

On A Class of Soc-Injective Modules

Akeel Ramadan Mehdi

Department of Mathematics, College of Education,
University of Al-Qadisiyah, Al-Diwaniyah City, Iraq

E-mail: akeel.mehdi@qu.edu.iq

ABSTRACT. Let R be a ring. The class of SA -injective right R -modules (SAI_R) is introduced as a class of soc-injective right R -modules. Let N be a right R -module. A right R -module M is said to be SA - N -injective if every R -homomorphism from a semi-artinian submodule of N into M extends to N . A module M is called SA -injective, if M is SA - R -injective. We characterize rings over which every right module is SA -injective. Conditions under which the class SAI_R is closed under quotient (resp. direct sums, pure homomorphic images) are given. The definability of the class SAI_R is studied. Finally, relations between SA -injectivity and certain generalizations of injectivity are given.

Keywords: Semi-artinian submodule, Definable class, Injective module, Noetherian module, Flat module.

2020 Mathematics subject classification: 16D50, 16D40, 16D10.

1. INTRODUCTION

Throughout R is an associative ring with identity and all modules are unitary R -modules. If not otherwise specified, by a module (resp. homomorphism) we will mean a right R -module (resp. right R -homomorphism). We use $R\text{-Mod}$ (resp. $\text{Mod-}R$) to denote the class of left (resp. right) R -modules. We will use M^* to denote the character module $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of a right module

Received 02 June 2019; Accepted 26 May 2021
©2023 Academic Center for Education, Culture and Research TMU

M . Let $\mathcal{G}$ (resp. $\mathcal{F}$) be a class of right (resp. left) R -modules. A pair $(\mathcal{F}, \mathcal{G})$ is called almost dual pair if $\mathcal{G}$ is closed under summands and direct products, and for any left R -module M , $M \in \mathcal{F}$ if and only if $M^* \in \mathcal{G}$ [12, p. 66]. An exact sequence $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ of right R -modules is said to be pure if the sequence $0 \rightarrow \text{Hom}_R(N, A) \rightarrow \text{Hom}_R(N, B) \rightarrow \text{Hom}_R(N, C) \rightarrow 0$ is exact, for every finitely presented right R -module N and we called that $\alpha(A)$ is a pure submodule of B [21]. A right R -module M is called FP -injective if every monomorphism $\alpha: M \rightarrow N$ is pure. A right R -module M is called pure injective if M is injective with respect to all pure short exact sequences [21]. If a subclass $\mathcal{G}$ of $\text{Mod-}R$ is closed under pure submodules, direct limits and direct products, then it is called a definable class [16]. We denote by $\text{Soc}(M)$ to the socle of a module M . A right R -module M is called semi-artinian if for any proper submodule N of M we have $\text{Soc}(M/N) \neq 0$ [9, p. 238]. We will denote to the sum of all semi-artinian submodules of a right R -module M by $\text{Sa}(M)$. If N is a submodule of a right R -module M , the notation $N \subseteq^{sa} M$ means that N is a semi-artinian submodule of M .

We refer the reader to [2], [9], [16], [18] and [21], for general background materials.

Injective modules have been studied extensively, and several generalizations for these modules are given, for example, soc-injective modules [1], $\mathcal{L}$ -injective Modules [13], and n - FP -injective modules [5]. If $\text{Ext}^1(R/K, M) = 0$, for any semisimple right ideal K of R , then a right R -module M is called soc-injective [1], where $\text{Ext}^1(A, B)$ is defined as the first right derived functor of $\text{Hom}_R(A, B)$, for any two right R -modules A, B (see [4, Ch. VI] for more details).

In section 2 of this paper, we introduce the class of SA -injective modules. This class of modules lies between injective modules and soc-injective modules. We first give examples to show that the notion of SA -injectivity is distinct from that of injectivity and soc-injectivity. We characterize rings over which every module is SA -injective. We prove the equivalence of the following statements: (1) Every right R -module is SA -injective; (2) Every semi-artinian module is SA -injective; (3) Every semi-artinian right ideal of R is SA -injective; (4) Every semi-artinian right ideal of R is a direct summand of R . Conditions under which the class of SA -injective right R -modules (SAI_R) is closed under quotient are given. For instance, we prove that the equivalence of the following: (1) The class SAI_R is closed under quotient; (2) Sums of any two SA -injective submodules of any right R -module is SA -injective; (3) All semi-artinian right ideals of R are projective. Finally, we give conditions such that the class SAI_R is closed under direct sums. For instance, we prove that the following are equivalent. (1) $\text{Sa}(R_R)$ is noetherian; (2) Any direct sum of SA -injective right R -modules is SA -injective; (3) The class SAI_R is closed under pure submodules; (4) All FP -injective modules are SA -injective.

In section 3, we study the definability of the class SAI_R . It is shown that the following assertions are equivalent: (1) SAI_R is definable; (2) The class SAI_R is closed under pure submodules and pure homomorphic images; (3) Every semi-artinian right ideal in R is finitely presented; (4) A module $M \in SAI_R$ iff $M^* \in (SAI_R)^\ominus$; (5) A module $M \in SAI_R$ iff $M^{**} \in SAI_R$. Finally, we prove that if the class SAI_R is a definable, then the class of flat left R -modules and the class $(SAI_R)^\ominus$ are coincide iff all modules in SAI_R are FP -injective iff all pure-injective modules in SAI_R are injective.

In section 4, we give relations between SA -injectivity and certain generalizations of injectivity (in particular, quasi-injectivity and F -injectivity). Firstly, we prove that a ring R is a right semi-artinian ring iff every SA -injective right R -module is quasi-injective iff every cyclic SA -injective right R -module is quasi-injective. Then, we prove that a commutative ring R is semisimple if and only if R is a semi-artinian ring and every quasi-injective R -module is SA -injective. Also, we prove that $Sa(R_R)$ is a noetherian right R -module if and only if every F -injective right R -module is SA -injective. Finally, we prove that a ring R is a (von Neumann) regular and every P -injective right R -module is SA -injective if and only if every SA -injective right R -module is P -injective and every semi-artinian right ideal of R is a direct summand of R_R .

2. SA -INJECTIVE MODULES

Definition 2.1. Let N be a module. A module M is called SA - N -injective, if for any semi-artinian submodule K of N , any homomorphism $f : K \rightarrow M$ extends to N . M is called SA -injective if M is SA - R -injective. A ring R is called SA -injective if the module R_R is SA -injective.

We will use SAI_R to denote the class of SA -injective right R -modules.

EXAMPLES 2.2. (1) All injective modules are SA -injective. Since 0 is the only semi-artinian right ideal in $\mathbb{Z}$, we have the right $\mathbb{Z}$ -module $\mathbb{Z}$ is a SA -injective but it is not injective. Hence SA -injectivity is a proper generalization of injectivity.

(2) Since every semisimple module is semi-artinian, we have every SA -injective module is soc-injective. The converse is not true in general, for example: let $R = \mathbb{Z}_2[x_1, x_2, \dots]$ where $x_i^3 = 0$ for all i , $x_i^2 = x_j^2 \neq 0$ for all i and j and $x_i x_j = 0$ for all $i \neq j$. By [1, Example 5.7], R is a semiprimary commutative and soc-injective ring but it is not self injective. By [18, Example 1, p. 184], R is a right semi-artinian ring, so that Proposition 2.5 in [18, p. 183] implies that $I \subseteq^{sa} R_R$ for any right ideal I in R and hence R is not SA -injective ring.

(3) Clearly, if $\text{Soc}(N_R) = 0$, then 0 is the only semi-artinian submodule of N and hence every module is SA - N -injective. Particularly, all $\mathbb{Z}$ -modules are SA -injective.

(4) All modules with zero socles are SA -injective, this follows from the fact that $\text{Soc}(M) = 0$ if and only if $\text{Sa}(M) = 0$, for any module M .

Proposition 2.3. *Let N be a module. Then following statements hold:*

- (1) *The class of SA - N -injective modules is closed under isomorphic copies, direct products, direct summands and finite direct sums.*
- (2) *For any submodule K of N , if M is SA - N -injective module, then M is SA - K -injective.*
- (3) *If M is SA - N -injective module, then M is SA - K -injective, for any module K isomorphic to N .*

Proof. Clear. □

Corollary 2.4. *The class of SA -injective right R -modules (SAI_R) is closed under isomorphic copies, direct products, direct summands and finite direct sums.*

Proposition 2.5. *Let M be a module and $\{N_i : i \in I\}$ be a family of modules. If $\bigoplus_{i \in I} N_i$ is a multiplication module, then M is SA - $\bigoplus_{i \in I} N_i$ -injective iff M is SA - N_i -injective, for all $i \in I$.*

Proof. ($\Rightarrow$) By Proposition 2.3((2),(3)).

($\Leftarrow$) Let $K \subseteq {}^{s.a} \bigoplus_{i \in I} N_i$. Since $\bigoplus_{i \in I} N_i$ is a multiplication module (by hypothesis), we have from [20, Theorem 2.2, p. 3844] that $K = \bigoplus_{i \in I} K_i$ with K_i a submodule of N_i , for all $i \in I$. By [9, p. 238], $K_i \subseteq {}^{s.a} N_i$. For $i \in I$, consider the following diagram:

$$\begin{array}{ccc}
 K_i & \xhookrightarrow{i_2} & N_i \\
 \downarrow i_{K_i} & & \downarrow i_{N_i} \\
 K = \bigoplus_{i \in I} K_i & \xhookrightarrow{i_1} & \bigoplus_{i \in I} N_i \\
 \downarrow f & & \\
 M & &
 \end{array}$$

where i_{K_i} , i_{N_i} are injection maps and i_1 , i_2 are inclusion maps. The hypothesis implies that there exists homomorphism $h_i : N_i \rightarrow M$ such that $h_i \circ i_2 = f \circ i_{K_i}$. By [9, Theorem 4.1.6(2)], there exists exactly one homomorphism $h : \bigoplus_{i \in I} N_i \rightarrow M$ satisfying $h_i = h \circ i_{N_i}$. Thus $f \circ i_{K_i} = h_i \circ i_2 = h \circ i_{N_i} \circ i_2 = h \circ i_1 \circ i_{K_i}$ for all $i \in I$. Let $(a_i)_{i \in I} \in \bigoplus_{i \in I} K_i$, thus $a_i \in K_i$, for all $i \in I$ and $f((a_i)_{i \in I}) = f(\sum_{i \in I} i_{K_i}((a_i)_{i \in I})) = (h \circ i_1)((a_i)_{i \in I})$. Thus $f = h \circ i_1$ and the proof is complete. □

Recall that a ring R is called a right invariant if each of its right ideals is an ideal of R [20, p. 3839].

Corollary 2.6. (1) Let M be a module over a right invariant ring R and $1 = \lambda_1 + \lambda_2 + \dots + \lambda_m$ in R such that λ_j are orthogonal idempotent. Then M is SA-injective iff M is SA- $\lambda_j R$ -injective for every $j = 1, 2, \dots, m$.

(2) If M is SA- aR -injective module and $aR \cong bR$, where a and b are idempotents of R , then M is SA- bR -injective.

Proof. (1) By [2, Corollary 7.3], $R = \bigoplus_{j=1}^m \lambda_j R$. Since R is a right invariant ring, we get from [20, Proposition 3.1, p. 3855] that R is a multiplication module and hence Proposition 2.5 implies that M is SA-injective iff M is SA- $\lambda_j R$ -injective for all $1 \leq j \leq m$.

(2) By Proposition 2.3(3). $\square$

Proposition 2.7. The following statements are equivalent for a module M .

- (1) All modules are SA- M -injective.
- (2) All semi-artinian modules are SA- M -injective.
- (3) All semi-artinian submodules of M are SA- M -injective.
- (4) All semi-artinian submodules of M are direct summands of M .

Proof. Straightforward. $\square$

Proposition 2.7 implies the next result.

Corollary 2.8. For a ring R , the following conditions are equivalent.

- (1) $\text{Mod-}R = \text{SAI}_R$.
- (2) All semi-artinian modules are SA-injective.
- (3) All semi-artinian right ideals of R are SA-injective.
- (4) If $I \subseteq^{\text{sa}} R_R$, then I is a direct summand of R_R .

Corollary 2.9. A module M is semisimple if and only if M is semi-artinian and all modules are SA- M -injective.

Proof. ($\Rightarrow$) It is obvious.

($\Leftarrow$) If K is a submodule of M , then K is semi-artinian by [9, p. 238] and hence Proposition 2.7 implies that K is a direct summand of M . Thus M is a semisimple module. $\square$

As a special case of Corollary 2.9, we have the following corollary.

Corollary 2.10. A ring R is a right semisimple ring if and only if it is a right semi-artinian ring and $\text{Mod-}R = \text{SAI}_R$.

In general, not every semi-artinian submodule of a projective module is projective, for example, if $M = \mathbb{Z}_4$ as $\mathbb{Z}_4$ -module and $K = 2\mathbb{Z}_4$, then $K \subseteq^{\text{sa}} M$ but K is not a projective $\mathbb{Z}_4$ -module.

Theorem 2.11. *The following conditions are equivalent for a projective module M .*

- (1) *The class of $SA-M$ -injective modules is closed under quotient.*
- (2) *Every quotient of an injective module is $SA-M$ -injective.*
- (3) *If K_1 and K_2 are two $SA-M$ -injective submodules of a module N , then $K_1 + K_2$ is $SA-M$ -injective.*
- (4) *If K_1 and K_2 are two injective submodules of a module N , then $K_1 + K_2$ is $SA-M$ -injective.*
- (5) *If $K \subseteq^{sa} M$, then K is projective.*

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are obvious.

(2) $\Rightarrow$ (5) Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xhookrightarrow{i} & M \\ & & \downarrow f & & \\ E & \xrightarrow{h} & N & \longrightarrow & 0 \end{array}$$

where N and E are modules, K is a semi-artinian submodule of M , h is an epimorphism and f is a homomorphism. We can assume that E is injective (see, e.g. [3, Proposition 5.2.10]). By $SA-M$ -injectivity of N , f can be extended to a homomorphism $g : M \rightarrow N$. By projectivity of M , there is a homomorphism $\tilde{g} : M \rightarrow E$ such that $h \circ \tilde{g} = g$. Let $\tilde{f} : K \rightarrow E$ be the restriction of $\tilde{g}$ over K . It is clear that $h \circ \tilde{f} = f$. Then K is projective.

(5) $\Rightarrow$ (1) Let L and N be modules such that N is $SA-M$ -injective and $h : N \rightarrow L$ is an epimorphism. If $K \subseteq^{sa} M$ and $f : K \rightarrow L$ is any homomorphism, then the hypothesis implies that K is projective and hence there is a homomorphism $g : K \rightarrow N$ with $h \circ g = f$. By $SA-M$ -injectivity of N , there is a homomorphism $\tilde{g} : M \rightarrow N$ with $\tilde{g} \circ i = g$. Let $\beta = h \circ \tilde{g} : M \rightarrow L$. Then $\beta \circ i = h \circ \tilde{g} \circ i = h \circ g = f$. and hence L is an $SA-M$ -injective module.

(1) $\Rightarrow$ (3) Let K_1 and K_2 be two $SA-M$ -injective submodules of a module K . Thus $K_1 + K_2$ is a homomorphic image of the direct sum $K_1 \oplus K_2$. $SA-M$ -injectivity of $K_1 \oplus K_2$ and the hypothesis imply that $K_1 + K_2$ is $SA-M$ -injective.

(4) $\Rightarrow$ (2) Let F be an injective module with submodule D . Let $B = F \oplus F$, $L = \{(x, x) \mid x \in D\}$, $\bar{B} = B/L$, $K_1 = \{b + L \in \bar{B} \mid b \in F \oplus 0\}$, $K_2 = \{b + L \in \bar{B} \mid b \in 0 \oplus F\}$. Then $\bar{B} = K_1 + K_2$. Since $(F \oplus 0) \cap L = 0$ and $(0 \oplus F) \cap L = 0$, $F \cong K_i$, $i = 1, 2$. Since $K_1 \cap K_2 = \{b + L \in \bar{B} \mid b \in D \oplus 0\} = \{b + L \in \bar{B} \mid b \in 0 \oplus D\}$, $K_1 \cap K_2 \cong D$ under $b \mapsto b + L$ for all $b \in D \oplus 0$. By hypothesis, $\bar{B}$ is $SA-M$ -injective. Injectivity of K_1 implies that $\bar{B} = K_1 \oplus A$ for some submodule A of $\bar{B}$, so $A \cong (K_1 + K_2)/K_1 \cong K_2/K_1 \cap K_2 \cong F/D$. By Proposition 2.3(5), F/D is $SA-M$ -injective. $\square$

Theorem 2.11 implies the following result.

Corollary 2.12. *The following statements are equivalent.*

- (1) *The class SAI_R is closed under quotient.*
- (2) *Every quotient of an injective module is SA-injective.*
- (3) *For any module N , if N_1 and N_2 are submodules of N with $N_1, N_2 \in SAI_R$, then $N_1 + N_2 \in SAI_R$.*
- (4) *For any module N , if N_1 and N_2 are injective submodules of N , then $N_1 + N_2 \in SAI_R$.*
- (5) *If $I \subseteq^{sa} R_R$, then I is projective.*

Theorem 2.13. *If M is a finitely generated module, then the following statements are equivalent.*

- (1) *$Sa(M)$ is noetherian.*
- (2) *The class of SA- M -injective modules is closed under direct sums.*
- (3) *Direct sums of injective modules are SA- M -injective.*
- (4) *If K is injective module, then $K^{(S)}$ is SA- M -injective for any index set S ,*
- (5) *If K is injective module, then $K^{(\mathbb{N})}$ is SA- M -injective.*

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) Clear.

(1) $\Rightarrow$ (2) Let $E = \bigoplus_{i \in I} M_i$, where M_i are SA- M -injective modules and $f : K \rightarrow E$ be a homomorphism with $K \subseteq^{sa} M$. Since $Sa(M)$ is a noetherian module, we have K is finitely generated and hence $f(K) \subseteq \bigoplus_{j \in I_1} M_j$, for some finite subset I_1 of I and hence $\bigoplus_{j \in I_1} M_j$ is SA-injective. Then f can be extended to a homomorphism $g : M \rightarrow E$ and so E is SA-injective.

(5) $\Rightarrow$ (1) Let $K_1 \subseteq K_2 \subseteq \dots$ be a chain of submodules of $Sa(M)$. For each $i \geq 1$, let $F_i = E(M/K_i)$, $F = \bigoplus_{i=1}^{\infty} F_i$ and $M_i = \prod_{j=1}^{\infty} F_j = F_i \oplus (\prod_{j=1, j \neq i}^{\infty} F_j)$, then M_i is injective. By hypothesis, $\bigoplus_{i=1}^{\infty} M_i = (\bigoplus_{i=1}^{\infty} F_i) \oplus (\bigoplus_{i=1}^{\infty} \prod_{j=1, j \neq i}^{\infty} F_j)$ is SA- M -injective and hence Proposition 2.3(1) implies that F itself is SA- M -injective.

Define $f : H = \bigcup_{i=1}^{\infty} K_i \rightarrow F$ by $f(a) = (a + K_i)_i$. Clearly, f is a well defined homomorphism. Since $Sa(M) \subseteq^{sa} M$ (by [9, p. 238]), we have $\bigcup_{i=1}^{\infty} K_i \subseteq^{sa} M$ and hence f can be extended to a homomorphism $g : M \rightarrow F$. Since M is finitely generated, we have $g(M) \subseteq \bigoplus_{i=1}^n E(M/K_i)$ for some n and hence $f(\bigcup_{i=1}^{\infty} K_i) \subseteq \bigoplus_{i=1}^n E(M/K_i)$. Since $\pi_i f(x) = \pi_i(x + K_j)_{j \geq 1} = x + K_i$, for all $x \in H$ and $i \geq 1$, where $\pi_i : \bigoplus_{j \geq 1} E(M/K_j) \rightarrow E(M/K_i)$ is the projection map, $\pi_i f(H) = H/K_i$ for all $i \geq 1$. Since $f(H) \subseteq \bigoplus_{i=1}^n E(M/K_i)$, $H/K_i = \pi_i f(H) = 0$,

for all $i \geq n+1$, so $H = K_i$ for all $i \geq n+1$ and hence the chain $K_1 \subseteq K_2 \subseteq \dots$ terminates at K_{n+1} . Thus $\text{Sa}(M)$ is a noetherian module. $\square$

Proposition 2.14. *The following statements are equivalent.*

- (1) $\text{Sa}(R_R)$ is noetherian.
- (2) The class SAI_R is closed under direct sums.
- (3) Any direct sum of injective modules is SA -injective.
- (4) If K is injective module, then $K^{(S)}$ is SA -injective for any index set S .
- (5) If K is injective module, then $K^{(\mathbb{N})}$ is SA -injective.
- (6) The class SAI_R is closed under pure submodules.
- (7) All FP -injective modules are SA -injective.

Proof. By applying Theorem 2.13, we have the equivalent of (1), (2), (3), (4) and (5).

(1) $\Rightarrow$ (6). Let $N \in \text{SAI}_R$ and K a pure submodule of N . Let $C \subseteq^{sa} R_R$, thus the hypothesis implies that C is finitely generated and so R/C is a finitely presented. Hence the sequence $\text{Hom}_R(R/C, N) \rightarrow \text{Hom}_R(R/C, N/K) \rightarrow 0$ is exact. By [8, Theorem XII.4.4 (4), p. 491], the sequence $\text{Hom}_R(R/C, N) \rightarrow \text{Hom}_R(R/C, N/K) \rightarrow \text{Ext}^1(R/C, K) \rightarrow \text{Ext}^1(R/C, N)$ is exact. Thus $\text{Ext}^1(R/C, K) = 0$ and hence $K \in \text{SAI}_R$. Therefore, the class SAI_R is closed under pure submodules.

(6) $\Rightarrow$ (7). If M is any FP -injective module, then M is a pure submodule of a SA -injective module. By hypothesis, $M \in \text{SAI}_R$.

(7) $\Rightarrow$ (1). Let I be a submodule of $\text{Sa}(R_R)$, thus $I \subseteq^{sa} R_R$. Let $\alpha : I \rightarrow M$ be a homomorphism, where M is a FP -injective module. By hypothesis, M is SA -injective and hence α extends to R_R . By [6], I is finitely generated and hence $\text{Sa}(R_R)$ is a noetherian module. $\square$

3. DEFINABILITY OF THE CLASS SAI_R

If $\mathcal{X} \subseteq \text{Mod-}R$, then we write $\mathcal{X}^\ominus = \{M \in R\text{-Mod} \mid M^* = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \in \mathcal{X}\}$ and $\mathcal{X}^+ = \{M \in \text{Mod-}R \mid M \text{ is a pure submodule of a module in } \mathcal{X}\}$.

Lemma 3.1. *The pair $((\text{SAI}_R)^\ominus, \text{SAI}_R)$ is an almost dual pair over a ring R .*

Proof. By Corollary 2.4 and [12, Proposition 4.2.11, p. 72]. $\square$

Corollary 3.2. *Consider the following conditions for the class SAI_R over a ring R .*

- (1) The class SAI_R is definable.
- (2) $(\text{SAI}_R, (\text{SAI}_R)^\ominus)$ is an almost dual pair over a ring R .
- (3) $(\text{SAI}_R)^* \subseteq (\text{SAI}_R)^\ominus$.
- (4) $(\text{SAI}_R)^{**} \subseteq \text{SAI}_R$.
- (5) The class SAI_R is closed under pure homomorphic images.

Then (1) $\Leftrightarrow$ (2), (1) $\Rightarrow$ (3), (1) $\Rightarrow$ (5) and (3) $\Leftrightarrow$ (4). Moreover, if $\text{Sa}(R_R)$ is noetherian, then all five conditions are equivalent.

Proof. (1) $\Leftrightarrow$ (2). By Lemma 3.1 and [12, Proposition 4.3.8, p. 89].

(1) $\Rightarrow$ (3). Since SAI_R is a definable class, it is closed under pure submodules and hence $(SAI_R)^+ = SAI_R$. Since $((SAI_R)^\ominus, SAI_R)$ is an almost dual (by Lemma 3.1), it follows from [12, Theorem 4.3.2, p. 85], that $(SAI_R)^* \subseteq (SAI_R)^\ominus$.

(1) $\Rightarrow$ (5). By [16, 3.4.8, p. 109].

(3) $\Rightarrow$ (4). By Lemma 3.1 and [12, Theorem 4.3.2, p. 85].

(4) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1). Suppose that $\text{Sa}(R_R)$ is a noetherian module. By Proposition 2.14, the class SAI_R is closed under pure submodules and hence $(SAI_R)^+ = SAI_R$. Thus the results follow from [12, Theorem 4.3.2, p. 85]. $\square$

Corollary 3.3. *If every SA-injective modules is pure-injective, then the following statements are equivalent for a class SAI_R over a ring R .*

- (1) SAI_R is definable.
- (2) The class SAI_R is closed under direct sums.
- (3) $(SAI_R)^+ = SAI_R$
- (4) $\text{Sa}(R_R)$ is a noetherian module.

Proof. By Proposition 2.14, Lemma 3.1 and [12, Theorem 4.5.1, p. 103]. $\square$

If A is a right R -module and B is a left R -module, then $\text{Tor}_1(A, B)$ is defined as the first left derived functor of the tensor product $A \otimes_R B$ (see [4, Ch. VI] for more details).

Lemma 3.4. *A left R -module $M \in (SAI_R)^\ominus$ iff $\text{Tor}_1(R/I, M) = 0$, for any semi-artinian right ideal I of a ring R .*

Proof. Let M be a left R -module and $I \subseteq^{sa} R_R$. By [7, Theorem 3.2.1, p. 75], $\text{Ext}^1(R/I, M^*) \cong (\text{Tor}_1(R/I, M))^*$, so that $\text{Tor}_1(R/I, M) = 0$ if and only if $M^* \in SAI_R$. Hence $({}_R\text{SAF}, SAI_R)$ is an almost dual, where ${}_R\text{SAF} = \{M \in R\text{-Mod} \mid \text{Tor}_1(R/I, M) = 0, \text{ for any semi-artinian right ideal } I \text{ of a ring } R\}$. By [12, Proposition 4.2.11, p. 72], $(SAI_R)^\ominus = {}_R\text{SAF}$. $\square$

A module M is called n -presented if there is an exact sequence $F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$, with each F_i is a finitely generated free modules [5].

Theorem 3.5. *The following statements are equivalent for a class SAI_R over a ring R .*

- (1) SAI_R is definable.
- (2) The class SAI_R is closed under pure submodules and pure homomorphic images.
- (3) Every semi-artinian right ideal in R is finitely presented.
- (4) A module $M \in SAI_R$ iff $M^* \in (SAI_R)^\ominus$.
- (5) A module $M \in SAI_R$ iff $M^{**} \in SAI_R$.

Proof. (1) $\Rightarrow$ (2). By [16, 3.4.8, p. 109].

(2) $\Rightarrow$ (3). Let N be any FP -injective module, thus there is an injective module H with pure exact sequence $0 \rightarrow N \xrightarrow{i} H \xrightarrow{\pi} H/N \rightarrow 0$. By hypothesis, $H/N \in SAI_R$. Let $K \subseteq^{sa} R_R$, thus $\text{Ext}^1(R/K, H/N) = 0$. By [8, Theorem 4.4 (4), p. 491], the sequence $0 = \text{Ext}^1(R/K, H/N) \rightarrow \text{Ext}^2(R/K, N) \rightarrow \text{Ext}^2(R/K, H) = 0$ is exact and hence $\text{Ext}^2(R/K, N) = 0$. By [8, Theorem 4.4 (3), p. 491], the sequence $0 = \text{Ext}^1(R, N) \rightarrow \text{Ext}^1(K, N) \rightarrow \text{Ext}^2(R/K, N) = 0$ is exact, so that $\text{Ext}^1(K, N) = 0$. By hypothesis, SAI_R is closed under pure submodules, so that K is finitely generated by Proposition 2.14 and hence [6, Proposition, p. 361] implies that K is finitely presented.

(3) $\Rightarrow$ (1). Let $M \in SAI_R$. Let $K \subseteq^{sa} R_R$, thus K is finitely presented (by hypothesis) and hence there is an exact sequence $F_2 \xrightarrow{\alpha_2} F_1 \xrightarrow{\alpha_1} K \rightarrow 0$, where F_1, F_2 are finitely generated free modules. Let $\beta = i\alpha_1$, where $i : K \rightarrow R$ is the inclusion mapping, thus the sequence $F_2 \xrightarrow{\alpha_2} F_1 \xrightarrow{\beta} R \xrightarrow{\pi} R/K \rightarrow 0$ is exact, where $\pi : R \rightarrow R/K$ is the natural epimorphism. Hence R/K is a 2-presented module, so that from [5, Lemma 2.7 (2)] we have $\text{Tor}_1(R/K, M^*) \cong (\text{Ext}^1(R/K, M))^* = 0$. By Lemma 3.4, $M^* \in (SAI_R)^\ominus$ and hence $(SAI_R)^* \subseteq (SAI_R)^\ominus$. By hypothesis, every semi-artinian right ideal in R is finitely generated, so that $\text{Sa}(R_R)$ is noetherian. By Corollary 3.2, SAI_R is a definable class.

(1) $\Rightarrow$ (4). By Corollary 3.2, $(SAI_R, (SAI_R)^\ominus)$ is an almost dual pair and hence a module $M \in SAI_R$ iff $M^* \in (SAI_R)^\ominus$.

(4) $\Rightarrow$ (5). By hypothesis, $(SAI_R)^* \subseteq (SAI_R)^\ominus$. By Corollary 3.2, $(SAI_R)^{**} \subseteq SAI_R$. Hence for any module M , if $M \in SAI_R$, then $M^{**} \in SAI_R$.

Conversely, if $M^{**} \in SAI_R$, then $M^* \in (SAI_R)^\ominus$. By hypothesis, $M \in SAI_R$.

(5) $\Rightarrow$ (1). Let N be a FP -injective module, thus there is a pure exact sequence $0 \rightarrow N \rightarrow E \rightarrow E/N \rightarrow 0$, where E is an injective module. By [21, 34.5, p. 286], the sequence $0 \rightarrow N^{**} \rightarrow E^{**} \rightarrow (E/N)^{**} \rightarrow 0$ is split. By hypothesis, $E^{**} \in SAI_R$ and hence $N^{**} \in SAI_R$. By hypothesis, $N \in SAI_R$ so that $\text{Sa}(R_R)$ is noetherian by Proposition 2.14. Thus SAI_R is definable class by Corollary 3.2. $\square$

Note that if the class SAI_R is closed under pure submodules, then $(SAI_R)^+ = SAI_R$. Thus we have the following corollary.

Corollary 3.6. *The class SAI_R is a definable if and only if it is closed under pure submodules and the class $(SAI_R)^+$ is a definable.*

Corollary 3.7. *If the class SAI_R is a definable, then the following are equivalent.*

- (1) *The class of flat left R -modules and the class $(SAI_R)^\ominus$ are coincide.*
- (2) *Every module in SAI_R is FP -injective.*
- (3) *Every pure-injective module in SAI_R is injective.*

Proof. (1) $\Rightarrow$ (2). Let $M \in \text{SAI}_R$, thus $M^* \in (\text{SAI}_R)^\ominus$ by Corollary 3.2. By hypothesis, M^* is a flat left R -module and hence [10, Theorem, p. 239] implies that M^{**} is injective. Since M is a pure submodule in M^{**} , we have M is FP -injective by [21, 35.8, p. 301].

(2) $\Rightarrow$ (3). Let M be any pure-injective module in SAI_R . Let $\mathcal{E} : 0 \rightarrow M \rightarrow N \rightarrow K \rightarrow 0$ be an exact sequence. By hypothesis, M is FP -injective. By [17, Proposition 2.6], the sequence $\mathcal{E}$ is pure and hence pure-injectivity of M implies that the sequence $\mathcal{E}$ is split by [21, 33.7, p. 279]. Therefore, M is injective.

(3) $\Rightarrow$ (1). Let M be a flat left R -module, thus $\text{Tor}_1(N, M) = 0$, for any right R -module N . By Lemma 3.4, $M \in (\text{SAI}_R)^\ominus$. Conversely, if $M \in (\text{SAI}_R)^\ominus$, then $M^* \in \text{SAI}_R$. By [16, Proposition 4.3.29, p. 149], M^* is a pure injective module. By hypothesis, M^* is injective and hence M is flat by [10, Theorem, p. 239]. $\square$

4. RELATIONS BETWEEN SA -INJECTIVITY AND CERTAIN GENERALIZATIONS OF INJECTIVITY

A right R -module M is called quasi-injective if, for every submodule N of M , every right R -homomorphism from N to M can be extended to a right R -endomorphism of M [3, p. 169].

In general, if M is SA -injective right R -module, then M need not be quasi-injective, for example $\mathbb{Z}$ as $\mathbb{Z}$ -module is SA -injective (by Example 2.2(1)) but it is not quasi-injective. Also, the converse is not true in general, for example in the ring $\mathbb{Z}_4$, the ideal $I = \langle 2 \rangle$ is a quasi-injective $\mathbb{Z}_4$ -module but it is not SA -injective $\mathbb{Z}_4$ -module.

The following theorem gives a relation between SA -injective modules and quasi-injective modules.

Theorem 4.1. *The following statements are equivalent for a ring R .*

- (1) R is a right semi-artinian ring.
- (2) Every SA -injective right R -module is injective.
- (3) Every SA -injective right R -module is quasi-injective.
- (4) Every cyclic SA -injective right R -module is quasi-injective.

Proof. (1) $\Rightarrow$ (2) Let M be any SA -injective right R -module. Let I be any right ideal of a ring R and $f : I \rightarrow M$ be any right R -homomorphism. Since R is a right semi-artinian ring (by hypothesis), it follows from [9, Exercise 7(8), p. 238] that I is a semi-artinian right ideal of R . Since M is an SA -injective right R -module (by hypothesis), f extends to R and hence M is an injective right R -module.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (1) Let M be any nonzero cyclic right R -module. We will prove that $\text{Soc}(M) \neq 0$. Assume that $\text{Soc}(M) = 0$. Let N be a nonzero submodule of M . Thus $\text{Soc}(N) = 0$ and hence from Example 2.2(4) that M and N are SA -injective right R -modules. By Corollary 2.4, $N \oplus M$ is an SA -injective right R -module. By hypothesis, $N \oplus M$ is a quasi-injective right R -module. By [15, Proposition 1.17, p. 8], N is an M -injective right R -module and hence N is a direct summand of M . Thus M is semisimple and hence $M = \text{Soc}(M) = 0$ and this is a contradiction. Thus $\text{Soc}(M) \neq 0$ for any nonzero cyclic right R -module M and hence from [18, p. 183] we have that R is a right semi-artinian ring. $\square$

Since every left perfect ring is right semi-artinian [9, Theorem 11.6.3, p. 294], we have the following corollary immediately from Theorem 4.1.

Corollary 4.2. *If R is a left perfect ring, then every SA -injective right R -module is injective (quasi-injective).*

In the following proposition, we give another connection between SA -injective modules and quasi-injective modules.

Proposition 4.3. *A commutative ring R is semisimple if and only if R is a semi-artinian ring and every quasi-injective R -module is SA -injective.*

Proof. ($\Rightarrow$) By Corollary 2.10.

($\Leftarrow$) Let M be any quasi-injective R -module. By hypothesis, M is SA -injective. Since R is a semi-artinian ring (by hypothesis), it follows from Theorem 4.1 that M is injective and hence from [19, Corollary 2.2] we get that R is a semisimple ring. $\square$

The following corollary is immediately from Theorem 4.1 and Proposition 4.3.

Corollary 4.4. *The following statements are equivalent for a commutative ring R .*

- (1) R is semisimple.
- (2) For each R -module M , M is SA -injective if and only if it is quasi-injective.

A right R -module M is called P -injective (resp. F -injective) if, for every principally (resp. finitely generated) right ideal I of R , every right R -homomorphism from I to M can be extended to a right R -homomorphism from R into M (see, for example [11] and [22]).

If M is SA -injective right R -module, then M need not be P -injective (resp. F -injective) in general, for example $\mathbb{Z}$ as $\mathbb{Z}$ -module is SA -injective (by Example 2.2(1)) but it is not P -injective (resp. F -injective). Also, the converse is not true in general, for example: let $F = \mathbb{Z}_2$ be the field of two elements, $F_n = F$ for $n = 1, 2, \dots$, $Q = \prod_{i=1}^{\infty} F_i$, $S = \bigoplus_{i=1}^{\infty} F_i$. If R is the subring of Q generated

by 1 and S , then R is a F-injective right R -module (by [1, Example 4.5]) and hence R_R is a P-injective module. Thus Example 4.5 in [1] implies that R is not a soc-injective right R -module and so R is not a SA -injective module. Thus R is F-injective (P-injective) right R -module but it is not SA -injective.

The following proposition gives a condition under which every F-injective right R -module is SA -injective.

Proposition 4.5. *Let R be a ring. Then $Sa(R_R)$ is a noetherian right R -module if and only if every F-injective right R -module is SA -injective.*

Proof. ($\Rightarrow$) Let M be any F-injective right R -module. Let I be a semi-artinian right ideal of R and let $f : I \rightarrow M$ be any right R -homomorphism. Since $Sa(R_R)$ is noetherian and $I \subseteq Sa(R_R)$, it follows that I is a finitely generated right ideal. By F-injectivity of M , f extends to a right R -homomorphism from R into M and hence M is SA -injective.

($\Leftarrow$) Let $\{M_i\}_{i \in I}$ be a family of injective right R -modules. Thus M_i are F-injective modules. By [22, Proposition 2.1(c)], $\bigoplus_{i \in I} M_i$ is an F-injective module. By hypothesis, $\bigoplus_{i \in I} M_i$ is a SA -injective module and hence from Proposition 2.14 we get that $Sa(R_R)$ is a noetherian right R -module. $\square$

Directly from Proposition 4.5 and Proposition 2.14, we have the following corollary.

Corollary 4.6. *Let R be a ring. Then every F-injective right R -module is SA -injective if and only if every FP-injective right R -module is SA -injective.*

A ring R is called (von Neumann) regular if for any $a \in R$, there is $b \in R$ such that $a = aba$ [9, p. 38].

Proposition 4.7. *The following statements are equivalent.*

- (1) R is a (von Neumann) regular ring and every P-injective right R -module is SA -injective.
- (2) R is a (von Neumann) regular ring and $Sa(R_R)$ is a noetherian right R -module.
- (3) Every SA -injective right R -module is P-injective and every semi-artinian right ideal of R is a direct summand of R_R .

Proof. (1) $\Rightarrow$ (2) Since every F-injective right R -module is P-injective, we have from hypothesis that every F-injective right R -module is SA -injective. By Proposition 4.5, $Sa(R_R)$ is a noetherian right R -module.

(2) $\Rightarrow$ (3) Since R is a (von Neumann) regular ring, it follows from [14, Lemma 2] that every SA -injective right R -module is P-injective. Let I be any semi-artinian right ideal of R . Thus $I \subseteq Sa(R_R)$. Since $Sa(R_R)$ is a noetherian right R -module (by hypothesis), we have that I is a finitely generated right ideal. By [9, Exercise 13, p. 38], I is a direct summand of R_R .

(3) $\Rightarrow$ (1) Since every semi-artinian right ideal of R is a direct summand of R_R (by hypothesis), it follows that from Corollary 2.8 that every right R -module is SA -injective and hence every P -injective right R -module is SA -injective. Since every SA -injective right R -module is P -injective (by hypothesis), we have that every right R -module is P -injective. By [14, Lemma 2], R is a (von Neumann) regular ring. $\square$

ACKNOWLEDGMENTS

The author is grateful to the referee for his/her useful comments which greatly improved the paper.

REFERENCES

1. I. Amin, M. Yousif, N. Zeyada, Soc-injective Rings and Modules, *Comm. Algebra*, **33**, (2005), 4229-4250.
2. F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, 1974.
3. P. E. Bland, *Rings and Their Modules*, Walter de Gruyter and Co., Berlin, 2011.
4. H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, 1956.
5. J. L. Chen, N. Q. Ding, On n -coherent Rings, *Comm. Algebra*, **24**, (1996), 3211-3216.
6. E. E. Enochs, A Note on absolutely Pure Modules, *Canad. Math. Bull.*, **19**, (1976), 361-362.
7. E. E. Enochs, O. M. G. Jenda, *Relative Homological Algebra*, Walter de Gruyter, 2000.
8. P. A. Grillet, *Abstract Algebra*, 2nd edition, GTM 242, Springer, 2007.
9. F. Kasch, *Modules and Rings*, Academic Press, New York, 1982.
10. J. Lambek, A Module is Flat if and only if Its Character Module Is Injective, *Canad. Math. Bull.*, **7**, (1964), 237-243.
11. W. K. Nicholson, M. F. Yousif, *Quasi-Frobenius Rings*, Cambridge Tracts in Mathematics 158, Cambridge Univ. Press, Cambridge, 2003.
12. A. R. Mehdi, *Purity Relative to Classes of Finitely Presented Modules*, PhD Thesis, Manchester University, 2013.
13. A. R. Mehdi, On $\mathcal{L}$ -injective Modules, *Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki*, **28**(2), (2018), 176-192.
14. R. Y. C. Ming, On (von Neumann) Regular Rings, *Proc. Edinburgh Math. Soc.*, **19**, (1974), 89-91.
15. S. H. Mohamed, B. J. Müller, *Continuous and Discrete Modules*, Cambridge Univ. Press, UK: Cambridge, 1990.
16. M. Prest, *Purity, Spectra and Localisation*, Cambridge University Press, 2009.
17. B. Stenström, Coherent Rings and FP -injective Modules, *J. London. Math. Soc.*, **2**, (1970), 323-329.
18. B. Stenström, *Rings of Quotients*, Springer-Verlag, New York, 1975.
19. A. K. Tiwary, B. M. Pandeya, Pseudo Projective and Pseudo Injective Modules, *Indian J. Pure Appl. Math.*, **9**(9), (1978), 941-949.
20. A. A. Tuganbaev, Multiplication Modules, *J. Mathematical Sciences*, **123**, (2004), 3839-3905.

21. R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, 1991.
22. M. Zayed, On f -injective Modules, *Arch. Math.*, *Birkhäuser Verlag, Basel*, **78**, (2002), 345-349.