

On bi–bases of Γ –semihypergroups

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ABSTRACT. This paper focuses on the Γ –semihypergroups. Our goal seeks to find the conditions of sub– Γ –semihypergroup using bi–bases properties. We provide definitions and explain some properties of bi–bases in Γ –semihypergroups. The findings extend the results from bi–bases of Γ –semigroups. The findings demonstrate that if B is a bi–bases of a Γ –semihypergroup H ; then, B is a sub– Γ –semihypergroup of H if and only if for any $b, c \in B$ and $\gamma \in \Gamma$, $b \in b\gamma c$ or $c \in b\gamma c$.

Keywords: Γ –semihypergroup, bi–bases, sub– Γ –semihypergroup.

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1. INTRODUCTION AND PRELIMINARIES

F. Marty [1] created hyperstructure theory. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Fabrici [2, 3] introduced the concepts of one–sided and two–sided bases of a semigroup which were extended to ordered semigroups by T. Changpas and P. Summaprab [9]. Later,

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P. Kummoon and T. Changpas [6] defined and identified some properties of bi-bases in semigroups, and they [7] extended the results to Γ -semigroups. This paper attempts to find the condition of sub- Γ -semihypergroup using bi-bases properties in Γ -semihypergroups and begins by introducing the concept of bi-bases of Γ -semihypergroup and extend the results of bi-bases in Γ -semigroups to Γ -semihypergroups. In this section, the authors begin by recalling terminologies of Γ -semihypergroups as follows:

Let H be a nonempty set. Then, the map $\circ : H \times H \rightarrow P^*(H)$ where $P^*(H)$ is the family of nonempty subset of H . The system (H, \circ) is called a semihypergroup if for every $x, y, z \in H$, $x \circ (y \circ z) = (x \circ y) \circ z$. If A and B are two nonempty subsets of H , then, we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b; \quad x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\} \text{ for all } x \in H.$$

A nonempty subset A of semihypergroup H is called a subsemihypergroup of H if $A \circ A \subseteq A$.

Definition 1.1. [8] Let H and Γ be two nonempty sets. Then, H is called a Γ -semihypergroup if Γ is a set of hyperoperation on H and for every $\alpha, \beta \in \Gamma$, $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in H$.

If A and B are two nonempty subsets of H , we denote

$$A\Gamma B = \bigcup_{\gamma \in \Gamma} A\gamma B = \cup \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

Let (H, \circ) be a semihypergroup and $\Gamma = \{\circ\}$. Then, H is a Γ -semihypergroup. Clearly, every semihypergroup is a Γ -semihypergroup.

Definition 1.2. [5] Let H be a Γ -semihypergroup. A nonempty subset A of H is called a sub- Γ -semihypergroup of H if $A\Gamma A \subseteq A$. A sub- Γ -semihypergroup A of H is called a bi- Γ -hyperideal of H if $A\Gamma H\Gamma A \subseteq A$.

Proposition 1.3. [8] Let H be a Γ -semihypergroup and B_i be a bi- Γ -hyperideal of H for any $i \in I$. If $\bigcap_{i \in I} B_i \neq \emptyset$; then, $\bigcap_{i \in I} B_i$ is a bi- Γ -hyperideal of H .

Let A be a nonempty subset of a Γ -semihypergroup H and define the set of all bi- Γ -hyperideal of H containing A as follows:

$$K = \{B \mid B \text{ is a bi-}\Gamma\text{-hyperideal of } H \text{ containing } A\}.$$

Clearly, $K \neq \emptyset$, because $H \in K$. Suppose $(A)_b = \bigcap_{B \in K} B$. This indicates seen that $A \subseteq (A)_b$. By proposition 1.3, $(A)_b$ is a bi- Γ -hyperideal of H . Moreover, $(A)_b$ is the smallest bi- Γ -hyperideal of H containing A .

Proposition 1.4. Let A be a nonempty subset of a Γ -semihypergroup H . Then,

$$(A)_b = A \cup A\Gamma A \cup A\Gamma H\Gamma A$$

Proof. Suppose $B = A \cup A\Gamma A \cup A\Gamma H\Gamma A$. Clearly, $A \subseteq B$. Consider $B\Gamma B = (A \cup A\Gamma A \cup A\Gamma H\Gamma A)\Gamma(A \cup A\Gamma A \cup A\Gamma H\Gamma A) \subseteq A\Gamma A \cup A\Gamma H\Gamma A \subseteq B$. Hence, B is a sub- Γ -semihypergroup of H containing A . Consider $B\Gamma H\Gamma B = (A \cup A\Gamma A \cup A\Gamma H\Gamma A)\Gamma H\Gamma(A \cup A\Gamma A \cup A\Gamma H\Gamma A) \subseteq A\Gamma H\Gamma A \subseteq B$. Therefore, B is a bi- Γ -hyperideal of H containing A . Let C be any bi- Γ -hyperideal of H containing A . Thus, $A \subseteq C$. Since C is a sub- Γ -semihypergroup of H ; so, $A\Gamma A \subseteq C\Gamma C \subseteq C$. $A\Gamma H\Gamma A \subseteq C\Gamma H\Gamma C \subseteq C$ because C is bi- Γ -hyperideal of H . Hence, $B = A \cup A\Gamma A \cup A\Gamma H\Gamma A \subseteq C$. Thus, B is the smallest bi- Γ -hyperideal of H containing A . Therefore, $(A)_b = A \cup A\Gamma A \cup A\Gamma H\Gamma A$. \square

$(A)_b$ is called the bi- Γ -hyperideal of H generated by A .

Definition 1.5. Let H be a Γ -semihypergroup. A nonempty subset B of H is called a bi-bases of H if it satisfies the following two conditions.

1. $H = (B)_b$ (i.e., $H = B \cup B\Gamma B \cup B\Gamma H\Gamma B$).
2. If A is a nonempty subset of B such that $H = (A)_b$; then, $A = B$.

EXAMPLE 1.6. Let $H = \{x, y, z, w\}$ and $\Gamma = \{\beta, \alpha\}$ be the sets of hyperoperations defined below

β	x	y	z	w	α	x	y	z	w
x	$\{x\}$	$\{x, y\}$	$\{z, w\}$	$\{w\}$	x	$\{x, y\}$	$\{x, y\}$	$\{z, w\}$	$\{w\}$
y	$\{x, y\}$	$\{x, y\}$	$\{z, w\}$	$\{w\}$	y	$\{x, y\}$	$\{y\}$	$\{z, w\}$	$\{w\}$
z	$\{z, w\}$	$\{z, w\}$	$\{z\}$	$\{w\}$	z	$\{z, w\}$	$\{z, w\}$	$\{z\}$	$\{w\}$
w	$\{w\}$	$\{w\}$	$\{w\}$	$\{w\}$	w	$\{w\}$	$\{w\}$	$\{w\}$	$\{w\}$

N. Yaqoob [4] showed that H is a Γ -semihypergroup. Consider $(A)_1 = \{x\}$ and $(A)_2 = \{y\}$; so, $(A)_1$ and $(A)_2$ are bi-bases of H .

2. MAIN RESULTS

In this section, we characterize bi-bases of Γ -semihypergroups and show conditions of sub- Γ -semihypergroup using bi-bases properties.

Lemma 2.1. Let B be a bi-bases of a Γ -semihypergroup H and $a, b \in B$. If $a \in b\Gamma b \cup b\Gamma H\Gamma b$; then, $a = b$.

Proof. Let $a, b \in B$. Suppose $a \in b\Gamma b \cup b\Gamma H\Gamma b$ and $a \neq b$. Setting $A = B \setminus \{a\}$; then, $A \subseteq B$. From $a \neq b$, so $b \in A$. Hence, $(A)_b \subseteq (B)_b = H$. Let $x \in H = (B)_b$. Then, $x \in B \cup B\Gamma B \cup B\Gamma H\Gamma B$. There are three cases to consider.

Case 1: $x \in B$.

Subcase 1.1: $x \neq a$. Then, $x \in B \setminus \{a\} = A \subseteq (A)_b$.

Subcase 1.2: $x = a$. So, $x = a \in b\Gamma b \cup b\Gamma H\Gamma b \subseteq A\Gamma A \cup A\Gamma H\Gamma A \subseteq (A)_b$.

Case 2 : $x \in B\Gamma B$. Hence, $x \in b_1\gamma b_2$ for some $b_1, b_2 \in B$ and $\gamma \in \Gamma$.

Subcase 2.1: $b_1 = a$ and $b_2 = a$. By assumption,

$$\begin{aligned}
 x \in b_1\gamma b_2 &= a\gamma a \\
 &\subseteq (b\Gamma b \cup b\Gamma H\Gamma b)\Gamma(b\Gamma b \cup b\Gamma H\Gamma b) \\
 &= b\Gamma b\Gamma b \cup b\Gamma b\Gamma b\Gamma H\Gamma b \cup b\Gamma H\Gamma b\Gamma b \cup b\Gamma H\Gamma b\Gamma b\Gamma H\Gamma b \\
 &\subseteq A\Gamma A\Gamma A \cup A\Gamma A\Gamma A\Gamma A \cup A\Gamma H\Gamma A\Gamma A \cup A\Gamma H\Gamma A\Gamma A\Gamma H\Gamma A \\
 &\subseteq A\Gamma H\Gamma A \subseteq (A)_b
 \end{aligned}$$

Subcase 2.2: $b_1 \neq a$ and $b_2 = a$. By assumption and $A = B \setminus \{a\}$,

$$\begin{aligned}
 x \in b_1\gamma b_2 &= b_1\gamma a \subseteq (B \setminus \{a\})\Gamma(b\Gamma b \cup b\Gamma H\Gamma b) \\
 &= (B \setminus \{a\})\Gamma b\Gamma b \cup (B \setminus \{a\})\Gamma b\Gamma H\Gamma b \\
 &\subseteq A\Gamma A\Gamma A \cup A\Gamma A\Gamma H\Gamma A \subseteq A\Gamma H\Gamma A \subseteq (A)_b
 \end{aligned}$$

Subcase 2.3: $b_1 = a$ and $b_2 \neq a$. By assumption and $A = B \setminus \{a\}$; then,

$$\begin{aligned}
 x \in b_1\gamma b_2 &= a\gamma b_2 \subseteq (b\Gamma b \cup b\Gamma H\Gamma b)\Gamma(B \setminus \{a\}) \\
 &= b\Gamma b\Gamma(B \setminus \{a\}) \cup b\Gamma H\Gamma b\Gamma(B \setminus \{a\}) \\
 &\subseteq A\Gamma A\Gamma A \cup A\Gamma A\Gamma H\Gamma A \subseteq A\Gamma H\Gamma A \subseteq (A)_b.
 \end{aligned}$$

Subcase 2.4: $b_1 \neq a$ and $b_2 \neq a$. From $A = B \setminus \{a\}$, so $x \in b_1\gamma b_2 \subseteq (B \setminus \{a\})\Gamma(B \setminus \{a\}) = A\Gamma A \subseteq (A)_b$.

Case 3: $x \in B\Gamma H\Gamma B$. Hence, $x \in b_3\gamma_1 h\gamma_2 b_4$ for some $b_3, b_4 \in B$, $\gamma_1, \gamma_2 \in \Gamma$ and $h \in H$.

Subcase 3.1: $b_3 = a$ and $b_4 = a$, consider

$$\begin{aligned}
 x \in b_3\gamma_1 h\gamma_2 b_4 &= a\gamma_1 h\gamma_2 a \\
 &= (b\Gamma b \cup b\Gamma H\Gamma b)\Gamma H\Gamma(b\Gamma b \cup b\Gamma H\Gamma b) \\
 &= b\Gamma b\Gamma H\Gamma b \cup b\Gamma b\Gamma H\Gamma b\Gamma H\Gamma b \cup b\Gamma H\Gamma b\Gamma H\Gamma b\Gamma b \cup b\Gamma H\Gamma b\Gamma H\Gamma b\Gamma b \\
 &\subseteq A\Gamma A\Gamma H\Gamma A \cup A\Gamma A\Gamma A\Gamma H\Gamma A \cup A\Gamma H\Gamma A\Gamma H\Gamma A \cup A\Gamma H\Gamma A\Gamma H\Gamma A \\
 &\subseteq A\Gamma H\Gamma A \subseteq (A)_b.
 \end{aligned}$$

Subcase 3.2: $b_3 \neq a$ and $b_4 = a$. By assumption and $A = B \setminus \{a\}$; so,

$$\begin{aligned}
 x \in b_3\gamma_1 h\gamma_2 b_4 &= b_3\gamma_1 h\gamma_2 a \\
 &\subseteq (B \setminus \{a\})\Gamma H\Gamma(b\Gamma b \cup b\Gamma H\Gamma b) \\
 &= (B \setminus \{a\})\Gamma H\Gamma b\Gamma b \cup (B \setminus \{a\})\Gamma H\Gamma b\Gamma H\Gamma b \\
 &\subseteq A\Gamma H\Gamma A\Gamma A \cup A\Gamma H\Gamma A\Gamma H\Gamma A \subseteq A\Gamma H\Gamma A \subseteq (A)_b.
 \end{aligned}$$

Subcase 3.3 : $b_3 = a$ and $b_4 \neq a$. By assumption and $A = B \setminus \{a\}$; hence,

$$\begin{aligned} x \in b_3\gamma_1h\gamma_2b_4 &= a\gamma_1h\gamma_2b_4 \\ &\subseteq (b\Gamma b \cup b\Gamma H\Gamma b)\Gamma H\Gamma(B \setminus \{a\}) \\ &= b\Gamma b\Gamma H\Gamma(B \setminus \{a\}) \cup b\Gamma H\Gamma b\Gamma H\Gamma(B \setminus \{a\}) \\ &\subseteq A\Gamma A\Gamma H\Gamma A \cup A\Gamma H\Gamma A\Gamma H\Gamma A \subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.4: $b_3 \neq a$ and $b_4 \neq a$. From $A = B \setminus \{a\}$; then, $x \in b_3\gamma_1h\gamma_2b_4 \subseteq (B \setminus \{a\})\Gamma H\Gamma(B \setminus \{a\}) = A\Gamma H\Gamma A \subseteq (A)_b$. Thus, $(A)_b = H$. This is a contradiction. Therefore, $a = b$. \square

Lemma 2.2. *Let B be a bi-bases of a Γ -semihypergroup H and $a, b, c \in B$. If $a \in c\Gamma b \cup c\Gamma H\Gamma b$, then $a = b$ or $a = c$.*

Proof. Let $a, b, c \in B$ and $h \in H$. Assume $a \in c\Gamma b \cup c\Gamma H\Gamma b$ such that $a \neq b$ and $a \neq c$. Setting $A = (B \setminus \{a\})$. Hence, $A \subseteq B$, then $(A)_b \subseteq (B)_b = H$. From $a \neq b$ and $a \neq c$; so, $b, c \in A$. Let $x \in H$. Since $(B)_b = H$; thus, $x \in B \cup B\Gamma B \cup B\Gamma H\Gamma B$. There are three cases to consider.

Case 1: $x \in B$.

Subcase 1.1: $x \neq a$. Then, $x \in B \setminus \{a\} = A \subseteq (A)_b$.

Subcase 1.2: $x = a$. Thus, $x = a \in c\Gamma b \cup c\Gamma H\Gamma b \subseteq A\Gamma A \cup A\Gamma H\Gamma A \subseteq (A)_b$.

Case 2: $x \in B\Gamma B$. Then, $x \in b_1\gamma b_2$ for some $b_1, b_2 \in B$ and $\gamma \in \Gamma$.

Subcase 2.1: $b_1 = a$ and $b_2 = a$. By assumption,

$$x \in b_1\gamma b_2 = a\gamma a \subseteq (c\Gamma b \cup c\Gamma H\Gamma b)\Gamma(c\Gamma b \cup c\Gamma H\Gamma b) \subseteq A\Gamma H\Gamma A \subseteq (A)_b.$$

Subcase 2.2: $b_1 \neq a$ and $b_2 = a$. By assumption and $A = B \setminus \{a\}$; then,

$$x \in b_1\gamma b_2 = b_1\gamma a \subseteq (B \setminus \{a\})\Gamma(c\Gamma b \cup c\Gamma H\Gamma b) \subseteq A\Gamma H\Gamma A \subseteq (A)_b.$$

Subcase 2.3: $b = a$ and $b \neq a$. By assumption and $A = B \setminus \{a\}$; so,

$$x \in b_1\gamma b_2 = a\gamma b_2 \subseteq (c\Gamma b \cup c\Gamma H\Gamma b)\Gamma(B \setminus \{a\}) \subseteq A\Gamma H\Gamma A \subseteq (A)_b.$$

Subcase 2.4: $b_1 \neq a$ and $b_2 \neq a$. By assumption and $A = B \setminus \{a\}$; thus,

$$x \in b_1\gamma b_2 \subset (B \setminus \{a\})\Gamma(B \setminus \{a\}) = A\Gamma A \subseteq (A)_b.$$

Case 3: $x \in B\Gamma H\Gamma B$. Hence, $x \in b_3\gamma_1h\gamma_2b_4$ for some $b_3, b_4 \in B$, $\gamma_1, \gamma_2 \in \Gamma$ and $h \in H$.

Subcase 3.1: $b_3 = a$ and $b_4 = a$. Then,

$$x \in b_3\gamma_1h\gamma_2b_4 = a\gamma_1h\gamma_2a \subseteq (c\Gamma b \cup c\Gamma H\Gamma b)\Gamma H\Gamma(c\Gamma b \cup c\Gamma H\Gamma b) \subseteq A\Gamma H\Gamma A \subseteq (A)_b.$$

Subcase 3.2: $b_3 \neq a$ and $b_4 = a$. By assumption and $A = B \setminus \{a\}$; so,

$$x \in b_3\gamma_1h\gamma_2b_4 = b_3\gamma_1h\gamma_2a \subseteq (B \setminus \{a\})\Gamma H\Gamma(c\Gamma b \cup c\Gamma H\Gamma b) \subseteq A\Gamma H\Gamma A \subseteq (A)_b.$$

Subcase 3.3: $b_3 = a$ and $b_4 \neq a$. By assumption and $A = B \setminus \{a\}$; thus,

$$x \in b_3\gamma_1h\gamma_2b_4 = a\gamma_1h\gamma_2b_4 \subseteq (c\Gamma b \cup c\Gamma H\Gamma b)\Gamma H\Gamma(B \setminus \{a\}) \subseteq A\Gamma H\Gamma A \subseteq (A)_b.$$

Subcase 3.4: $b_3 \neq a$ and $b_4 \neq a$. From $A = B \setminus \{a\}$; then,

$$x \in b_3\gamma_1h\gamma_2b_4 \in (B \setminus \{a\})\Gamma H\Gamma(B \setminus \{a\}) = A\Gamma H\Gamma A \subseteq (A)_b.$$

From all cases, $(A)_b = H$. This is a contradiction. Therefore, $a = b$ or $a = c$. \square

Definition 2.3. Let H be a Γ -semihypergroup. Define a quasi-order on H by, for any $a, b \in H$, $a \leq_b b \Leftrightarrow (a)_b \subseteq (b)_b$.

In example 1.6, $A_1 = \{x\}$ and $A_2 = \{y\}$ are bi-bases of H . Since $(x)_b \subseteq (y)_b$, so $x \leq_b y$ and since $(y)_b \subseteq (x)_b$, then $y \leq_b x$. From $x \leq_b y$ and $y \leq_b x$, but $x \neq y$. Therefore, \leq_b is not a partial order on H . This shows that the order \leq_b defined above is not, in general, a partial order.

Lemma 2.4. Let B be a bi-bases of a Γ -semihypergroup H and $a, b \in B$. If $a \neq b$, then neither $a \leq_b b$ or $b \leq_b a$.

Proof. Let $a, b \in B$. Suppose $a \neq b$. If $a \leq_b b$; hence, $(a)_b \subseteq (b)_b$. Thus, $a \in (a)_b \subseteq (b)_b = \{b\} \cup b\Gamma b \cup b\Gamma H\Gamma b$. By Lemma 2.1, $a = b$. This is a contradiction. If $b \leq_b a$, can be proved similarly. \square

Lemma 2.5. Let B be a bi-bases of a Γ -semihypergroup H . For all $a, b, c \in B$, $\gamma_1, \gamma_2 \in \Gamma$ and $h \in H$.

1. If $a \in b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c$; then, $a = b$ or $a = c$.
2. If $a \in b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma H\Gamma b\gamma_1 h\gamma_2 c$; then, $a = b$ or $a = c$.

Proof. Let $a, b, c \in B$, $\gamma \in \Gamma$ and $h \in H$.

(1.) Assume $a \in b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c$ such that $a \neq b$ and $a \neq c$. Consider $A = B \setminus \{a\}$. Clearly, $A \subseteq B$, thus $(A)_b \subseteq (B)_b = H$. From $a \neq b$ and $a \neq c$; so, $b, c \in A$. Let $x \in H$. Since $(B)_b = H$, so $x \in B \cup B\Gamma B \cup B\Gamma H\Gamma B$. There are three cases to consider.

Case 1: $x \in B$.

Subcase 1.1: $x \neq a$. Then, $x \in B \setminus \{a\} = A \subseteq (A)_b$.

Subcase 1.2: $x = a$. By assumption, so

$$x = a \in b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c \subseteq A\Gamma A \cup A\Gamma H\Gamma A \subseteq (A)_b.$$

Case 2: $x \in B\Gamma B$. Thus, $x \in b_1\gamma b_2$ for some $b_1, b_2 \in B$ and $\gamma \in \Gamma$.

Subcase 2.1 : $b_1 = a$ and $b_2 = a$. By assumption, then

$$x \in b_1\gamma b_2 = a\gamma a$$

$$\subseteq (b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c)\Gamma(b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c) \\ \subseteq A\Gamma H\Gamma A \subseteq (A)_b$$

Subcase 2.2: $b_1 \neq a$ and $b_2 = a$. By assumption and $A = B \setminus \{a\}$; then,

$$x \in b_1\gamma b_2 = b_1\gamma a \subseteq (B \setminus \{a\})\Gamma(b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c) \\ \subseteq A\Gamma H\Gamma A \subseteq (A)_b.$$

Subcase 2.3: $b_1 = a$ and $b_2 \neq a$. By assumption and $A = B \setminus \{a\}$; so,

$$x \in b_1\gamma b_2 = a\gamma b_2 \subseteq (b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c)\Gamma(B \setminus \{a\}) \\ \subseteq A\Gamma H\Gamma A \subseteq (A)_b.$$

Subcase 2.4: $b_1 \neq a$ and $b_2 \neq a$. By assumption and $A = B \setminus \{a\}$; thus, $x \subseteq b_1\gamma b_2 \subseteq (B \setminus \{a\})\Gamma(B \setminus \{a\}) = A\Gamma A \subseteq (A)_b$.

Case 3: $x \in B\Gamma H\Gamma B$. Hence, $x \in b_3\gamma_1 h\gamma_2 b_4$ for some $b_3, b_4 \in B$ and for some $\gamma_1, \gamma_2 \in \Gamma$ and for some $h \in H$.

Subcase 3.1: $b_3 = a$ and $b_4 = a$. By assumption, then

$$\begin{aligned} x &\in b_3\gamma_1 h\gamma_2 b_4 \\ &= a\gamma_1 h\gamma_2 a \\ &\subseteq (b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c)\Gamma H\Gamma(b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.2: $b_3 \neq a$ and $b_4 = a$. By assumption and $A = B \setminus \{a\}$; so,

$$\begin{aligned} x &\in b_3\gamma_1 h\gamma_2 b_4 = b_3\gamma_1 h\gamma_2 a \subseteq (B \setminus \{a\})\Gamma H\Gamma(b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.3: $b_3 = a$ and $b_4 \neq a$. By assumption and $A = B \setminus \{a\}$; thus,

$$\begin{aligned} x &\in b_3\gamma_1 h\gamma_2 b_4 = b_3\gamma_1 h\gamma_2 a \subseteq (b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c)\Gamma H\Gamma(B \setminus \{a\}) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.4: $b_3 \neq a$ and $b_4 \neq a$. By assumption and $A = B \setminus \{a\}$; hence, $x \in b_3\gamma b_4 \subseteq (B \setminus \{a\})\Gamma(B \setminus \{a\}) = A\Gamma A \subseteq (A)_b$.

From all cases, $(A)_b = H$. This is a contradiction. Therefore, $a = b$ and $a = c$.

(2.) Assume $a \in b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma H\Gamma b\gamma_1 h\gamma_2 c$ such that $a \neq b$ and $a \neq c$. Setting $A = B \setminus \{a\}$. Then, $A \subseteq B$. Thus, $(A)_b \subseteq (B)_b = H$. From $a \neq b$ and $a \neq c$; so, $b, c \in A$. Let $x \in H$. Since $(B)_b = H$; then, $x \in B \cup B\Gamma B \cup B\Gamma H\Gamma B$. There are three cases to consider.

Case 1: $x \in B$.

Subcase 1.1: $x \neq a$. Then, $x \in B \setminus \{a\} = A \subseteq (A)_b$.

Subcase 1.2: $x = a$. By assumption, then

$$x = a \in b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma H\Gamma b\gamma_1 h\gamma_2 c \subseteq A\Gamma H\Gamma A \subseteq (A)_b.$$

Case 2: $x \in B\Gamma B$. Thus, $x \in b_1\gamma b_2$ for some $b_1, b_2 \in B$ and $\gamma \in \Gamma$.

Subcase 2.1: $b_1 = a$ and $b_2 = a$. By assumption, thus

$$\begin{aligned} x &\in b_1\gamma b_2 = a\gamma a \\ &\subseteq (b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma H\Gamma b\gamma_1 h\gamma_2 c)\Gamma \\ &\quad (b\gamma_1 h\gamma_2 c \cup a\gamma b\gamma_1 h\gamma_2 c\Gamma a\gamma b\gamma_1 h\gamma_2 c \cup a\gamma b\gamma_1 h\gamma_2 c\Gamma H\Gamma a\gamma b\gamma_1 h\gamma_2 c) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 2.2: $b_1 \neq a$ and $b_2 = a$. By assumption and $A = B \setminus \{a\}$; so,

$$\begin{aligned} x &\in b_1\gamma b_2 = b_1\gamma a \\ &\subseteq (B \setminus \{a\})\Gamma(b\gamma_1 h\gamma_2 c \cup a\gamma b\gamma_1 h\gamma_2 c\Gamma a\gamma b\gamma_1 h\gamma_2 c \cup a\gamma b\gamma_1 h\gamma_2 c\Gamma H\Gamma a\gamma b\gamma_1 h\gamma_2 c) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 2.3: $b_1 = a$ and $b_2 \neq a$. By assumption $A = B \setminus \{a\}$; hence,

$$\begin{aligned} x \in b_1\gamma b_2 &= a\gamma b_2 \\ &\subseteq (b\gamma_1 h\gamma_2 c \cup a\gamma b\gamma_1 h\gamma_2 c \Gamma b\gamma_1 h\gamma_2 c \cup a\gamma b\gamma_1 h\gamma_2 c \Gamma H\Gamma a\gamma b\gamma_1 h\gamma_2 c) \Gamma (B \setminus \{a\}) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 2.4: $b_1 \neq a$ and $b_2 \neq a$. From $A = B \setminus \{a\}$; then,

$$x \in b_1\gamma b_2 \in (B \setminus \{a\})\Gamma(B \setminus \{a\}) = A\Gamma A \subseteq (A)_b.$$

Case 3: $x \in B\Gamma H\Gamma B$. Hence, $x \in b_3\gamma h\gamma_2 b_4$ for some $b_3, b_4 \in B$, $\gamma_1, \gamma_2 \in \Gamma$ and $h \in H$.

Subcase 3.1: $b_1 = a$ and $b_2 = a$. By assumption, so

$$\begin{aligned} x \in b_3\gamma_1 h\gamma_2 b_4 &= a\gamma_1 h\gamma_2 a \\ &\subseteq (b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c \Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c \Gamma H\Gamma b\gamma_1 h\gamma_2 c) \Gamma H\Gamma \\ &\quad (b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c \Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c \Gamma H\Gamma b\gamma_1 h\gamma_2 c) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.2: $b_3 \neq a$ and $b_4 = a$. By assumption and $A = B \setminus \{a\}$; thus,

$$\begin{aligned} x \in b_3\gamma_1 h\gamma_2 b_4 &= b_3\gamma h\gamma_2 a \\ &\subseteq (B \setminus \{a\})\Gamma H\Gamma (b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c \Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c \Gamma H\Gamma b\gamma_1 h\gamma_2 c) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.3: $b_3 = a$ and $b_4 \neq a$. By assumption $A = B \setminus \{a\}$; hence,

$$\begin{aligned} x \in b_3\gamma h\gamma_2 b_4 &= a\gamma_1 h\gamma_2 b_4 \\ &\subseteq (b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c \Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c \Gamma H\Gamma b\gamma_1 h\gamma_2 c) \Gamma H\Gamma (B \setminus \{a\}) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.4: $b_3 \neq a$ and $b_4 \neq a$. From $A = B \setminus \{a\}$; then,

$$x \in b_3\gamma_1 h\gamma_2 b_4 \in (B \setminus \{a\})\Gamma H\Gamma (B \setminus \{a\}) = A\Gamma H\Gamma A \subseteq (A)_b.$$

From all cases, $(A)_b = H$. This is a contradiction. Therefore, $a = b$ or $a = c$. \square

Lemma 2.6. *Let B be a bi-bases of a Γ -semihypergroup H .*

1. *For any $a, b, c \in B, \gamma_1 \in \Gamma$, if $a \neq b$ and $a \neq c$; then, $a \not\leq_b b\gamma_1 c$.*
2. *For any $a, b, c \in B, \gamma_2, \gamma_3 \in \Gamma$ and $h \in H$, if $a \neq b$ and $a \neq c$; then, $a \not\leq_b b\gamma_2 h\gamma_3 c$.*

Proof. (1.) Assume $a \neq b$ and $a \neq c$. Suppose $a \leq_b b\gamma_1 c$, thus $(a)_b \subseteq (b\gamma_1 c)_b$. Hence, $a \in (a)_b \subseteq (b\gamma_1 c)_b = b\gamma_1 c \cup b\gamma_1 c \Gamma b\gamma_1 c \cup b\gamma_1 c \Gamma H\Gamma b\gamma_1 c$. By Lemma 2.5(1.), it follows $a = b$ and $a = c$. This contradicts the assumption.

(2.) Assume $a \neq b$ and $a \neq c$. Suppose $a \leq_b b\gamma_2 h\gamma_3 c$, then $(a)_b \subseteq (b\gamma_2 h\gamma_3 c)_b$. Thus, $a \in (a)_b \subseteq (b\gamma_2 h\gamma_3 c)_b = b\gamma_2 h\gamma_3 c \cup b\gamma_2 h\gamma_3 c \Gamma b\gamma_2 h\gamma_3 c \cup b\gamma_2 h\gamma_3 c \Gamma H\Gamma b\gamma_2 h\gamma_3 c$. By Lemma 2.5(2.), it follows $a = b$ or $a = c$. This contradicts the assumption. \square

Theorem 2.7. *A nonempty subset B of a Γ -semihypergroup H is a bi-bases of H if and only if B satisfies the following conditions.*

1. For any $x \in H$,
 - 1.1. there exists $b \in B$ such that $x \leq_b b$ or
 - 1.2. there exists $b_1, b_2 \in B$ and $\gamma \in \Gamma$ such that $x \leq_b b_1\gamma b_2$ or
 - 1.3. there exists $b_3, b_4 \in B$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $x \leq_b b_3\gamma_1 h\gamma_2 b_4$.
2. For any $a, b, c \in B$ and $\gamma_1 \in \Gamma$, if $a \neq b$ and $a \neq c$; then, $a \not\leq_b b\gamma_1 c$.
3. For any $a, b, c \in B, \gamma_2, \gamma_3 \in \Gamma$ and $h \in H$, if $a \neq b$ and $a \neq c$; then, $a \not\leq_b b\gamma_2 h\gamma_3 c$.

Proof. Assume B is a bi-bases of H . Then, $H = (B)_b$. To show that (1.) holds, let $x \in H$. Thus, $x \in B \cup B\Gamma B \cup B\Gamma H\Gamma B$. We consider three cases.

case 1: $x \in B$. Then, $x = b$ for some $b \in B$. This implies $(x)_b \subseteq (b)_b$. Therefore, $x \leq_b b$.

case 2: $x \in B\Gamma B$. Then, $x \in b_1\gamma b_2$ for some $b_1, b_2 \in B$ and $\gamma \in \Gamma$. This implies $(x)_b \subseteq (b_1\gamma b_2)_b$. Hence, $x \leq_b b_1\gamma b_2$.

case 3: $x \in B\Gamma H\Gamma B$. Then, $x \in b_3\gamma_1 h\gamma_2 b_4$ for some $b_3, b_4 \in B, h \in H$ and $\gamma_1, \gamma_2 \in \Gamma$. This implies $(x)_b \subseteq (b_3\gamma_1 h\gamma_2 b_4)_b$. Hence, $x \leq_b b_3\gamma_1 h\gamma_2 b_4$.

The validity of (2.) and (3.) follow from Lemma 2.6(1.) and Lemma 2.6(2.), respectively. Conversely, we will show that B is a bi-bases of H . Clearly, $(B)_b \subseteq H$. Let $x \in H$. By assumption, there exists $b \in B$ such that $x \leq_b b$, thus $(x)_b \subseteq (b)_b$. Thus, $x \in (x)_b \subseteq (b)_b = b \cup b\Gamma b \cup b\Gamma H\Gamma b \subseteq B \cup B\Gamma B \cup B\Gamma H\Gamma B = (B)_b$. Then, $H \subseteq (B)_b$. Hence, $H = (B)_b$. Suppose $H = (A)_b$. Since $A \subset B$, there exists $b \in B \setminus A$. Since $b \in B \subseteq H = (A)_b$, so $b \in (A)_b$. Thus, $b \in A \cup A\Gamma A \cup A\Gamma H\Gamma A$. Since $b \notin A$, we have $b \in A\Gamma A \cup A\Gamma H\Gamma A$. There are two cases to consider.

case 1: $b \in A\Gamma A$. Thus, $b \in a_1\gamma_1 a_2$ for some $a_1, a_2 \in A$ and $\gamma_1 \in \Gamma$. From $A \subseteq B$, so $a_1, a_2 \in B$. Since $b \notin A$; hence, $b \neq a_1$ and $b \neq a_2$. From $b \in a_1\gamma_1 a_2$; then, $(b)_b \subseteq (a_1\gamma_1 a_2)_b$. Hence, $b \leq_b a_1\gamma_1 a_2$. This contradicts to (2).

case 2: $b \in A\Gamma H\Gamma A$. Hence, $b \in a_3\gamma_2 h\gamma_3 a_4$ for some $a_3, a_4 \in A, h \in H$ and $\gamma_2, \gamma_3 \in \Gamma$. From $A \subset B$, so $a_3, a_4 \in B$. Since $b \notin A$, then $b \neq a_3$ and $b \neq a_4$. From $b \in a_3\gamma_2 h\gamma_3 a_4$, so $(b)_b \subseteq (a_3\gamma_2 h\gamma_3 a_4)_b$. Therefore, $b \leq_b a_3\gamma_2 h\gamma_3 a_4$. This contradicts to (3.). \square

Theorem 2.8. *Let B be a bi-bases of a Γ -semihypergroup H . Then B is a sub- Γ -semihypergroup of H if and only if for any $b, c \in B$ and $\gamma \in \Gamma$, $b \in b\gamma c$ or $c \in b\gamma c$*

Proof. Suppose B is a sub- Γ -semihypergroup of H and $b \notin b\gamma c$ and $c \notin b\gamma c$ for any $b, c \in B$ and $\gamma \in \Gamma$. Assume $a \in b\gamma c$, then $a \neq b$ and $a \neq c$. Hence, $a \in b\gamma c \cup b\gamma c\Gamma b\gamma c \cup b\gamma c\Gamma H\Gamma b\gamma c$. By Lemma 2.5(1.), $a = b$ or $a = c$. This is a contradiction. Conversely, assume $b \in b\gamma c$ or $c \in b\gamma c$ for any $b, c \in B$. We will show that B is a sub- Γ -semihypergroup of H . Let $a \in B\Gamma B$. Hence, $a \in b\gamma c$ for some $b, c \in B$ and $\gamma \in \Gamma$. This implies $a \in b\gamma c \cup b\gamma c\Gamma b\gamma c \cup b\gamma c\Gamma H\Gamma b\gamma c$.

By Lemma 2.5(1.), $a = b$ or $a = c$. Hence, $a \in \{b, c\} \subseteq B$. Therefore B is a sub- Γ -semihypergroup of H . \square

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