$(\tilde{\mathrm{R}})_{ij}^{n+\frac{1}{2}}=\mathcal{H}(\mathrm{R}_t)_i^{n+\frac{1}{2}}+\mathcal{H}_y(\mathrm{R}_x)_{ij}^{n+\frac{1}{2}}+\mathcal{H}_x(\mathrm{R}_y)_{ij}^{n+\frac{1}{2}},$$

noticing that

$$\big|(R_t)_{ij}^{n+\frac{1}{2}}\big|\le\frac{1}{\alpha-1}\max_{0\le k\le n}\big|u_t^{(3)}(x_i,y_j,c_k)\big|\exp\big(\frac{2\alpha-2}{2-\alpha}\big)\tau^2+O(\tau^2).$$

Then it holds that

$$|\tilde{\mathrm{R}}_{ij}^{n+\frac{1}{2}}|\le C_R(\tau^2+h_x^4+h_y^4),\quad (x_i,y_j)\in\Omega_h,\ 0\le n\le N-1, \tag{4.25}$$

where C_R is a positive constant, that is independent of the time step size τ and grid spacing h_x, h_y.

Adding small term $-\frac{\big((\alpha-1)\tau\big)^2\tau^2}{4((\alpha-1)\tau+M_0)}\delta_x^2\delta_y^2\delta_tU_{ij}^{n+\frac{1}{2}}$ into right hand side of Eq. (4.24), we obtain:

$$(\alpha-1)\tau\mathcal{H}\delta_tU_{ij}^{n+\frac{1}{2}}+\mathcal{H}\Big[M_0\delta_tU_{ij}^{n+\frac{1}{2}}-\sum_{k=1}^{n}\big(M_{n-k}-M_{n-k+1}\big)\delta_tU_{ij}^{k-\frac{1}{2}}-M_n\psi_{ij}\Big]$$
$$=(\alpha-1)\tau\Lambda_hU_i^{n+\frac{1}{2}}-\frac{\big((\alpha-1)\tau\big)^2\tau^2}{4((\alpha-1)\tau+M_0)}\delta_x^2\delta_y^2\delta_tU_{ij}^{n+\frac{1}{2}}$$
$$+(\alpha-1)\tau\mathcal{H}f_{ij}^{n+\frac{1}{2}}+(\mathrm{R})_{ij}^{n+\frac{1}{2}},\quad (x_i,y_j)\in\Omega_h,\quad 0\le n\le N-1, \tag{4.26}$$

where

$$(\mathrm{R})_{ij}^{n+\frac{1}{2}}=(\tilde{\mathrm{R}})_{ij}^{n+\frac{1}{2}}+\frac{\big((\alpha-1)\tau\big)^2\tau^2}{4((\alpha-1)\tau+M_0)}\delta_x^2\delta_y^2\delta_tU_{ij}^{n+\frac{1}{2}}.$$

In addition, it follows from the initial and boundary value conditions that

$$U_{ij}^0=\phi(x_i,y_j),\quad (x_i,y_j)\in\overline{\Omega}_h, \tag{4.27}$$

$$U_{ij}^n=\varphi(x_i,y_j,t_n),\quad (x_i,y_j)\in\partial\Omega_h,\ 1\le n\le N. \tag{4.28}$$

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Replacing U_{ij}^n with it's numerical approximation u_{ij}^n in Eqs. (4.26)-(4.28) and neglecting the last two terms of Eq. (4.26), we can derive the following difference scheme:

$$(\alpha-1)\tau\mathcal{H}\delta_t u_{ij}^{n+\frac{1}{2}} + \mathcal{H}\Big[M_0\delta_t u_{ij}^{n+\frac{1}{2}} - \sum_{k=1}^{n}\big(M_{n-k}-M_{n-k+1}\big)\delta_t u_{ij}^{k-\frac{1}{2}} - M_n\psi_{ij}\Big]$$
$$= (\alpha-1)\tau\Lambda_h u_i^{n+\frac{1}{2}} + (\alpha-1)\tau\mathcal{H}f_{ij}^{n+\frac{1}{2}} - \frac{\big((\alpha-1)\tau\big)^2\tau^2}{4((\alpha-1)\tau+M_0)}\delta_x^2\delta_y^2\delta_t u_{ij}^{n+\frac{1}{2}},$$
$$(x_i,y_j)\in\Omega_h,\quad 0\le n\le N-1, \tag{4.29}$$

$$u_{ij}^0 = \phi(x_i,y_j),\quad (x_i,y_j)\in\overline{\Omega}_h, \tag{4.30}$$

$$u_{ij}^n = \varphi(x_i,y_j,t_n),\quad (x_i,y_j)\in\partial\Omega_h,\ 1\le n\le N. \tag{4.31}$$

Define $\mu=(\alpha-1)\tau$, we now rewrite Eq. (4.29) as follows:

$$\mu\mathcal{H}\delta_t u_{ij}^{n+\frac{1}{2}} + \mathcal{H}\Big[M_0\delta_t u_{ij}^{n+\frac{1}{2}} - \sum_{k=1}^{n}\big(M_{n-k}-M_{n-k+1}\big)\delta_t u_{ij}^{k-\frac{1}{2}} - M_n\psi_{ij}\Big]$$
$$= \mu\Lambda_h u_i^{n+\frac{1}{2}} + \mu\mathcal{H}f_{ij}^{n+\frac{1}{2}} - \frac{\mu^2\tau^2}{4(\mu+M_0)}\delta_x^2\delta_y^2\delta_t u_{ij}^{n+\frac{1}{2}},$$
$$(x_i,y_j)\in\Omega_h,\quad 0\le n\le N-1. \tag{4.32}$$

After simplification of terms and using the approximate factorization technique, we get as follows:

$$\big(\mathcal{H}_x(\mu+M_0) - \frac{\mu\tau}{2}\delta_x^2\big)\big(\mathcal{H}_y - \frac{\mu\tau}{2(\mu+M_0)}\delta_y^2\big)u_{ij}^{n+1}$$
$$= \big(\mathcal{H}_x(\mu+M_0) + \frac{\mu\tau}{2}\delta_x^2\big)\big(\mathcal{H}_y + \frac{\mu\tau}{2(\mu+M_0)}\delta_y^2\big)u_{ij}^{n}$$
$$+ \sum_{k=1}^{n}\big(M_{n-k}-M_{n-k+1}\big)\mathcal{H}\delta_t u_{ij}^{k-\frac{1}{2}} + \tau M_n\mathcal{H}\psi_{ij} + \mu\tau\mathcal{H}f_{ij}^{n+\frac{1}{2}},$$
$$(x_i,y_j)\in\Omega_h,\quad 0\le n\le N-1. \tag{4.33}$$

Difference scheme Eq. (4.33) for solving the two-dimensional problem can be divided into two sets of independent one-dimensional problems. For this purpose, we apply the D'Yakonov and Peaceman-Rachford ADI methods [20].

Introducing an intermediate level u^*, an ADI scheme of D'Yakonov-type is designed as follows:

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$$
\begin{cases}
\big(\mathcal{H}_x(\mu+M_0)-\frac{\mu\tau}{2}\delta_x^2\big)u_{ij}^{*}=\big(\mathcal{H}_x(\mu+M_0)+\frac{\mu\tau}{2}\delta_x^2\big)\big(\mathcal{H}_y+\frac{\mu\tau}{2(\mu+M_0)}\delta_y^2\big)u_{ij}^{n}\\
+\sum_{k=1}^{n}\big(M_{n-k}-M_{n-k+1}\big)\mathcal{H}\delta_t u_{ij}^{k-\frac{1}{2}}+\tau M_n\mathcal{H}\psi_{ij}+\mu\tau\mathcal{H}f_{ij}^{n+\frac{1}{2}},\\
1\le i\le M_1-1,\\
\\
\big(\mathcal{H}_y-\frac{\mu\tau}{2(\mu+M_0)}\delta_y^2\big)u_{ij}^{n+1}=u_{ij}^{*}, \qquad 1\le j\le M_2-1,\ n=0,1,\dots,N-1,\\
u_{0,j}^{*}=\big(\mathcal{H}_y-\frac{\mu\tau}{2(\mu+M_0)}\delta_y^2\big)u_{0j}^{n+1},\ u_{M_1,j}^{*}=\big(\mathcal{H}_y-\frac{\mu\tau}{2(\mu+M_0)}\delta_y^2\big)u_{Mj}^{n+1},\\
u_{i,0}^{n}=\varphi(x_i,y_0,t_n),\ u_{i,M_2}^{n}=\varphi(x_i,y_{M_2},t_n).
\end{cases}
\tag{4.34}
$$

At first, we solve the first and third system of equations of (4.34) for fixed $j\in\{1,...,M_2-1\}$ to compute u_{ij}^{*}. Having computed u_{ij}^{*}, we then solve the second and fourth system of equations of (4.34) for fixed $i\in\{1,...,M_1-1\}$ to compute u_{ij}^{n} .

The scheme Eq. (4.33) can be decomposed into an ADI scheme of Peaceman-Rachford type as follows:

$$
\begin{cases}
\big(\mathcal{H}_x(\mu+M_0)-\frac{\mu\tau}{2}\delta_x^2\big)u_{ij}^{*}=\big(\mathcal{H}_y+\frac{\mu\tau}{2(\mu+M_0)}\delta_y^2\big)u_{ij}^{n}\\
+\frac{\tau}{2(\mu+M_0)}\sum_{k=1}^{n}\big(M_{n-k}-M_{n-k+1}\big)\mathcal{H}_y\delta_t u_{ij}^{k-\frac{1}{2}}\\
+\frac{\tau}{2(\mu+M_0)}M_n\mathcal{H}_y\psi_{ij}+\frac{\tau}{2(\mu+M_0)}\mathcal{H}_y f_{ij}^{n+\frac{1}{2}},\\
1\le i\le M_1-1,\ n=0,1,\dots,N-1,\\
\\
\big(\mathcal{H}_y-\frac{\mu\tau}{2}\delta_y^2\big)u_{ij}^{n+1}=\big(\mathcal{H}_x(\mu+M_0)+\frac{\mu\tau}{2}\delta_x^2\big)u_{ij}^{*}\\
+\frac{\tau}{2(\mu+M_0)}\sum_{k=1}^{n}\big(M_{n-k}-M_{n-k+1}\big)\mathcal{H}_y\delta_t u_{ij}^{k-\frac{1}{2}}\\
+\frac{\tau}{2(\mu+M_0)}M_n\mathcal{H}_y\psi_{ij}+\frac{\tau}{2(\mu+M_0)}\mathcal{H}_y f_{ij}^{n+\frac{1}{2}},\\
1\le j\le M_2-1,\ n=0,1,\dots,N-1.
\end{cases}
\tag{4.35}
$$

5. Stability and convergence of the compact ADI scheme

To analyse the stability and convergence of the Compact ADI scheme (4.29)-(4.31), we introduce some lemmas. We first introduce the following space of grid functions as follows:

$$S_h=\{v|v=\{v_{ij}|(x_i,y_j)\in\bar{\Omega}_h\}\text{ and }v_{ij}=0\text{ if }(x_i,y_j)\in\partial\Omega_h\}.$$

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For any grid function $w, v \in S_h$, define the following inner products and norms.

$$\langle w^n, v^n\rangle_h = h_x h_y \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} w_{ij}^n v_{ij}^n, \quad \|v\| = \sqrt{\langle v, v\rangle_h},$$

$$\|\delta_x v\| = \left[h_x h_y \sum_{i=3}^{M_1-2}\sum_{j=2}^{M_2-1} |\delta_x v_{i-\frac{1}{2},j}|^2\right]^{\frac{1}{2}},$$

$$\|\delta_x\delta_y v\| = \left[h_x h_y \sum_{i=1}^{M_1}\sum_{j=1}^{M_2} |\delta_x\delta_y v_{i-\frac{1}{2},j-\frac{1}{2}}|^2\right]^{\frac{1}{2}},$$

$$\|\delta_y\delta_x^2 v\| = \left[h_x h_y \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2} |\delta_y\delta_x^2 v_{i,j-\frac{1}{2}}|^2\right]^{\frac{1}{2}}, \quad \|\delta_x^2 v\| = \left[h_x^2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} |\delta_x^2 v_{i,j}|^2\right]^{\frac{1}{2}},$$

and we denote $\|\delta_y^2 v\|$, $\|\delta_y v\|$, and $\|\delta_x\delta_y^2 v\|$ similarly. The discrete semi-norm H^1 and H^1 norm of the space S_h are defined by

$$\|\nabla_h v\| = \left(\|\delta_x v\|^2 + \|\delta_y v\|^2\right)^{\frac{1}{2}}, \qquad \|v\|_{H^1} = \left(\|v\|^2 + \|\nabla_h v\|^2\right)^{\frac{1}{2}},$$

and

$$\|v\|_{\tilde{H}^1} = \left(\|\mathcal{H}_y v\|_A^2 + \|\mathcal{H}_x v\|_B^2\right)^{\frac{1}{2}},$$

where

$$\|v\|_A = \left(\|\delta_x v\|^2 - \frac{h_x^2}{12}\|\delta_x^2 v\|^2\right)^{\frac{1}{2}}, \qquad \|v\|_B = \left(\|\delta_y v\|^2 - \frac{h_y^2}{12}\|\delta_y^2 v\|^2\right)^{\frac{1}{2}}.$$

Lemma 5.1. *See ([36]). For any grid function $v \in S_h$, we have*

$$\sqrt{\frac{48(L_1^2+L_2^2)}{27\left[6(L_1^2+L_2^2)+L_1^2L_2^2\right]}}\|v\|_{H^1} \le \|v\|_{\tilde{H}^1} \le \frac{4}{3}\|v\|_{H^1}. \tag{5.1}$$

From Lemma 5.1 one can result that the norm $\|.\|_{\tilde{H}}$ is equivalent to the standard H^1 norm. Beside, it is more convenient than the standard H^1 norm for stability and convergence analysis.

Lemma 5.2. *See ([36]). For any grid function $v \in S_h$, it holds that*

$$\langle \mathcal{H}_x\delta_t v^{n+\frac{1}{2}}, \delta_x^2 v^{n+\frac{1}{2}}\rangle_h = -\frac{1}{2\tau}\left(\|v^{n+1}\|_A^2 - \|v^n\|_A^2\right), \tag{5.2}$$

$$\langle \mathcal{H}_y\delta_t v^{n+\frac{1}{2}}, \delta_y^2 v^{n+\frac{1}{2}}\rangle_h = -\frac{1}{2\tau}\left(\|v^{n+1}\|_B^2 - \|v^n\|_B^2\right). \tag{5.3}$$

Lemma 5.3. *See ([36]). For any grid function $v \in S_h$, it holds that*

$$\langle \Lambda_h v^{n+\frac{1}{2}}, \mathcal{H}\delta_t v^{n+\frac{1}{2}}\rangle_h = -\frac{1}{2\tau}\left(\|v^{n+1}\|_{\tilde{H}^1}^2 - \|v^n\|_{\tilde{H}^1}^2\right), \tag{5.4}$$

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and

$$\langle \delta_x^2\delta_y^2 v, \mathcal{H}v\rangle_h \geq \frac{1}{3}\|\delta_x\delta_y v\|^2. \quad (5.5)$$

Theorem 5.4. *Suppose* $\{u_{ij}^n|(x_i,y_j)\in\Omega,\ 1\leq n\leq N\}$ *is the solution of the difference scheme (4.29)-(4.31), where* $u_{ij}^n = 0$ *on* $\partial\Omega_h$*; then it holds that*

$$\|u^m\|_{\tilde{\mathcal{H}}^1}^2 \leq \|u^0\|_{\tilde{\mathcal{H}}^1}^2 + \frac{1}{2}\big(1-\exp(\frac{1-\alpha}{2-\alpha}t_n)\big)\|\mathcal{H}\psi\|^2 + (\alpha-1)\tau\sum_{k=0}^{m-1}\|\mathcal{H}f^{n+\frac{1}{2}}\|^2,$$
$$0\leq m\leq N.$$

Proof. Multiplying the Eq. (4.29) by $2h_xh_y\mathcal{H}\delta_t u_i^{n+\frac{1}{2}}$, summing over i and j for $(x_i,y_j)\in\Omega_h$, we have:

$$\begin{aligned}
&2(\alpha-1)\tau\|\mathcal{H}\delta_t u^{n+\frac{1}{2}}\|^2 + 2M_0\|\mathcal{H}\delta_t u^{n+\frac{1}{2}}\|^2\\
&-2\sum_{k=1}^{n}(M_{n-k}-M_{n-k+1})\big\langle\mathcal{H}\delta_t u^{k-\frac{1}{2}},\mathcal{H}\delta_t u^{n+\frac{1}{2}}\big\rangle_h - 2M_n\big\langle\mathcal{H}\psi,\mathcal{H}\delta_t u^{n+\frac{1}{2}}\big\rangle_h\\
&=2(\alpha-1)\tau\big\langle\Lambda_h u^{n+\frac{1}{2}},\mathcal{H}\delta_t u^{n+\frac{1}{2}}\big\rangle_h - \frac{\big((\alpha-1)\tau\big)^2\tau^2}{2((\alpha-1)\tau+M_0)}\big\langle\delta_x^2\delta_x^2\delta_t u^{n+\frac{1}{2}},\mathcal{H}\delta_t u^{n+\frac{1}{2}}\big\rangle_h\\
&+2(\alpha-1)\tau\big\langle\mathcal{H}f^{n+\frac{1}{2}},\delta_t u^{n+\frac{1}{2}},\mathcal{H}\delta_t u^{n+\frac{1}{2}}\big\rangle_h. \quad (5.6)
\end{aligned}$$

By Lemma (5.3) it follows that

$$\begin{aligned}
&-\frac{\big((\alpha-1)\tau\big)^2\tau^2}{2((\alpha-1)\tau+M_0)}\big\langle\delta_x^2\delta_x^2\delta_t u^{n+\frac{1}{2}},\mathcal{H}\delta_t u^{n+\frac{1}{2}}\big\rangle_h\\
&\leq -\frac{\big((\alpha-1)\tau\big)^2\tau^2}{6((\alpha-1)\tau+M_0)}\|\delta_x\delta_y\delta_t u^{n+\frac{1}{2}}\|^2 \leq 0, \quad (5.7)
\end{aligned}$$

and

$$2(\alpha-1)\tau\big\langle\Lambda_h u^{n+\frac{1}{2}},\mathcal{H}\delta_t u^{n+\frac{1}{2}}\big\rangle_h = -(\alpha-1)\big(\|u^{n+1}\|_{\tilde{H}^1}^2 - \|u^n\|_{\tilde{H}^1}^2\big). \quad (5.8)$$

Substituting Eqs. (5.7) and (5.8) into Eq. (5.6) and noticing that both M_n and $(M_{n-k}-M_{n-k+1})$ are positive, we obtain:

$$\begin{aligned}
&2(\alpha-1)\tau\|\mathcal{H}\delta_t u^{n+\frac{1}{2}}\|^2 + 2M_0\|\mathcal{H}\delta_t u^{n+\frac{1}{2}}\|^2 + \alpha-1\big(\|u^{n+1}\|_{\tilde{H}^1}^2 - \|u^n\|_{\tilde{H}^1}^2\big)\\
&\leq \sum(M_{n-k}-M_{n-k+1})\|\mathcal{H}\delta_t u^{k+\frac{1}{2}}\|^2 + (M_0-M_n)\|\mathcal{H}\delta_t u^{n+\frac{1}{2}}\|^2\\
&+M_n\big(\|\psi\|^2+\|\mathcal{H}\delta_t u^{n+\frac{1}{2}}\|^2\big) + (\alpha-1)\tau\big(\|\mathcal{H}f^{n+\frac{1}{2}}\|^2+\|\mathcal{H}\delta_t u^{n+\frac{1}{2}}\|^2\big), \quad (5.9)
\end{aligned}$$

the following relation can now be obtained in a similar way to the stability analysis in one-dimensional case

$$\Theta(u^m) \leq (\alpha-1)\|\phi\|_{\tilde{\mathcal{H}}_1}^2 + \big(1-\exp(\frac{1-\alpha}{2-\alpha}T)\big)\|\mathcal{H}\psi\|^2 + (\alpha-1)\tau\sum_{n=0}^{m-1}\|\mathcal{H}f^{n+\frac{1}{2}}\|^2,$$
$$0\leq m\leq N.$$

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□

Theorem 5.5. *Suppose that the equation (2.1) has smooth solution* $u(x,y,t) \in C^{6,4,4}_{x,y,t}$, *and let* $\{u^n_{ij} | (x_i, y_j) \in \Omega_h, 1 \le n \le N\}$ *be the solution of the difference scheme (4.29)-(4.31), then it holds that*

$$\|e^n\|^2_{\mathcal{H}} = O(\tau^2 + h_x^4 + h_y^4).$$

6. Numerical results

To show the efficiency of the proposed scheme for the time fractional cattaneo equation, we present two numerical examples in the one and two-dimensional cases. We test the accuracy and the stability of the presented scheme in the paper for different values of τ, h and $h = h_x = h_y$. In order to carry out our numerical examples, we have used the Maple 18 software with a PC of 4 GHz CPU and 6 GB memory. To show the accuracy of the proposed scheme, we use the following error norms

$$e(\tau,h) = \max_{1\le i\le M-1} \left|U(x_i,t_n) - u_i^N\right|,$$
$$e(\tau,h) = \max_{1\le i,j\le M-1} \left|U(x_i,y_j,t_n) - u_{ij}^N\right|.$$

We denote the numerical convergence orders by

$$Rate_I = \log_2\left(\frac{e(2\tau,h)}{e(\tau,h)}\right),\ Rate_{II} = \log_2\left(\frac{e(\tau,2h)}{e(\tau,h)}\right).$$

Example 6.1. Consider the time fractional Cattaneo equation in the form of [16]

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + {}_0^{\mathtt{CF}}\mathcal{D}_t^{\alpha} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), & 0 < x < 1,\ 0 < t \le 1, \\ u(x,0) = \sin(\pi x), & 0 \le x \le 1, \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin(\pi x), & 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0, & 0 < t \le 1, \end{cases} \tag{6.1}$$

in which $f(x,t) = \sin(\pi x)\left(\pi^2 \exp(t) + 2\exp(t) - \exp(\frac{1-\alpha}{2-\alpha}t)\right)$. The exact solution is $u(x,t) = \exp(t)\sin(\pi x)$.

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τ	$\alpha = 1.25$		$\alpha = 1.5$		$\alpha = 1.75$	
	$e(\tau, h)$	$Rate_I$	$e(\tau, h)$	$Rate_I$	$e(N, M)$	$Rate_I$
$\frac{1}{5}$	1.17×10^{-2}	–	1.11×10^{-2}	–	9.21×10^{-3}	–
$\frac{1}{10}$	2.93×10^{-3}	2.00	2.78×10^{-3}	2.00	2.31×10^{-3}	2.00
$\frac{1}{20}$	7.34×10^{-4}	2.00	6.97×10^{-4}	2.00	5.79×10^{-4}	1.99
$\frac{1}{40}$	1.81×10^{-4}	2.01	1.72×10^{-4}	2.02	1.42×10^{-4}	2.03
$\frac{1}{80}$	4.63×10^{-5}	1.97	4.40×10^{-5}	1.96	3.84×10^{-5}	1.88

Table 1: Numerical convergence orders in temporal direction with $h = \frac{1}{100}$ for Example 6.1.

h	$\alpha = 1.25$		$\alpha = 1.5$		$\alpha = 1.75$	
	$e(\tau, h)$	$Rate_{II}$	$e(\tau, h)$	$Rate_{II}$	$e(N, M)$	$Rate_{II}$
$\frac{1}{4}$	3.66×10^{-3}	–	3.70×10^{-3}	–	3.86×10^{-3}	–
$\frac{1}{8}$	2.25×10^{-4}	4.02	2.27×10^{-4}	4.03	2.47×10^{-4}	3.97
$\frac{1}{16}$	1.38×10^{-5}	4.02	1.39×10^{-5}	4.03	1.45×10^{-5}	4.09
$\frac{1}{32}$	7.35×10^{-7}	4.23	6.08×10^{-7}	4.51	6.44×10^{-7}	4.49

Table 2: Numerical convergence orders in spatial direction with $\tau = \frac{1}{1000}$ for Example 6.1.

Table 1 shows that the numerical convergence orders of the new developed difference scheme (2.17)-(2.19) is approximately $O(\tau^2)$ in temporal direction. Having seen Table 1, we conclude that the accuracy of the preseneted method is not dependent on α. Tables 1 and 2 confirm the theoretical analysis in temporal and spatial directions, respectively. The plots of the exact and approximate solutions at final time $T = 1$ for different values of $\alpha = 1.25, 1.5$ and 1.75 with $h = \frac{1}{100}, \tau = \frac{1}{50}$ for Example 6.1 is shown in Figure 1. Figure 2 exhibits the plots of absolute error with different values $\alpha = 1.1, 1.5, 1.6$ and 1.8, respectively.

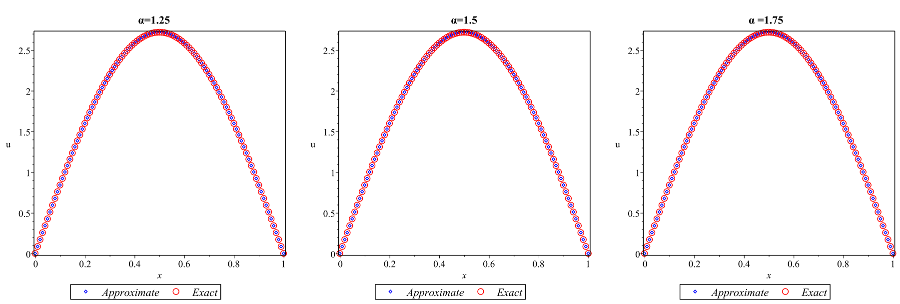

FIGURE 1. The solution curves at $T = 1$ with $h = \frac{1}{100}, \tau = \frac{1}{50}$ for Example 6.1.

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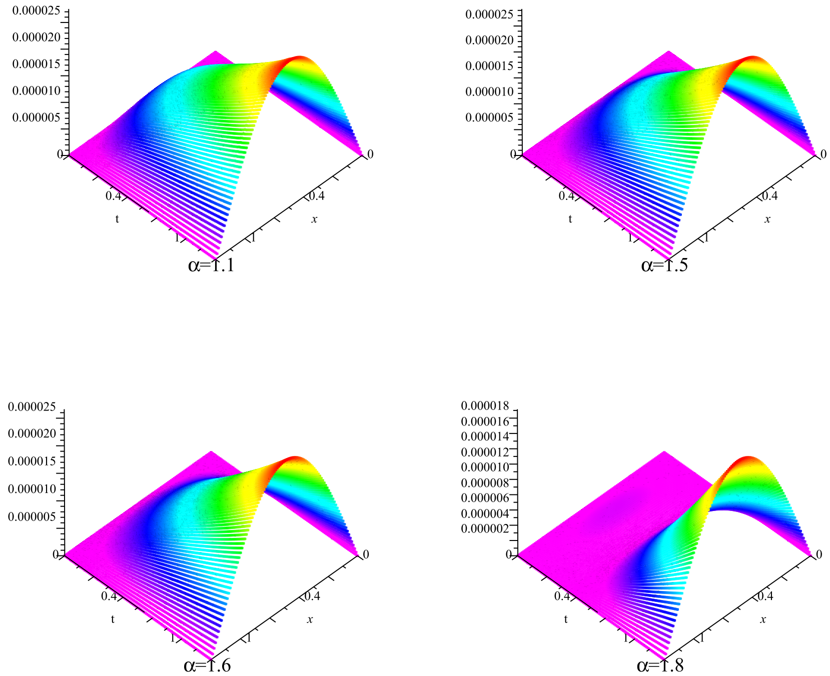

FIGURE 2. Plots of absolute error with different α for Example 6.1.

EXAMPLE 6.2. Consider the time fractional cattaneo equation in two-dimensional domain $\Omega = (0,1) \times (0,1)$

$$\begin{cases} \frac{\partial u(x,y,t)}{\partial t} + {}_{0}^{\mathtt{CF}}\mathcal{D}_t^{\alpha} u(x,y,t) = \Delta u(x,y,t) + f(x,t), & (x,y) \in \Omega,\ 0 < t \leq 1, \\ u(x,y,0) = \sin(\pi x)\sin(\pi y), & (x,y) \in \bar{\Omega}, \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin(\pi x)\sin(\pi y), & (x,y) \in \bar{\Omega}, \\ u(x,y,t) = \exp(t)\sin(\pi x)\sin(\pi y), & (x,y) \in \partial\Omega, \end{cases} \tag{6.2}$$

in which $f(x,t) = \sin(\pi x)\sin(\pi y)\left(2\pi^2\exp(t) + 2\exp(t) - \exp(\frac{1-\alpha}{2-\alpha}t)\right)$. The exact solution is $u(x,y,t) = \exp(t)\sin(\pi x)\sin(\pi y)$.

α	τ	D'Yakonov ADI scheme			P-R ADI scheme		
		$e(\tau,h)$	$Rate_I$	$CPU(s)$	$e(\tau,h)$	$Rate_I$	$CPU(s)$
1.25	$\frac{1}{5}$	6.44×10^{-2}	–	1	6.44×10^{-2}	–	2
	$\frac{1}{10}$	1.63×10^{-2}	1.99	4	1.63×10^{-2}	1.99	6
	$\frac{1}{20}$	4.06×10^{-3}	2.00	11	4.06×10^{-3}	2.00	14
	$\frac{1}{40}$	1.01×10^{-3}	2.01	30	1.01×10^{-3}	2.01	46
	$\frac{1}{80}$	2.47×10^{-4}	2.03	97	2.47×10^{-4}	2.03	157
1.5	$\frac{1}{5}$	5.51×10^{-2}	–	2	5.51×10^{-2}	–	2
	$\frac{1}{10}$	1.36×10^{-2}	2.02	5	1.36×10^{-2}	2.02	5
	$\frac{1}{20}$	3.37×10^{-3}	2.02	9	3.37×10^{-3}	2.02	15
	$\frac{1}{40}$	8.33×10^{-4}	2.02	30	8.33×10^{-4}	2.02	44
	$\frac{1}{80}$	2.04×10^{-4}	2.04	93	2.03×10^{-4}	2.04	153
1.75	$\frac{1}{5}$	4.29×10^{-2}	–	2	4.29×10^{-2}	–	2
	$\frac{1}{10}$	1.00×10^{-2}	2.10	5	1.00×10^{-2}	2.10	5
	$\frac{1}{20}$	2.41×10^{-3}	2.06	11	2.41×10^{-3}	2.06	14
	$\frac{1}{40}$	5.85×10^{-4}	2.04	30	5.85×10^{-4}	2.04	45
	$\frac{1}{80}$	1.40×10^{-4}	2.06	95	1.40×10^{-4}	2.06	150

Table 3: The errors and CPU time (seconds) of the D'Yakonov ADI scheme and the P-R ADI scheme for Example 6.2.

Table 3 describes the error and CPU time of the proposed scheme at final time $T = 1$ and $h = \frac{1}{20}$. It is clear to see that the two schemes produce the same accuracy for the same temporal grid size, while the D'Yakonov ADI scheme needs less CPU time. Hence, one can conclude that the D'Yakonov ADI scheme is more efficient than the P-R ADI scheme. From Table 3, this fact is extracted that the computational orders of our schemes is independent on the fractional order α.

h	$e(\tau,h)$	$Rate_{II}$
$\frac{1}{4}$	4.05×10^{-3}	–
$\frac{1}{8}$	2.48×10^{-4}	4.03
$\frac{1}{16}$	1.52×10^{-5}	4.03
$\frac{1}{32}$	8.26×10^{-7}	4.20

Table 4: Numerical convergence orders in spatial direction with $\alpha = 1.5$ and $\tau = \frac{1}{2000}$ for Example 6.2.

From Table 4, this fact is extracted that the numerical convergence order in spatial direction is very close to the theoretical order. Figures 3 and 4 show plots of exact solution, numerical solution and absolute error with different values $\alpha = 1.25, 1.5$ and 1.75 for Example 6.2, where $\tau = \frac{1}{50}$ and $h = \frac{1}{50}$.

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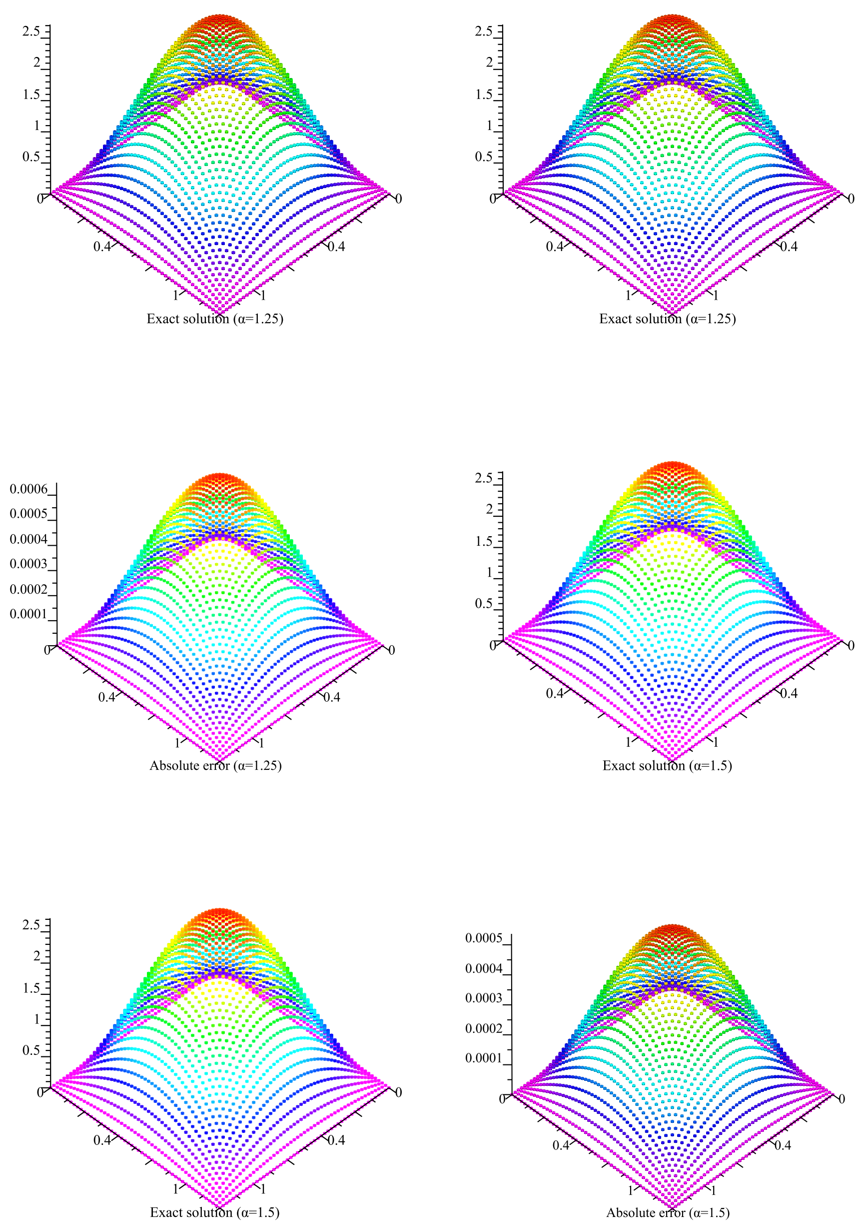

FIGURE 3. Plots of the exact solution, numerical solution and absolute error at $T = 1$ with $h = \frac{1}{50}, \tau = \frac{1}{50}$ and $\alpha = 1.25, 1.5$ for Example 6.2.

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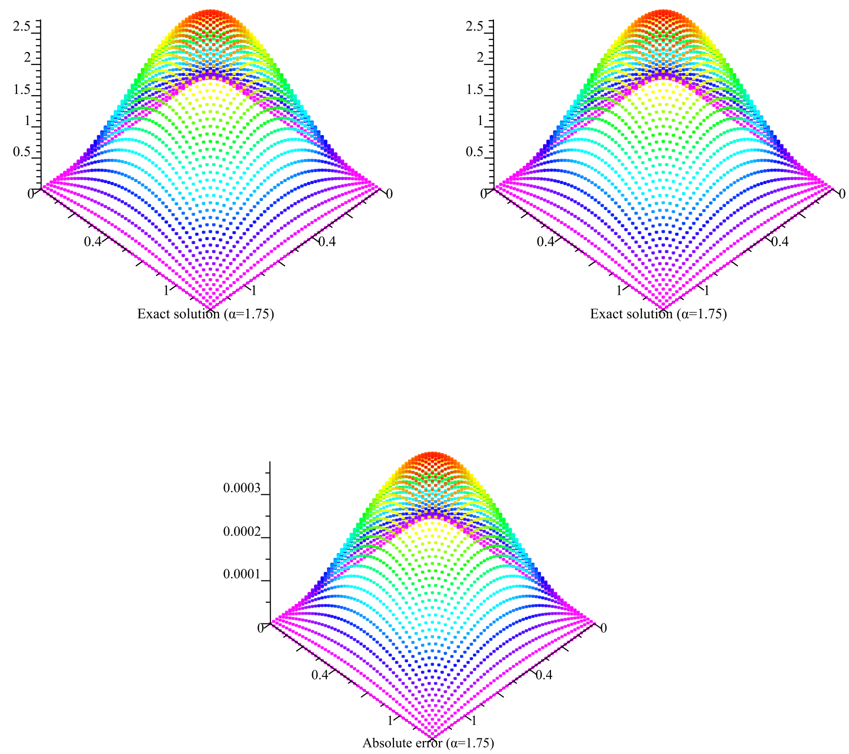

Figure 4. Plots of the exact solution, numerical solution and absolute error at $T = 1$ with $h = \frac{1}{50}, \tau = \frac{1}{50}$ and $\alpha = 1.75$ for Example 6.2.

7. Conclusions

In this paper, we have presented a high-order CFD and two ADI scheme for the solution of fractional Cattaneo equation in one and two-dimensions, respectively. The time fractional derivative has been described in the Caputo-Fabrizio sense. The D'Yakonov ADI scheme decreases CPU time in comparison with the P-R ADI scheme. The solvability, unconditional stability and $\tilde{H}^1$ convergence of the presented scheme have been proved. Numerical results confirm that the presented scheme has approximately $O(h^4)$ in space variables and $O(\tau^2)$ in the fractional time step which are compatible with theoretical results. What distinguishes this paper from our previous studies is its accuracy aspect because the accuracy of the suggested scheme is not dependent on the fractional order α.

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