

## Coefficient Estimates for a General Subclass of m-fold Symmetric Bi-univalent Functions by Using Faber Polynomials

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ABSTRACT. In the present paper, we introduce a new subclass  $\mathcal{H}_{\Sigma_m}(\lambda, \beta)$  of the m-fold symmetric bi-univalent functions. Also, we find the estimates of the Taylor-Maclaurin initial coefficients  $|a_{m+1}|, |a_{2m+1}|$  and general coefficients  $|a_{mk+1}| (k \geq 2)$  for functions in this new subclass. The results presented in this paper would generalize and improve some recent works of several earlier authors.

**Keywords:** Bi-univalent functions, m-fold symmetric bi-univalent functions, Coefficient estimates, Faber polynomials.

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## 1. INTRODUCTION

Let  $\mathcal{A}$  denote the class of functions  $f$  which are analytic in the open unit disk  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$  and normalized by the conditions  $f(0) = f'(0) - 1 = 0$  showing in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

We let  $\mathcal{S}$  to denote the class of functions  $f \in \mathcal{A}$  which are univalent in  $\Delta$  (see [5, 7, 9]). Every function  $f \in \mathcal{S}$  has an inverse  $f^{-1}$ , which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \quad \left( |w| < r_0(f), r_0(f) \geq \frac{1}{4} \right).$$

In fact, the inverse function  $f^{-1}$  is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \quad (1.2)$$

A function  $f \in \mathcal{A}$  is said to be bi-univalent in  $\Delta$ , if both  $f$  and  $f^{-1}$  are univalent in  $\Delta$  (see [21]).

We denote  $\sigma_{\mathcal{B}}$  the class of bi-univalent functions in  $\Delta$  given by (1.1). For a brief history and interesting examples of functions in the class  $\sigma_{\mathcal{B}}$ , see [21]. In fact that this widely-cited work by Srivastava et al. [21] actually revived the study of analytic and bi-univalent functions in recent years and that it has led to a flood of papers on the subject by (for example) Srivastava et al. [7, 8, 17, 18, 19, 20, 22, 23, 24, 28, 29] and others [30, 31].

Some of the important and well-investigated subclasses  $\sigma_{\mathcal{B}}$  of bi-univalent function include (for example) the subclass  $\mathcal{H}_{\Sigma}(\beta)$  ( $0 \leq \beta < 1$ ) and the subclass  $\mathcal{S}^*(\beta)$  bi-starlike functions of order  $\beta$  ( $0 \leq \beta < 1$ ) (see [6, 21]). By definition, we have

$$\mathcal{H}_{\Sigma}(\beta) = \{f \in \sigma_{\mathcal{B}} : \operatorname{Re}(f'(z)) > \beta, \operatorname{Re}(g'(w)) > \beta \quad (z, w \in \Delta)\}$$

and

$$\mathcal{S}^*(\beta) = \left\{ f \in \sigma_{\mathcal{B}} : \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta, \operatorname{Re} \left\{ \frac{wg'(w)}{g(w)} \right\} > \beta \quad (z, w \in \Delta) \right\},$$

where the function  $g = f^{-1}$  given by (1.2).

For each function  $f \in \mathcal{S}$  function

$$h(z) = \sqrt[m]{f(z^m)} \quad (1.3)$$

is univalent and maps unit disk  $\Delta$  into a region with  $m$ -fold symmetry. A function  $f$  is said to be  $m$ -fold symmetric (see [13, 14]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \quad (z \in \Delta, m \in \mathbb{N}). \quad (1.4)$$

We denote by  $\mathcal{S}_m$  the class of m-fold symmetric univalent functions in  $\Delta$ , which are normalized by the series expansion (1.4). In fact, the functions in class  $\mathcal{S}$  are one-fold symmetric.

In [22] Srivastava et al. defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. They gave some important results, such as each function  $f \in \sigma_{\mathcal{B}}$  generates an m-fold symmetric bi-univalent function for each  $m \in \mathbb{N}$ . Furthermore, for the normalized form of  $f$  given by (1.4), they obtained the series expansion for  $f^{-1}$  as follows:

$$f^{-1}(w) = w + \sum_{k=1}^{\infty} A_{mk+1} w^{mk+1} \tag{1.5}$$

$$= w - a_{m+1} w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}] w^{2m+1} - \left[ \frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1} \right] w^{3m+1} + \dots \tag{1.6}$$

We denote by  $\Sigma_m$  the class of m-fold symmetric bi-univalent functions in  $\Delta$ . For  $m = 1$ , formula (1.6) coincides with formula (1.2) of the class  $\sigma_{\mathcal{B}}$ . Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left( \frac{z^m}{1-z^m} \right)^{\frac{1}{m}}, \left[ \frac{1}{2} \log \left( \frac{1+z^m}{1-z^m} \right)^{\frac{1}{m}} \right] \text{ and } [-\log(1-z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left( \frac{w^m}{1+w^m} \right)^{\frac{1}{m}}, \left( \frac{e^{2w^m}-1}{e^{2w^m}+1} \right)^{\frac{1}{m}} \text{ and } \left( \frac{e^{w^m}-1}{e^{w^m}} \right)^{\frac{1}{m}},$$

respectively.

Recently, many authors investigated bounds for various subclasses of m-fold bi-univalent functions (see [3, 10, 11, 12, 15, 16, 18, 24, 25, 26]).

The aim of the this paper is to introduce new subclass of  $\Sigma_m$  and obtain estimates on initial coefficients  $|a_{m+1}|$ ,  $|a_{2m+1}|$  and general coefficients  $|a_{mk+1}| (k \geq 2)$  for functions in the subclass and improve some recent works of many authors.

## 2. PRELIMINARY RESULTS

In the present paper by using the Faber polynomial expansions we obtain estimates of general coefficients  $|a_{mk+1}|$  where  $k \geq 2$ , of functions in the subclass  $\mathcal{H}_{\Sigma_m}(\lambda, \beta)$  of  $\Sigma_m$ . For this purpose we need the following lemmas.

**Lemma 2.1.** [1, 2] *Let  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ , be univalent function in  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ . Then we can write,*

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{k=1}^{\infty} F_k(a_2, a_3, \dots, a_{k+1}) z^k, \tag{2.1}$$

where  $F_k(a_2, a_3, \dots, a_{k+1})$  is a Faber polynomial of degree  $k$ ,

$$F_k(a_2, a_3, \dots, a_{k+1}) = \sum_{i_1+2i_2+\dots+ki_k=k} A_{(i_1, i_2, \dots, i_k)} a_2^{i_1} a_3^{i_2} \cdots a_{k+1}^{i_k} \quad (2.2)$$

and

$$A_{(i_1, i_2, \dots, i_k)} := (-1)^{k+2i_1+3i_2+\dots+(k+1)i_k} \frac{(i_1 + i_2 + \dots + i_k - 1)!k}{i_1!i_2!\dots i_k!}.$$

The first Faber polynomials  $F_k(a_2, a_3, \dots, a_{k+1})$  are given by:

$$F_1(a_2) = -a_2, \quad F_2(a_2, a_3) = a_2^2 - 2a_3 \quad \text{and} \quad F_3(a_2, a_3, a_4) = -a_2^3 + 3a_2a_3 - 3a_4.$$

**Lemma 2.2.** Let  $f(z) = z + \sum_{k=1}^{\infty} a_{m+k+1}z^{m+k+1}$ , be univalent function in  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ . Then we can write,

$$\frac{zf'(z)}{f(z)} = 1 - F_m(a_{m+1})z^m - F_{2m}(a_{m+1}, a_{2m+1})z^{2m} - \dots - F_{mk}(a_{m+1}, a_{2m+1}, \dots, a_{mk+1})z^{mk} - \dots, \quad (2.3)$$

where

$$\begin{aligned} F_m(a_{m+1}) &= F_m(0, \dots, 0, a_{m+1}), \\ F_{2m}(a_{m+1}, a_{2m+1}) &= F_{2m}(0, \dots, 0, a_{m+1}, 0, \dots, 0, a_{2m+1}), \dots, \\ F_{mk}(a_{m+1}, a_{2m+1}, \dots, a_{mk+1}) &= \\ F_{mk}(0, \dots, 0, a_{m+1}, 0, \dots, 0, a_{2m+1}, \dots, 0, \dots, 0, a_{(k-1)m+1}, 0, \dots, 0, a_{mk+1}), \dots \end{aligned}$$

*Proof.* By using Lemma 2.1 for function  $f(z) = z + a_{m+1}z^{m+1} + a_{2m+1}z^{2m+1} + \dots \in \Sigma_m$ , we have

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{k \geq 1} F_k(a_2, a_3, \dots, a_{k+1})z^k.$$

Noting that (2.2), for  $t \in \mathbb{N}$ , we have

$$\begin{aligned} &F_{tm}(0, \dots, 0, a_{m+1}, 0, \dots, 0, a_{2m+1}, 0, \dots, 0, a_{tm+1}) = \\ &\sum_{mi_m+2mi_{2m}+\dots+tm i_{tm}=tm} A_{(i_1, i_2, \dots, i_{tm})} a_{m+1}^{i_m} a_{2m+1}^{i_{2m}} \cdots a_{tm+1}^{i_{tm}}. \end{aligned}$$

Also, for  $1 \leq j \leq m - 1$  and  $t \in \mathbb{N}$ , the equation

$$mi_m + 2mi_{2m} + \dots + tm i_{tm} = tm + j$$

doesn't have positive integer solution, so

$$\begin{aligned} &F_{tm+j}(0, \dots, 0, a_{m+1}, 0, \dots, 0, a_{2m+1}, 0, \dots, 0, a_{tm+1}, 0, \dots, 0) = \\ &\sum_{mi_m+2mi_{2m}+\dots+tm i_{tm}=tm+j} A_{(i_1, i_2, \dots, i_{tm+j})} a_{m+1}^{i_m} a_{2m+1}^{i_{2m}} \cdots a_{tm+1}^{i_{tm}} = 0. \end{aligned}$$

□

By applying Lemma 2.2 for the function  $zf'(z) = z + \sum_{k=1}^{\infty} (mk+1)a_{mk+1}z^{mk+1}$ , we can obtain the following lemma.

**Lemma 2.3.** *Let  $f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1}$ , be univalent function in  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ . Then we can write,*

$$1 + \frac{zf''(z)}{f'(z)} = \frac{z(zf'(z))'}{zf'(z)} = 1 - F_m\left((m+1)a_{m+1}\right)z^m - F_{2m}\left((m+1)a_{m+1}, (2m+1)a_{2m+1}\right)z^{2m} - \dots - F_{mk}\left((m+1)a_{m+1}, (2m+1)a_{2m+1}, \dots, (mk+1)a_{mk+1}\right)z^{mk} - \dots \quad (2.4)$$

**Lemma 2.4.** [14] *If  $h \in \mathcal{P}$ , then  $|c_k| \leq 2$  for each  $k$ , where  $\mathcal{P}$  is the family of all functions  $h$  analytic in  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$  for which  $Re(h(z)) > 0$  where  $h(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \dots$ .*

### 3. THE SUBCLASS $\mathcal{H}_{\Sigma_m}(\lambda, \beta)$

In this section, we introduce the general subclass  $\mathcal{H}_{\Sigma_m}(\lambda, \beta)$ .

**Definition 3.1.** For  $0 \leq \beta < 1$  and  $\lambda \geq 0$ , a function  $f \in \Sigma_m$  given by (1.4) is said to be in the subclass  $\mathcal{H}_{\Sigma_m}(\lambda, \beta)$ , if the following two conditions are satisfied:

$$Re \left\{ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \beta \quad (z \in \Delta) \quad (3.1)$$

and

$$Re \left\{ (1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right\} > \beta \quad (w \in \Delta), \quad (3.2)$$

where  $g = f^{-1}$  given by (1.5).

*Remark 3.2.* By setting  $\lambda = 0$ , the subclass  $\mathcal{H}_{\Sigma_m}(\beta, \lambda)$  reduces to the subclass  $\mathcal{S}_{\Sigma_m}^\beta$  considered by Altinkaya and Yalcin [4].

*Remark 3.3.* For one-fold symmetric bi-univalent functions, we denote the subclass  $\mathcal{H}_{\Sigma_1}(\lambda, \beta) = \mathcal{H}_\Sigma(\lambda, \beta)$ . Special cases of this subclass illustrated below:

- The subclass  $\mathcal{H}_\Sigma(\lambda, \beta)$  is the same the subclass  $B_\Sigma(\beta, \lambda)$  studied by Li and Wang [27].
- By putting  $\lambda = 0$ , then the subclass  $\mathcal{H}_\Sigma(\lambda, \beta)$  reduces to the subclass  $\mathcal{S}_{\sigma_B}(\beta)$  of bi-starlike functions of order  $\beta (0 \leq \beta < 1)$ .
- By putting  $\lambda = 1$ , then the subclass  $\mathcal{H}_\Sigma(\lambda, \beta)$  reduces to the subclass  $\mathcal{K}_{\sigma_B}(\beta)$  of bi-convex functions of order  $\beta (0 \leq \beta < 1)$ .

**Theorem 3.4.** Let  $f$  given by (1.4) be in the subclass  $\mathcal{H}_{\Sigma_m}(\lambda, \beta)$  ( $0 \leq \beta < 1$ ,  $\lambda \geq 0$ ). If  $a_{mt+1} = 0$ ,  $1 \leq t \leq k-1$ , then

$$|a_{mk+1}| \leq \frac{2(1-\beta)}{mk(1+\lambda mk)}, \quad k \geq 2.$$

*Proof.* By applying Lemmas 2.2 and 2.3 for function  $f \in \mathcal{H}_{\Sigma_m}(\lambda, \beta)$  of the form (1.4), we can write

$$\begin{aligned} (1-\lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) &= 1 - \left[ (1-\lambda)F_m(a_{m+1}) + \lambda F_m((m+1)a_{m+1}) \right] z^m \\ &- \left[ (1-\lambda)F_{2m}(a_{m+1}, a_{2m+1}) + \lambda F_{2m}((m+1)a_{m+1}, (2m+1)a_{2m+1}) \right] z^{2m} - \dots \\ &- \left[ (1-\lambda)F_{mk}(a_{m+1}, \dots, a_{mk+1}) + \lambda F_{mk}((m+1)a_{m+1}, \dots, (mk+1)a_{mk+1}) \right] z^{mk} \\ &- \dots \end{aligned} \quad (3.3)$$

and similarly for  $g(w) = f^{-1}(w) = w + \sum_{k=1}^{\infty} A_{mk+1}w^{mk+1}$ , we have

$$\begin{aligned} (1-\lambda) \frac{wg'(w)}{g(w)} + \lambda \left( 1 + \frac{wg''(w)}{g'(w)} \right) &= 1 - \left[ (1-\lambda)F_m(A_{m+1}) + \lambda F_m((m+1)A_{m+1}) \right] w^m \\ &- \left[ (1-\lambda)F_{2m}(A_{m+1}, A_{2m+1}) + \lambda F_{2m}((m+1)A_{m+1}, (2m+1)A_{2m+1}) \right] w^{2m} - \dots \\ &- \left[ (1-\lambda)F_{mk}(A_{m+1}, \dots, A_{mk+1}) + \lambda F_{mk}((m+1)A_{m+1}, \dots, (mk+1)A_{mk+1}) \right] w^{mk}, \\ &- \dots \end{aligned} \quad (3.4)$$

On the other hand, since  $f \in \mathcal{H}_{\Sigma_m}(\lambda, \beta)$  by definition, there exist two positive real part functions  $p(z) = 1 + \sum_{k \geq 1} p_{mk}z^{mk}$  and  $q(w) = 1 + \sum_{k \geq 1} q_{mk}w^{mk}$  where  $Re(p(z)) > 0$  and  $Re(q(w)) > 0$  in  $\Delta$ . So that

$$(1-\lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) = \beta + (1-\beta)p(z) \quad (3.5)$$

and

$$(1-\lambda) \frac{wg'(w)}{g(w)} + \lambda \left( 1 + \frac{wg''(w)}{g'(w)} \right) = \beta + (1-\beta)q(w). \quad (3.6)$$

Comparing the corresponding coefficients of (3.3) and (3.5), we get

$$\begin{aligned} - \left[ (1-\lambda)F_{mk}(a_{m+1}, \dots, a_{mk+1}) + \lambda F_{mk}((m+1)a_{m+1}, \dots, (mk+1)a_{mk+1}) \right] \\ = (1-\beta)p_{mk} \quad (k \geq 1). \end{aligned} \quad (3.7)$$

Comparing the corresponding coefficients of (3.4) and (3.6), we get

$$\begin{aligned} - \left[ (1-\lambda)F_{mk}(A_{m+1}, \dots, A_{mk+1}) + \lambda F_{mk}((m+1)A_{m+1}, \dots, (mk+1)A_{mk+1}) \right] \\ = (1-\beta)q_{mk} \quad (k \geq 1). \end{aligned} \quad (3.8)$$

Note that for  $a_{mt+1} = 0$  ( $1 \leq t \leq k-1$ ), we get

$$\begin{aligned} A_{mk+1} &= -a_{mk+1}, \quad F_{mk}(0, \dots, 0, a_{mk+1}) = -mka_{mk+1} \\ \text{and } F_{mk}(0, \dots, 0, (mk+1)a_{mk+1}) &= -mk(mk+1)a_{mk+1}. \end{aligned} \quad (3.9)$$

So from (3.7), (3.8) and (3.9), we have

$$\begin{aligned} mk(1 + \lambda mk)a_{mk+1} &= (1 - \beta)p_{mk}, \\ mk(1 + \lambda mk)A_{mk+1} &= -mk(1 + \lambda mk)a_{mk+1} = (1 - \beta)q_{mk}. \end{aligned}$$

Taking the absolute values of the above equalities and using Lemma 2.4, we gain

$$|a_{mk+1}| = \frac{(1 - \beta)|p_{mk}|}{mk(1 + \lambda mk)} = \frac{(1 - \beta)|q_{mk}|}{mk(1 + \lambda mk)} \leq \frac{2(1 - \beta)}{mk(1 + \lambda mk)}.$$

So completes the proof.  $\square$

**Theorem 3.5.** Let  $f$  given by (1.4) be in the subclass  $\mathcal{H}_{\Sigma_m}(\lambda, \beta)$  ( $0 \leq \beta < 1$ ,  $\lambda \geq 0$ ). Then

$$|a_{m+1}| \leq \min \left\{ \frac{2(1 - \beta)}{m(1 + \lambda m)}, \sqrt{\frac{2(1 - \beta)}{m^2(1 + \lambda m)}} \right\}$$

and

$$|a_{2m+1}| \leq \min \left\{ \frac{1 - \beta}{m(1 + 2\lambda m)} + \frac{2(m+1)(1 - \beta)^2}{m^2(1 + \lambda m)^2}, \frac{(1 + m)(1 - \beta)}{m^2(1 + \lambda m)} \right\}.$$

*Proof.* By putting  $k = 1, 2$  in (3.7), we get:

$$m(1 + \lambda m)a_{m+1} = (1 - \beta)p_m, \quad (3.10)$$

$$2m(1 + 2\lambda m)a_{2m+1} - m(1 + 2\lambda m + \lambda m^2)a_{m+1}^2 = (1 - \beta)p_{2m}. \quad (3.11)$$

Similarly, by putting  $k = 1, 2$  in (3.8), we get:

$$-m(1 + \lambda m)a_{m+1} = (1 - \beta)q_m, \quad (3.12)$$

$$-2m(1 + 2\lambda m)a_{2m+1} + m(1 + 2m + 2\lambda m + 3\lambda m^2)a_{m+1}^2 = (1 - \beta)q_{2m}. \quad (3.13)$$

From (3.10) and (3.12), we get

$$p_m = -q_m \quad (3.14)$$

and

$$a_{m+1}^2 = \frac{(1 - \beta)^2(p_m^2 + q_m^2)}{2m^2(1 + \lambda m)^2}. \quad (3.15)$$

Adding (3.11) and (3.13), we get

$$a_{m+1}^2 = \frac{(1 - \beta)(p_{2m} + q_{2m})}{2m^2(1 + \lambda m)}. \quad (3.16)$$

From the equations (3.15), (3.16) and by using Lemma 2.4, we get:

$$|a_{m+1}| \leq \frac{2(1 - \beta)}{m(1 + \lambda m)} \quad \text{and} \quad |a_{m+1}| \leq \sqrt{\frac{2(1 - \beta)}{m^2(1 + \lambda m)}},$$

respectively. So we get the desired estimate on the coefficient  $|a_{m+1}|$ .

Next, in order to find the bound on the coefficient  $|a_{2m+1}|$ , we subtract (3.13) from (3.11), we get

$$a_{2m+1} = \frac{(1-\beta)(p_{2m} - q_{2m})}{4m(1+2\lambda m)} + \frac{(m+1)}{2} a_{m+1}^2. \quad (3.17)$$

Therefore, we find from (3.15) and (3.17) that

$$a_{2m+1} = \frac{(1-\beta)(p_{2m} - q_{2m})}{4m(1+2\lambda m)} + \frac{(m+1)(1-\beta)^2(p_m^2 + q_m^2)}{4m^2(1+\lambda m)^2}. \quad (3.18)$$

Also, from (3.16) and (3.17), we have

$$a_{2m+1} = \frac{[1+2m+2\lambda m+3\lambda m^2]p_{2m} + [1+2\lambda m+\lambda m^2]q_{2m}}{4m^2(1+\lambda m)(1+2\lambda m)}(1-\beta). \quad (3.19)$$

So, from the equations (3.18), (3.19) and applying Lemma 2.4, we get

$$|a_{2m+1}| \leq \frac{1-\beta}{m(1+2\lambda m)} + \frac{2(m+1)(1-\beta)^2}{m^2(1+\lambda m)^2}$$

and

$$|a_{2m+1}| \leq \frac{(1+m)(1-\beta)}{m^2(1+\lambda m)}.$$

□

**Theorem 3.6.** *Let  $f$  given by (1.4) be in the subclass  $\mathcal{H}_{\Sigma_m}(\lambda, \beta)$  ( $0 \leq \beta < 1$ ,  $\lambda \geq 0$ ). Also let  $\rho$  be real number. Then*

$$|a_{2m+1} - \rho a_{m+1}^2| \leq \begin{cases} \frac{(1-\beta)}{m(1+2\lambda m)} ; & |T(\rho)| \leq 1 \\ \frac{(1-\beta)|T(\rho)|}{m(1+2\lambda m)} ; & |T(\rho)| \geq 1 \end{cases}$$

where

$$T(\rho) = \frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)}.$$

*Proof.* From the equation (3.17), we get

$$a_{2m+1} - \rho a_{m+1}^2 = \frac{m-2\rho+1}{2} a_{m+1}^2 + \frac{(1-\beta)(p_{2m} - q_{2m})}{4m(1+2\lambda m)}. \quad (3.20)$$

From the equation (3.16) and (3.20), we have

$$a_{2m+1} - \rho a_{m+1}^2 = \frac{(1-\beta)}{4m(1+2\lambda m)} \left\{ \left[ \frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)} + 1 \right] p_{2m} + \left[ \frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)} - 1 \right] q_{2m} \right\}.$$

Next, taking the absolute values we obtain

$$|a_{2m+1} - \rho a_{m+1}^2| \leq \frac{(1-\beta)}{4m(1+2\lambda m)} \left\{ \left| \frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)} + 1 \right| |p_{2m}| \right.$$

$$+ \left| \frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)} - 1 \right| |q_{2m}| \Big\}.$$

Then, by using Lemma 2.4, we conclude that

$$|a_{2m+1} - \rho a_{m+1}^2| \leq \begin{cases} \frac{(1-\beta)}{m(1+2\lambda m)}; & |T(\rho)| \leq 1 \\ \frac{(1-\beta)|T(\rho)|}{m(1+2\lambda m)}; & |T(\rho)| \geq 1. \end{cases}$$

□

#### 4. COROLLARIES AND CONSEQUENCES

By setting  $\lambda = 0$  in Theorem 3.4, we conclude the following result.

**Corollary 4.1.** *Let  $f$  given by (1.4) be in the subclass  $\mathcal{S}_{\Sigma_m}^\beta$  ( $0 \leq \beta < 1$ ). If  $a_{mt+1} = 0$ ,  $1 \leq t \leq k-1$ , then*

$$|a_{mk+1}| \leq \frac{2(1-\beta)}{mk}, \quad k \geq 2.$$

By setting  $\lambda = 0$  in Theorem 3.5, we conclude the following result.

**Corollary 4.2.** *Let  $f$  given by (1.4) be in the subclass  $\mathcal{S}_{\Sigma_m}^\beta$  ( $0 \leq \beta < 1$ ). Then*

$$|a_{m+1}| \leq \begin{cases} \frac{\sqrt{2(1-\beta)}}{m}; & 0 \leq \beta \leq \frac{1}{2} \\ \frac{2(1-\beta)}{m}; & \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leq \begin{cases} \frac{(m+1)(1-\beta)}{m^2}; & 0 \leq \beta \leq \frac{1+2m}{2(1+m)} \\ \frac{2(m+1)(1-\beta)^2}{m^2} + \frac{1-\beta}{m}; & \frac{1+2m}{2(1+m)} \leq \beta < 1. \end{cases}$$

*Remark 4.3.* The bounds on  $|a_{m+1}|$  and  $|a_{2m+1}|$  given in Corollary 4.2 are better than those given in [4, Corolary 7].

By setting  $\lambda = 0$  in Theorem 3.6, we conclude the following result.

**Corollary 4.4.** *Let  $f$  given by (1.4) be in the subclass  $\mathcal{S}_{\Sigma_m}^\beta$  ( $0 \leq \beta < 1$ ). Also let  $\rho$  be real number. Then*

$$|a_{2m+1} - \rho a_{m+1}^2| \leq \begin{cases} \frac{(1-\beta)}{m}; & |m-2\rho+1| \leq m \\ \frac{(1-\beta)|m-2\rho+1|}{m^2}; & |m-2\rho+1| \geq m. \end{cases}$$

By setting  $m = 1$  in Theorem 3.5, we conclude the following result.

**Corollary 4.5.** *Let  $f$  given by (1.1) be in the subclass  $B_{\Sigma}(\beta, \lambda)$  ( $0 \leq \beta < 1$ ,  $\lambda \geq 0$ ). Then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{1+\lambda}}; & \lambda + 2\beta \leq 1 \\ \frac{2(1-\beta)}{1+\lambda}; & \lambda + 2\beta \geq 1 \end{cases}$$

and

$$|a_3| \leq \begin{cases} \frac{2(1-\beta)}{1+\lambda}; & 0 \leq \beta \leq \frac{3+4\lambda-3\lambda^2}{4(1+2\lambda)} \\ \frac{1-\beta}{1+2\lambda} + \frac{4(1-\beta)^2}{(1+\lambda)^2}; & \frac{3+4\lambda-3\lambda^2}{4(1+2\lambda)} \leq \beta < 1. \end{cases}$$

*Remark 4.6.* The bounds on  $|a_2|$  and  $|a_3|$  given in Corollary 4.5 are better than those given in [27, Theorem 3.2].

By setting  $\lambda = 0$  in Corollary 4.5, we conclude the following result.

**Corollary 4.7.** *Let  $f$  given by (1.1) be in the subclass  $\mathcal{S}_{\sigma_B}(\beta)$  of bi-starlike functions of order  $\beta$  ( $0 \leq \beta < 1$ ). Then*

$$|a_2| \leq \begin{cases} \sqrt{2(1-\beta)}; & 0 \leq \beta \leq \frac{1}{2} \\ 2(1-\beta); & \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \begin{cases} 2(1-\beta); & 0 \leq \beta \leq \frac{3}{4} \\ (1-\beta) + 4(1-\beta)^2; & \frac{3}{4} \leq \beta < 1. \end{cases}$$

*Remark 4.8.* The bounds on  $|a_2|$  and  $|a_3|$  given in Corollary 4.7 are better than those given in [27, Corollary 3.3].

By setting  $\lambda = 1$  in Corollary 4.5, we conclude the following result.

**Corollary 4.9.** *Let  $f$  given by (1.1) be in the subclass  $\mathcal{K}_{\sigma_B}(\beta)$  of bi-convex functions of order  $\beta$  ( $0 \leq \beta < 1$ ). Then*

$$|a_2| \leq 1 - \beta$$

and

$$|a_3| \leq \begin{cases} 1 - \beta; & 0 \leq \beta \leq \frac{1}{3} \\ \frac{1-\beta}{3} + (1-\beta)^2; & \frac{1}{3} \leq \beta < 1. \end{cases}$$

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