Iranian Journal of Mathematical Sciences and Informatics
Vol. 19, No. 1 (2024), pp 161-174
DOI: 10.61186/ijmsi.19.1.161

Multiresolution Analysis on Sobolev Space over Local Fields of Positive Characteristic and Characterization of Scaling Function

Ashish Pathak[a*], Dileep Kumar[a,b]

[a]Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi-221005, India

[b]Department of Applied Science and Humanities (Mathematics), GL Bajaj Institute of Technology and Management, Greater Noida-201306, India

E-mail:ashishpathak@bhu.ac.in
E-mail:dkbhu07@gmail.com

Abstract. A general concept of wavelets and multiresolution analysis on Sobolev space over local fields of positive characteristic ($H^s(\mathbb{K})$) is given. A characterization of the scaling functions associated with an MRA is also given.

Keywords: Wavelets, Multiresolution analysis, Local fields, Sobolev space, Fourier transform.

2000 Mathematics subject classification: 42C40, 46E35, 11F87.

1. Introduction

The concept of wavelets was introduced in recent decades, at the beginning of the 1980s (See [1], [2], [6], [14]). Wavelet and its transform are a powerful tool for analyzing the problems arising in harmonic analysis, signal and image processing, sampling, filtering, and so on. Some applications of the Wavelet transform were provided in recent literature by Hari M. Srivastava, Firdous A. Shah et al. (See [17], [18], [19], [20]).

*Corresponding Author

Received 26 October 2018; Accepted 25 February 2024

Wavelets are constructed using the notion of multiresolution analysis, introduced by Meyer and Mallat ([5, 4]). They attracted considerable interest from the mathematical community and extended by several authors. S.Dahlke [7] introduced wavelets on Riemannian manifolds and locally compact abelian group. M.Papadakis developed the multiresolution analysis of abstract Hilbert spaces. Wavelet theory for local field has been developed by R.L. Benedetto and J.J. Benedetto ([8, 9]). Multiresolution analysis and wavelet on the p-adic fields $\mathbb{Q}_p$ are discussed in the paper ([10, 11, 12]). Jiang , Li and Jin ([13]) defined MRA on a local fields of positive characteristic and constructed the corresponding orthonormal wavelets. We are interested in Sobolev space over a local field $H^s(\mathbb{K})$ of positive characteristic. Sobolev space were introduced by Sobolev in the late thirties of the 20^{th} century and plays an important role in the modern mathematics and theoretical physics. The study of the wavelets in a Sobolev space $H^s(\mathbb{R})$ can date back to the work of Rieder, which extends the continuous wavelet transform to Sobolev spaces $H^s(\mathbb{R})$ for arbitrary reals. The paper aims to give a characterization of scaling functions of $H^s(\mathbb{K})$. This paper is divided into four sections. Section 2 contains general notations which are used further in this paper. This section also contains definitions of local field and Sobolev space on a local field. In section 3, we define an MRA on Sobolev space over a local field by specifying four properties. We show how to construct orthonormal basis from Riesz basis. Finally, in section 4, we give a necessary and sufficient condition on a function to be a scaling function for an MRA of $H^s(\mathbb{K})$.

2. Notation and definitions

A local field means an algebraic field and a topological space with the topological properties of locally compact, non-discrete, complete and totally disconnected, denoted by $\mathbb{K}$. The additive and multiplicative groups of $\mathbb{K}$ are denoted by $\mathbb{K}^+$ and $\mathbb{K}^*$, respectively. We may choose a Haar measure dx for $\mathbb{K}^+$. If $\alpha \neq 0 (\alpha \in \mathbb{K})$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$ and call $|\alpha|$ the absolute value or valuation of α. Let $|0| = 0$. The absolute value has the following properties:
(i) $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$;
(ii) $|xy| = |x||y|$;
(iii) $|x + y| \leq max(|x|, |y|)$.
The last one of these properties is called the ultrametric inequality. It follows that $|x + y| = max(|x|, |y|)$ if $|x| \neq |y|$.
The set $\mathfrak{D} = \{x \in \mathbb{K} : |x| \leq 1\}$ is called the ring of integers in $\mathbb{K}$. It is the unique maximal compact subring of $\mathbb{K}$. Define $\mathfrak{P} = \{x \in \mathbb{K} : |x| < 1\}$. The set $\mathfrak{P}$ is called the prime ideal in $\mathbb{K}$. The prime ideal in $\mathbb{K}$ is the unique maximal ideal in $\mathfrak{D}$. It is principal and prime. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{P}$. Such an element is called a prime element of $\mathbb{K}$ and $|\mathfrak{p}| = q^{-1}$ with that $q = p^c$, p is a prime number and c is a positive integer.

[DOI: 10.61186/ijmsi.19.1.161]

For a measurable subset E of $\mathbb{K}$, let $|E| = \int_K \zeta_E(x)dx$, where ζ_E is the characteristic function of E and dx is the Haar measure of K normalized so that $|\mathfrak{D}| = 1$. Then, it is easy to see that $|\mathfrak{P}| = q^{-1}$ and $|\mathfrak{p}| = q^{-1}$. It follows if $x \neq 0$, and $x \in \mathbb{K}$, then $|x| = q^k$ for some $k \in \mathbb{Z}$ (see [15]).
Let $\mathfrak{P}^k = \mathfrak{p}^k\mathfrak{D} = \{x \in K : |x| \leq q^{-k}\}, k \in \mathbb{Z}$. These are called fractional ideals. we have the fact that $|\mathfrak{P}^k| = q^{-k}$ and $\mathfrak{D} = \mathfrak{P}^0$. Each $\mathfrak{P}^k$ is compact and open and is a subgroup of $\mathbb{K}^+$. Therefore,the residue space $\mathfrak{D}\backslash\mathfrak{P}$ is isomorphic to $GF(q)$. Let $\mathfrak{D}^* = \mathfrak{D}\backslash\mathfrak{P} = \{x \in \mathbb{K} : |x| = 1\}$. $\mathfrak{D}^*$ is the group of units in $\mathbb{K}^*$. If $x \neq 0$, we can write $x = \mathfrak{p}^k x'$ with $x' \in \mathfrak{D}^*$. Let $\mathfrak{U} = \{a_i : i = 0, 1, ..., q-1\}$ be any fixed full set of coset representatives of $\mathfrak{P}$ in $\mathfrak{D}$.
If $\mathbb{K}$ is a local field, then there is a nontrivial, unitary, continuous character χ on $\mathbb{K}^+$. It can be proved that $\mathbb{K}^+$ is self dual.
Let χ be a fixed character on $\mathbb{K}^+$ that is trivial on $\mathfrak{D}$ but is nontrivial on $\mathfrak{P}^{-1}$. We can find such a character by starting with any nontrivial character and rescaling. We will define such a character for a local field of positive characteristic. It follows that χ is constant on cosets of $\mathfrak{D}$ and that if $y \in \mathfrak{P}^k$, then $\chi_y(\chi_y(x) = \chi(yx)$ is constant on cosets of $\mathfrak{P}^{-k}$.

Definition 2.1. If $f \in L^1(\mathbb{K})$, then the Fourier transform of f is the function $\hat{f}$ defined by

$$\hat{f}(y) = \int_{\mathbb{K}} f(x)\overline{\chi_y(x)}dx = \int_{\mathbb{K}} f(x)\chi(-yx)dx.$$

The Fourier transform in $L^p(\mathbb{K})$, $1 < p \leq 2$, can be defined similarly as in $L^p(\mathbb{R})$.
The inner product is denoted by

$$< f, g >= \int_{\mathbb{K}} f(x)\overline{g(x)}dx, \quad \text{for } f, g \in L^2(\mathbb{K}).$$

We will use the notation $\mathbb{N}_0 = \{0, 1, 2, ...\}$. Let χ_u be any character on $\mathbb{K}^+$. Since $\mathfrak{D}$ is a subgroup of $\mathbb{K}^+$, it follows that the restriction $\chi_u|_{\mathfrak{D}}$ is a character on $\mathfrak{D}$. Also, as characters on $\mathfrak{D}$, we have $\chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. That is, $\chi_u = \chi_v$ if $u + \mathfrak{D} = v + \mathfrak{D}$ and $\chi_u \neq \chi_v$ if $(u + \mathfrak{D}) \cap (v + \mathfrak{D}) = \phi$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in $\mathbb{K}^+$, then $\{\chi_{u(n) \,:\, n \in \mathbb{N}_0}\}$ is a list of distinct character on $\mathfrak{D}$. It is proved in [15] that this list is complete. That is, we have the following proposition.

Proposition 2.2. *Let $\{u(n) : n \in \mathbb{N}_0\}$ be complete list of (distinct) coset representatives of $\mathfrak{D}$ in $\mathbb{K}^+$. Then $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ is a complete list of (distinct) characters on $\mathfrak{D}$. Moreover, it is complete orthonormal system in $\mathfrak{D}$.*

Given such a list of character $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$, we define the Fourier coefficients of $f \in L^1(\mathfrak{D})$ as

$$\hat{f}(u(n)) = \int_{\mathfrak{D}} f(x)\bar{\chi}_{u(n)}(x)dx.$$

[DOI: 10.61186/ijmsi.19.1.161]

The series $\sum_{n=0}^{\infty}\hat{f}(u(n))\chi_{u(n)}(x)$ is called the Fourier series of$f$. From the standard L^2- theory for compact abelian groups we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds see ([15]):

$$\int_{\mathfrak{D}}|f(x)|^2dx=\sum_{n=0}^{\infty}|\hat{f}(u(n))|^2.$$

These results hold irrespective of the ordering of the characters. We describe the "natural" order on the sequence $\{u(n)\in\mathbb{K}\}_{n=0}^{\infty}$ as follows.
Recall that $\mathfrak{P}$ is the prime ideal in $\mathfrak{D}$, $\mathfrak{D}/\mathfrak{P}\cong GF(q)=\tau$, $q=p^c$, p is a prime, c a positive integer and $\Omega:\mathfrak{D}\to\tau$ the canonical homomorphism of $\mathfrak{D}$ on to τ. Note that $\tau=GF(q)$ is a c- dimensional vector space over $GF(p)\subset\tau$. We choose a set $\{1=\epsilon_0,\epsilon_1,...,\epsilon_{c-1}\}\subset\mathfrak{D}^*=\mathfrak{D}\backslash\mathfrak{P}$ such that $\{\Omega(\epsilon_k)\}_{k=0}^{c-1}$ is a basis of $GF(q)$ over $GF(p)$.

Definition 2.3. For k, $0\le k<q$, $k=a_0+a_1p+...+a_{c-1}p^{c-1}$, $0\le a_i<p$, $i=0,1,...,c-1$, we define

$$u(k)=(a_0+a_1\epsilon_1+...+a_{c-1}\epsilon_{c-1})\mathfrak{p}^{-1}\ \ (0\le k<q).$$

For $k=b_0+b_1q+...+b_sq^s$, $0\le b_i<q$, $k\ge 0$, we set

$$u(k)=u(b_0)+\mathfrak{p}^{-1}u(b_1)+...+\mathfrak{p}^{-s}u(b_s).$$

Note that for $k,l\ge 0$, $u(k+l)\ne u(k)+u(l)$. However, it is true that for all $r,s\ge 0$, $u(rq^s)=\mathfrak{p}^{-s}u(r)$, and for $r,s\ge 0$, $0\le t<q^s$, $u(rq^s+t)=u(rq^s)+u(t)=\mathfrak{p}^{-s}u(r)+u(t)$.
Then, it is easy to verify that $\{u(k):k\in\mathbb{N}_0\}=\{-u(k):k\in\mathbb{N}_0\}$, $\{u(k)+u(l):k\in\mathbb{N}_0\}=\{u(k):\mathbb{N}_0\}$, for a fixed $l\in\mathbb{N}_0$ and $u(n)=0$ if and only if $n=0$.

We will denote $\chi_{u(n)}=\chi_n$ for $n\in\mathbb{N}_0$. Since $\mathfrak{U}=\{a_i\}_{i=0}^{q-1}$ be a fixed set of coset representatives of $\mathfrak{P}$ in $\mathfrak{D}$. Then every $x\in\mathbb{K}$ can be expressed uniquely as

$$x=x_0+\sum_{k=1}^{n}a_k\mathfrak{p}^{-k},\quad x_0\in\mathfrak{D},\ a_k\in\mathfrak{U}.$$

Let $\mathbb{K}$ be a local field of positive characteristic p and $\epsilon_0,\epsilon_1,\epsilon_2,...,\epsilon_{c-1}$ be as above. we define a character χ on $\mathbb{K}$ as follows (see [16]) :

$$\chi(\epsilon_\nu\mathfrak{p}^{-k})=\begin{cases}exp(2\pi i/p), & \text{if } \nu=0\ and\ k=1,\\ 1, & \text{if } \nu=1,2,...,c-1\ or\ k\ne 1.\end{cases}$$

Note that χ is trivial on $\mathfrak{D}$ but nontrivial on $\mathfrak{P}^{-1}$.

Definition 2.4. We call a function f defined on $\mathbb{K}$ integral-periodic if $f(x+u(l))=f(x)$ for all $l\in\mathbb{N}_0$.

Proposition 2.5. *For all* $l,k\in\mathbb{N}_0$, $\chi_k(u(l))=1$.

[DOI: 10.61186/ijmsi.19.1.161]

2.1. **Distributions over local fields.** We denote $\mathscr{S}(\mathbb{K})$ the spaces of all finite linear combinations of characteristics functions of ball of $\mathbb{K}$. The space $\mathscr{S}(\mathbb{K})$ is an algebra of continuous function with compact support that is dense in $L^p(\mathbb{K}),\ 1 \leq p < \infty$. We observe that the Fourier transform is homeomorphism of $\mathscr{S}(\mathbb{K})$ onto $\mathscr{S}(\mathbb{K})$. The space $\mathscr{S}'(\mathbb{K})$ of continuous linear functional on $\mathscr{S}(\mathbb{K})$ is called the space of distributions.
The Fourier transform of $f \in \mathscr{S}(\mathbb{K})$ is denoted by $\hat{f}(\xi)$ and defined by the

$$\hat{f}(\xi) = \int_{\mathbb{K}} f(x)\overline{\chi_\xi(x)}dx = \int_{\mathbb{K}} f(x)\chi(\xi x)dx,\ \xi \in \mathbb{K}, \tag{2.1}$$

and the inverse Fourier transform by

$$f(x) = \int_{\mathbb{K}} \hat{f}(\xi)\chi_x(\xi)d\xi,\ x \in \mathbb{K}. \tag{2.2}$$

The Fourier transform and inverse Fourier transforms of a distributions $f \in \mathscr{S}'(\mathbb{K})$ is defined by

$$\langle \hat{f}, \phi\rangle = \langle f, \hat{\phi}\rangle,\ \langle f^\vee, \phi\rangle = \langle f, \phi^\vee\rangle,\ \text{for all } \phi \in \mathscr{S}(\mathbb{K}). \tag{2.3}$$

Definition 2.6. Sobolev space $H^s(\mathbb{K})$ over local fields.
Let $s \in \mathbb{R}$, we denote by $H^s(\mathbb{K})$ is the space of all $f \in \mathscr{S}'(\mathbb{K})$ such that

$$\hat{\nu}^{\frac{s}{2}}(\xi)\hat{f}(\xi) \in L^2(\mathbb{K}).$$

Obviously, for any real number s, $H^s(\mathbb{K})$ is a linear space. We equip $H^s(\mathbb{K})$ with the inner product

$$\langle f, g\rangle_s = \langle f, g\rangle_{H^s(\mathbb{K})} = \int_{\mathbb{K}} \hat{\nu}^s(\xi)\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi,$$

which induces the norm

$$||f||^2_{H^s(\mathbb{K})} = \int_{\mathbb{K}} \hat{\nu}^s(\xi)|\hat{f}(\xi)|^2 d\xi.$$

Example 2.7. The Dirac distributions $\delta \in H^s(\mathbb{K})$ for $s < -1$.

Theorem 2.8. *The space $\mathscr{S}(\mathbb{K})$ is dense in $H^s(\mathbb{K})$*

Proof. For the proof of above theorem see [3]. □

3. Multiresolution Analysis on $H^s(\mathbb{K})$

Pathak and Singh [3] modified the classical definition of multiresolution analysis on $L^2(\mathbb{K})$ and defined as follows.

Definition 3.1. Let s be any real number. A multiresolution analysis on $H^s(\mathbb{K})$ is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of the closed linear subspaces of $H^s(\mathbb{K})$ such that
(1) $V_j \subset V_{j+1}$;
(2) $\overline{\cup_{j\in\mathbb{Z}} V_j} = H^s(\mathbb{K})$;
(3) $\cap_{j\in\mathbb{Z}} V_j = 0$;

[DOI: 10.61186/ijmsi.19.1.161]

(4) For each $j \in \mathbb{Z}$, there exists a function $\phi^{(j)} \in H^s(\mathbb{K})$ such that $\{\phi^{(j)}_{j,k}\}_{k\in\mathbb{N}_0}$, forms an orthonormal basis of V_j, where

$$\phi^{(j)}_{j,k} = q^{\frac{j}{2}}\phi^{(j)}(\mathfrak{p}^{-j}. - u(k)),\ k \in \mathbb{N}_0\ \ and\ j \in \mathbb{Z}.$$

Such function $\phi^{(j)}$ are called scaling function or father wavelet. The condition $V_j \subset V_{j+1}$; for $j \in \mathbb{Z}$ is equivalent to the existence of integral-periodic function $m^{(j)}_0 \in L^2(\mathfrak{D})$ such that the following scale relation holds.

$$\hat{\phi}^{(j)}(\xi) = m^{(j+1)}_0(\mathfrak{p}\xi)\hat{\phi}^{(j+1)}(\mathfrak{p}\xi) \tag{3.1}$$

these functions $m^{(j+1)}_0$ are called low pass filter. Define $\psi^{(j)}_r, j \in \mathbb{Z}$ and $r \in D_1 = \{0, 1, 2, 3, 4...q - 1\}$, by the formula

$$\hat{\psi}^{(j)}_r(\xi) = m^{(j+1)}_r(\mathfrak{p}\xi)\hat{\phi}^{(j+1)}(\mathfrak{p}\xi), \quad j, k \in \mathbb{Z}, \quad r \in D_1. \tag{3.2}$$

These function $m^{(j+1)}_r$ are called high pass filters.
We get $\{\psi^{(j)}_{r,j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}_0,r\in D_1}$ form an orthonormal basis for $H^s(\mathbb{K})$, where

$$\psi^{(j)}_{r,j,k}(.) = q^{\frac{j}{2}}\psi^{(j)}_r(\mathfrak{p}^{-j}. - u(k)), j, k \in \mathbb{Z}, r \in D_1. \tag{3.3}$$

Theorem 3.2. *If $s \in \mathbb{R}$, $\phi^{(j)} \in H^s(\mathbb{K})$ then the distribution $\{q^{\frac{j}{2}}\phi^{(j)}(\mathfrak{p}^{-j}x - u(k)), k \in \mathbb{Z}\}$ are orthonormal in $H^s(\mathbb{K})$ if and only if*

$$\sum_{k=0}^{\infty} \hat{\nu}^s(\mathfrak{p}^{-j}(\xi + u(k)))|\hat{\phi}^{(j)}(\xi + u(k))|^2 = 1 \quad a.e. \tag{3.4}$$

Moreover, we also have

$$|\hat{\phi}^{(j)}(\mathfrak{p}^j\xi)| \le \hat{\nu}^{-\frac{s}{2}}(\xi). \tag{3.5}$$

Proof. See [3]. □

We will only assume that $\{q^{\frac{j}{2}}\phi^{(j)}(\mathfrak{p}^{-j}. - u(k) : k \in \mathbb{N}_0\}$ is a Riesz basis of V_j in some of our result, which is weaker than being an orthonormal basis. A sequence of functions $\{g_l : l \in \mathbb{N}_0\}$ is called a Riesz basis of Sobolev space $(H^s(\mathbb{K}), \|.\|_{H^s(\mathbb{K})})$. If for any $h \in H^s(\mathbb{K})$, there is a sequence $\{c_l : l \in \mathbb{N}_0\}$ such that $h = \sum_{l\in\mathbb{N}_0} c_l g_l$ which converges in $H^s(\mathbb{K})$ and

$$A^2 \sum_{l\in\mathbb{N}_0} |c_l|^2 \le \| \sum_{l\in\mathbb{N}_0} c_l g_l\|^2_{H^s(\mathbb{K})} \le B^2 \sum_{l\in\mathbb{N}_0} |c_l|^2, \tag{3.6}$$

where the constants A and B satisfy $0 < A \le B < \infty$ and independent of h.
A sequence $\{g_l : l \in \mathbb{N}_0\}$ of element of $H^s(\mathbb{K})$ is called a frame of $(H^s(\mathbb{K}), \|.\|_{H^s(\mathbb{K})})$, if there are $A, B > 0$ (these numbers called frame bounds) such that

$$A^2\|h\|^2_{H^s(\mathbb{K})} \le \sum_{l\in\mathbb{N}_0} |\langle h, g_l\rangle|^2 \le B^2\|h\|^2_{H^s(\mathbb{K})} \tag{3.7}$$

for every $h \in H^s(\mathbb{K})$.
If we assume $\{q^{\frac{j}{2}}\phi^{(j)}(\mathfrak{p}^{-j}x - u(k)) : k \in \mathbb{N}_0\}$ form a Riesz basis of V_j instead of an orthonormal basis in the definition of MRA. It will follows from theorem 3.2

[DOI: 10.61186/ijmsi.19.1.161]

that we can find another function $\phi_1^{(j)}$ such that $\{q^{\frac{j}{2}}\phi_1^{(j)}(\mathfrak{p}^{-j}. - u(k)) : k \in \mathbb{N}_0\}$ form an orthonormal basis of V_j, $j \in \mathbb{Z}$.

Lemma 3.3. *Let $\phi^{(j)} \in H^s(\mathbb{K})$ and $j \in \mathbb{Z}$. If the family $\{\phi_{j,k}^{(j)} = q^{\frac{j}{2}}\phi^{(j)}(\mathfrak{p}^{-j} - u(k)) \ : \ k \in \mathbb{N}_0\}$ satisfies the Riesz condition with bounds A_j, B_j , we have*

$$A_j^2 \sum_{k\in\mathbb{N}_0} |c_k|^2 \leq \| \sum_{k\in\mathbb{N}_0} c_k \phi_{j,k}^{(j)}\|_{H^s(\mathbb{K})}^2 \leq B_j^2 \sum_{k\in\mathbb{N}_0} |c_k|^2 \tag{3.8}$$

where the constants A_j, B_j satisfy $0 < A_j \leq B_j < \infty$ and are independent of $\{c_l\}_{l\in\mathbb{N}_0}$. Let

$$\sigma_{\phi^{(j)}}^2 = \sum_{l\in\mathbb{N}_0} \hat{\nu}^s(\mathfrak{p}^{-j}(\xi + u(k)))|\hat{\phi}^{(j)}(\xi + u(k))|^2. \tag{3.9}$$

Then,

$$A_j \leq \sigma_{\phi^{(j)}}^2 \leq B_j \ a.e. \ \ \xi \in \mathbb{K}. \tag{3.10}$$

Proof. Let $X_\rho = \{\xi \in D : \sum_{l\in\mathbb{N}_0} \hat{\nu}^s(\mathfrak{p}^{-j}(\xi + u(k)))|\hat{\phi}^{(j)}(\xi + u(k))|^2 > \rho\}$. Assume that $X\rho$ has positive measure , we shall show that $\rho \leq B_j$. Let $\mathbb{Q}_X$ denotes characteristic function of the set X. Consider the sequence $\{c_k\} \in l^2(\mathbb{N}_0)$ such that

$$\mathbb{Q}_{X_\rho}(\xi) = q^{-j} \sum_{k\in\mathbb{N}_0} c_k \overline{\chi}_k(\xi) \ \text{ for } a.e \ \ \xi \in \mathfrak{D}. \tag{3.11}$$

Then,

$$\begin{aligned}
\| \sum c_k q^{\frac{j}{2}} \phi^{(j)}(\mathfrak{p}^{-j} - u(k))\|_{H^s(\mathbb{K})} &= \int_{\mathbb{K}} | \sum_{k\in\mathbb{N}_0} c_k q^{-\frac{j}{2}} \overline{\chi}_k(\mathfrak{p}^j\xi)\phi^{(j)}(\mathfrak{p}^j\xi)|^2 \hat{\nu}^s(\xi) d\xi \\
&= \int_{\mathbb{K}} c_k \hat{\nu}^s(\mathfrak{p}^{-j}\xi)| \sum_{k\in\mathbb{N}_0} \overline{\chi}_k(\xi)|^2 |\hat{\phi}^{(j)}(\xi)|^2 d\xi \\
&= \int_{\mathfrak{D}} | \sum_{k\in\mathbb{N}_0} c_k \overline{\chi}_k(\xi)|^2 \ \sum_{k\in\mathbb{N}_0} \hat{\nu}^s(\mathfrak{p}^{-j}\xi + u(k)) \\
&\quad |\hat{\phi}^{(j)}(\xi + u(k))|^2 d\xi \\
&= \int_{X_\rho} \sigma_{\phi^{(j)}}^2(\xi) d\xi \\
&= \int_{X_\rho} \rho d\xi \\
&= \rho |X_\rho|.
\end{aligned}$$

Now by using Parseval's identity where

$$\| \sum_{k\in\mathbb{N}_0} c_k \phi_{j,k}^{(j)}(x)\|_{H^s(\mathbb{K})}^2 \geq \rho|X_\rho| = \rho \sum_{k\in\mathbb{N}_0} |c_k|^2. \tag{3.12}$$

Therefore by using (3.8) we get $\rho < B_j$. Hence the set $\{\xi \in \mathfrak{D} : \sigma_{\phi^{(j)}}^2 > c_2\}$ has measure zero. Therefore $\sigma_{\phi^{(j)}}^2 \leq B_j$ for a.e. $\xi \in \mathfrak{D}$.

[DOI: 10.61186/ijmsi.19.1.161]

Similarly we get left hand inequality of (3.10). Owing the periodicity of $\sigma_{\phi^{(j)}}$, we obtain the announced inequality. □

Theorem 3.4. *Suppose that* $\{q^{\frac{j}{2}}\phi^{(j)}(\mathfrak{p}^{-j}x-u(k)) : k \in \mathbb{N}_0\}$ *form a Riesz basis of* $\overline{V_j}$. *Then there is a function* $\phi_1^{(j)}$ *such that* $\{q^{\frac{j}{2}}\phi_1^{(j)}(\mathfrak{p}^{-j}x - u(k)) : k \in \mathbb{N}_0\}$ *form an orthonormal basis for* V_j.

Proof. It follows from lemma 3.3. □

4. Characterization of Scaling function

Now, in the present section, we will characterize those functions that are scaling functions for an MRA of $H^s(\mathbb{K})$.
A function $\phi^{(j)} \in H^s(\mathbb{K})$, we define the closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $H^s(\mathbb{K})$ as follows:

$$V_j = \overline{\{q^{j/2}\phi^{(j)}(\mathfrak{p}^{-j}. - u(k)) : k \in \mathbb{N}\}}.$$

If the sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ constitutes a multiresolution analysis of $H^s(\mathbb{K})$ then we can say that $\phi^{(j)}$ is a scaling function for a multiresolution analysis of $H^s(\mathbb{K})$.

Theorem 4.1. *A sequence of functions* $\phi^{(j)} \in H^s(\mathbb{K})$ *are a scaling function for MRA of* $H^s(\mathbb{K})$ *if and only if*

$$\sum_{k\in\mathbb{N}_0} \hat{\nu}^s(\mathfrak{p}^{-j}(\xi + u(k))|\hat{\phi}^{(j)}(\xi + u(k))|^2 = 1, \ a.e.\ \xi \in \mathfrak{D} \tag{4.1}$$

$$\lim_{j\to\infty} |\hat{\phi}^{(j)}(\mathfrak{p}^j\xi)| = \hat{\nu}^{\frac{-s}{2}}(\xi), \ a.e.\ \xi \in \mathbb{K} \tag{4.2}$$

and there exists integral periodic function $m_0^{(j+1)}$ *in* $L^2(\mathfrak{D})$ *such that*

$$\hat{\phi}^{(j)}(\xi) = m_0^{(j+1)}(\mathfrak{p}\xi)\hat{\phi}^{(j+1)}(\mathfrak{p}\xi), \ a.e.\ \xi \in \mathbb{K}. \tag{4.3}$$

To prove the above theorem we first prove the following two lemmas .

Lemma 4.2. *Let* $\{V_j : j \in \mathbb{Z}\}$ *be a sequence of closed subspaces of* $H^s(\mathbb{K})$ *satisfying MRA conditions. Then the following two conditions are equivalent*
(1) $\lim_{j\to\infty} \hat{\phi}^{(j)}(\mathfrak{p}^j\xi) = \hat{\nu}^{\frac{-s}{2}}(\xi)$,
(2) $\overline{\cup_{j\in\mathbb{Z}} V_j} = H^s(\mathbb{K})$.

Proof. Let $f \in (\cup_{j\in\mathbb{Z}} V_j)^{\perp}$. So that P_j is orthogonal projection from $H^s(\mathbb{K})$ onto V_j then $P_jf = 0$, for all $j \in \mathbb{Z}$. Assume that $\lim_{j\to\infty} \hat{\phi}^{(j)}(\mathfrak{p}^j\xi) = \hat{\nu}^{\frac{-s}{2}}(\xi)$. We shall show that $f = 0$, a.e. Let $\varepsilon > 0$. Since $\mathscr{S}(\mathbb{K})$ is dense in $H^s(\mathbb{K})$, there exist $h \in \mathscr{S}(\mathbb{K})$ such that

$$\|f - h\|_{H^s(\mathbb{K})} < \varepsilon. \tag{4.4}$$

[DOI: 10.61186/ijmsi.19.1.161]

Hence for all $j \in \mathbb{Z}$,we have

$$\|P_j h\|_{H^s(\mathbb{K})} = \|P_j(f-h)\|_{H^s(\mathbb{K})} \leq \|f-h\|_{H^s(\mathbb{K})} < \varepsilon. \tag{4.5}$$

$\{q^{\frac{j}{2}}\phi^{(j)}(\mathfrak{p}^{-j}x - u(k)) : k \in \mathbb{N}_0\}$ form a Riesz basis, and hence frame for V_j there exist $A_j, B_j > 0$ such that

$$A_j\|f\|^2_{H^s(\mathbb{K})} \leq \sum_{k\in\mathbb{N}_0} |\langle f, \phi^{(j)}_{j,k}\rangle|^2 \leq B_j\|f\|^2_{H^s(\mathbb{K})} \quad for\ all\ \ f \in V_j,\ \ j \in \mathbb{Z}. \tag{4.6}$$

We have,

$$\sum_{k\in\mathbb{N}_0} |\langle h, \phi^{(j)}_{j,k}\rangle|^2 \leq B_j\|P_j h\|^2_{H^s(\mathbb{K})}. \tag{4.7}$$

where

$$\begin{aligned}\sum_{k\in\mathbb{N}_0} |\langle h, \phi^{(j)}_{j,k}\rangle|^2 &= \int_K \hat{\nu}^{2s}(\xi)|\hat{\phi}^{(j)}(\mathfrak{p}^j\xi)|^2|\hat{h}(\xi)|^2 d\xi + \sum_{k=1}^{\infty}\int_K \hat{\nu}^s(\xi)\hat{\nu}^s(\xi+\mathfrak{p}^{-j}u(k)) \\ &\quad \hat{h}(\xi)\overline{\hat{\phi}^{(j)}}(\mathfrak{p}^{-j}\xi)\hat{h}(\xi+\mathfrak{p}^{-j}u(k))\hat{\phi}^{(j)}(\mathfrak{p}^j\xi+u(k))d\xi. \\ &= I_1 + I_2. \end{aligned} \tag{4.8}$$

Now by using (4.3), we have

$$\begin{aligned}|I_2| &\leq \int_{\mathbb{K}} \hat{\nu}^s(\xi)|\hat{h}(\xi)||\overline{\hat{\phi}^{(j)}}(\mathfrak{p}^j\xi)| \sum_{k=1}^{\infty} \hat{\nu}^s(\xi+\mathfrak{p}^{-j}u(k))|\overline{\hat{h}}(\xi+\mathfrak{p}^{-j}u(k))||\hat{\phi}^{(j)}(\mathfrak{p}^j\xi+u(k))|d\xi \\ &\leq \int_{\mathbb{K}} \hat{\nu}^{\frac{s}{2}}(\xi)|\hat{h}(\xi)| \sum_{k=1}^{\infty} \hat{\nu}^{\frac{s}{2}}(\xi+\mathfrak{p}^{-j}u(k))|\hat{h}(\xi+\mathfrak{p}^{-j}u(k))|d\xi.\end{aligned}$$

Again since $\hat{h} \in \mathscr{S}(\mathbb{K})$ there exists k, l such that $\hat{h}(\xi)$ is constant on cosets of $\mathfrak{P}^{-k}$ and is supported on $\mathfrak{P}^{-l}$. Now we show that for large j, each term of the sum that defines T_2 is zero. Let $\hat{h}(\xi) \neq 0$ then $\xi \in \mathfrak{P}^{-l}$, so $|\xi| \leq q^l$. For $j > l$ and for any $l \in \mathbb{N}$, we have

$$|\mathfrak{p}^{-j}u(l)| = q^j|u(l)| \geq q^j > q^l.$$

Therefore, for $j > l$, we have $|\xi| \neq |\mathfrak{p}^{-j}u(l)|$.
Hence

$$|\xi + \mathfrak{p}^{-j}u(k)| = max(|\xi|, |\mathfrak{p}^{-j}u(k)|) \geq q^j > q^l.$$

That is, $\xi + \mathfrak{p}^{-j}u(l) \notin \mathfrak{P}^{-l}$ and hence $\hat{h}(\xi + \mathfrak{p}^{-j}u(l)) = 0,\ \forall\ j > l$. This shows that,

$$\lim_{j\to\infty} |I_2| = 0. \tag{4.9}$$

Now, by using (4.7) and (4.9), we have

$$\int_{\mathbb{K}} \hat{\nu}^{2s}(\xi)|\hat{\phi}^{(j)}(\mathfrak{p}^j\xi)|^2|\hat{h}(\xi)|^2 d\xi + T_2 < \epsilon^2.$$

[DOI: 10.61186/ijmsi.19.1.161]

$$\lim |I_2| \to 0 \ \ as \ j \to \infty. \tag{4.10}$$

Now, by using (4.4), (4.7) and (4.8) we have,

$$\int_K \hat{\nu}^{2s}(\xi)|\hat{\phi}^{(j)}(\mathfrak{p}^j\xi)|^2|\hat{h}(\xi)|^2 d\xi < B_j\varepsilon^2. \tag{4.11}$$

$$\int_K \hat{\nu}^{2s}(\xi)|\hat{\phi}^{(j)}(\mathfrak{p}^j\xi)|^2|\hat{h}(\xi)|^2 d\xi < B_j\varepsilon^2 - I_2. \tag{4.12}$$

Using condition (4.4) and taking $\lim j \to \infty$. It follows that

$$\|h(\xi)\|_{H^s(\mathbb{K})} \leq B^{\frac{1}{2}}\varepsilon. \tag{4.13}$$

Hence

$$\|f\|_{H^s(\mathbb{K})} \leq (1 + B^{\frac{1}{2}})\varepsilon. \tag{4.14}$$

Since ε was arbitrary, we get that $f = 0$ a.e.
Assume now

$$\overline{\cup_{j\in\mathbb{Z}} V_j} = H^s(\mathbb{K}).$$

Consider

$$\hat{f}(\xi) = \hat{\nu}^{\frac{-s}{2}}(\xi)\varphi_0(\xi).$$

Then,

$$\|f\|_{H^s(\mathbb{K})} = 1.$$

We have $\|P_j f\|_{H^s(\mathbb{K})} \to \|f\|_{H^s(\mathbb{K})} = 1$ as $j \to \infty$.
Therefore we have

$$\|P_j f\|_{H^s(\mathbb{K})} = \sum_{k\in\mathbb{N}_0} |\langle f, \phi^{(j)}_{j,k}\rangle|^2. \tag{4.15}$$

Since $\{q^{\frac{j}{2}}\phi^{(j)}(\mathfrak{p}^{-j}x - u(k)) : k \in \mathbb{N}_0\}$ is an orthonormal basis of V_j. For j large enough , we have $\mathfrak{D} \subset \mathfrak{P}^{-j}$. Therefore by using Parseval's identity, we have

$$\begin{aligned}
\|P_j f\|_{H^s(\mathbb{K})} &= \sum_{k\in\mathbb{N}_0} |\int_{\mathbb{K}} \hat{\nu}^s(\xi)\hat{f}(\xi)q^{\frac{-j}{2}}\overline{\phi^{(j)}(\mathfrak{p}^j\xi)}\chi_k(\mathfrak{p}^j\xi)d\xi|^2 \\
&= \sum_{k\in\mathbb{N}_0} |\int_{\mathfrak{P}^{-j}} \hat{\nu}^s(\xi)\hat{f}(\xi)q^{\frac{-j}{2}}\overline{\phi^{(j)}(\mathfrak{p}^j\xi)}\chi_k(\mathfrak{p}^j\xi)d\xi|^2 \\
&= \sum_{k\in\mathbb{N}_0} |\int_{\mathfrak{D}} \hat{\nu}^s(\mathfrak{p}^{-j}\xi)\hat{f}(\mathfrak{p}^{-j}\xi)q^{\frac{j}{2}}\overline{\hat{\phi}^{(j)}(\xi)}\chi_k(\xi)d\xi|^2 \\
&= \int_{\mathfrak{D}} q^j|\hat{\nu}^s(\mathfrak{p}^{-j}\xi)|^2|\hat{f}(\mathfrak{p}^{-j}\xi)|^2|\overline{\hat{\phi}^{(j)}(\xi)}|^2 d\xi \\
&= \int_{\mathbb{K}} \hat{\nu}^s(\xi)|\hat{f}(\xi)|^2\hat{\nu}^s(\xi)|\hat{\phi}^{(j)}(\mathfrak{p}^j\xi)|^2 d\xi
\end{aligned}$$

since $\|P_j f\|_{H^s(\mathbb{K})} \to 1$ as $j \to \infty$

$$\lim_{j\to\infty} \int_{\mathbb{K}} \hat{\nu}^s(\xi)|\hat{f}(\xi)|^2 \hat{\nu}^s(\xi)|\hat{\phi}^{(j)}(\mathfrak{p}^j\xi)|^2 d\xi = 1. \tag{4.16}$$

From Dominated Convergence Theorem ,we get

$$\lim_{j\to\infty} \hat{\phi}^{(j)}(\mathfrak{p}^j\xi) = \hat{\nu}^{\frac{-s}{2}}(\xi).$$

□

Now, we show that the intersection triviality condition in the definition of MRA follows from the other properties .

Lemma 4.3. *Let $\{V_j : j \in \mathbb{Z}\}$ be the sequence of closed subspaces of $H^s(\mathbb{K})$ satisfying condition (1)and(2) of definition (3.1) with there exists a function $\phi^{(j)} \in H^s(\mathbb{K})$ such that $\{q^{\frac{j}{2}}\phi^{(j)}(\mathfrak{p}^{-j}x - u(k)) : k \in \mathbb{Z}\}$ form a Riesz basis of V_j, then*
$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

Proof. Since $\{q^{\frac{j}{2}}\phi^{(j)}(\mathfrak{p}^{-j}x - u(k)) : k \in \mathbb{Z}\}$ constitute a Riesz basis for V_j , so we can write

$$A_j\|f\|^2_{H^s(\mathbb{K})} \leq \sum_{k\in\mathbb{N}_0} |\langle f, \phi^{(j)}_{j,k}\rangle|^2 \leq B_j\|f\|^2_{H^s(\mathbb{K})} \quad \textit{for all} \;\; f \in V_j, \;\; j \in \mathbb{Z}. \tag{4.17}$$

Let $f \in \bigcap_{j\in\mathbb{Z}} V_j$ and $\varepsilon > 0$. We shall show that $f = 0$. Since $\mathscr{S}(\mathbb{K})$ is dense in $H^s(\mathbb{K})$. So there exists $g \in \mathscr{S}(\mathbb{K})$, where $g(x) = ((\hat{\nu}^{\frac{-s}{2}}\hat{h}(\xi))^2)^{\vee}$, $\hat{h} \in \mathscr{S}(\mathbb{K})$ such that

$$\|f - g\| < \varepsilon. \tag{4.18}$$

Hence for all $j \in \mathbb{Z}$

$$\|f - P_j g\|_{H^s(\mathbb{K})} = \|P_j(f - g)\|_{H^s(\mathbb{K})} \leq \|f - g\|_{H^s(\mathbb{K})} < \varepsilon. \tag{4.19}$$

Therefore

$$\|f\|_{H^s(\mathbb{K})} < \varepsilon + \|P_j g\|_{H^s(\mathbb{K})}. \tag{4.20}$$

From (4.17), we have

$$\|P_j g\|_{H^s(\mathbb{K})} \leq A_j^{\frac{-1}{2}} (\sum_{k\in\mathbb{N}_0} |\langle g, \phi^{(j)}_{j,k}\rangle|^2)^{\frac{1}{2}}. \tag{4.21}$$

Therefore there exist $j \in \mathbb{Z}$ such that

$$\sum_{k\in\mathbb{N}_0} |\langle g, \phi^{(j)}_{j,k}\rangle|^2 < \varepsilon^2 A_j. \tag{4.22}$$

Let $f \in \cap_{j\in\mathbb{Z}} V_j$. We know that $\mathscr{S}(\mathbb{K})$ is dense in $H^s(\mathbb{K})$ so there exists $\psi(\xi)$ such that,

$$||f - \psi||_{H^s(\mathbb{K})} < \epsilon, \text{ where } \psi(\xi) = (\hat{\nu}^{-\frac{s}{2}}(\xi)h(\xi))^{\vee}, \text{ and } \psi(\xi) \in \mathscr{S}(\mathbb{K}). \tag{4.23}$$

[DOI: 10.61186/ijmsi.19.1.161]

Now, for all $j \in \mathbb{Z}$,

$$||f - P_j\psi||_{H^s(\mathbb{K})} = ||P_j(f-\psi)||_{H^s(\mathbb{K})} \leq ||f-\psi||_{H^s(\mathbb{K})} < \epsilon.$$

Therefore,

$$||f||_{H^s(\mathbb{K})} \leq ||P_j\psi||_{H^s(\mathbb{K})} + \epsilon. \tag{4.24}$$

We only need to show that $\lim_{j\to\infty} ||P_j\psi||^2_{H^s(\mathbb{K})} = 0$.

Using equations (4.6), (4.15) and Hölder's inequality, we get

$$\begin{aligned} ||P_j\psi||_{H^s(\mathbb{K})} &= \sum_{k=0}^{\infty}\Big(\int_{\mathbb{K}} \hat{\nu}^s(\xi)|\hat{h}(\xi)|^2|\hat{\phi}^{(j)}(\mathfrak{p}^j\xi)|^2 d\xi\Big)^{\frac{1}{2}} \\ &\quad \Big(\int_{\mathbb{K}} \hat{\nu}^s(\xi+\mathfrak{p}^{-j}u(k))|\hat{h}(\xi+\mathfrak{p}^{-j}u(k))|^2|\hat{\phi}^{(j)}(\mathfrak{p}^j\xi+u(k))|^2 d\xi\Big)^{\frac{1}{2}}. \end{aligned}$$

Since $\hat{h} \in \mathscr{S}(\mathbb{K})$, so there exists a characteristic function $\varphi_r(\xi-\xi_0)$ of the set $\xi_0+\mathfrak{P}^r$, where r is some integers. Now $\hat{h}$ can be written as $\hat{h}(\xi) = q^{\frac{r}{2}}\varphi_r(\xi-\xi_0)$. If $\xi+\mathfrak{p}^{-j}u(k) \in \xi_0+\mathfrak{P}^r$, then $|\mathfrak{p}^{-j}u(k)| \leq q^{-r}$, hence $|u(k)| \leq q^{-r-j}$. Then summation index k is bounded by $q^{(-r-j)}$. So using this, we get

$$\begin{aligned} ||P_j\psi||_{H^s(\mathbb{K})} &\leq q^{-r-j}\Big(\int_{\mathbb{K}} \hat{\nu}^s(\xi)|\hat{h}(\xi)|^2|\hat{\phi}^{(j)}(\mathfrak{p}^j\xi)|^2 d\xi\Big)^{\frac{1}{2}} \\ &\leq q^{-r-j}\int_{\xi_0+\mathfrak{P}^r} \hat{\nu}^s(\xi)|\hat{\phi}^{(j)}(\mathfrak{p}^j\xi)|^2 d\xi \\ &= q^{-r}\int_{\mathfrak{p}^{-j}\xi_0+\mathfrak{P}^{-j+r}} \hat{\nu}^s(\mathfrak{p}^{-j}\xi)|\hat{\phi}^{(j)}(\xi)|^2 d\xi. \end{aligned}$$

Suppose that $\xi_0 \neq 0$. For any $\epsilon > 0$, choose $J < 0$ enough small satisfies the following two inequalities : $q^J < |\xi_0| = q^\rho$ such that $J+\rho < 0$, and

$$\int_{\mathfrak{P}^{(-J-\rho)}} \hat{\nu}^s(\mathfrak{p}^{-J}\xi)|\hat{\phi}^{(j)}(\xi)|^2 d\xi < \epsilon. \tag{4.25}$$

We have,

$$\mathfrak{p}^{-j}\xi_0 + \mathfrak{P}^{-j+r} \subset \mathfrak{P}^{(-J-\rho)} \text{ for all } j \leq J. \tag{4.26}$$

Since $|\mathfrak{p}^{-j}\xi_0| = q^j q^\rho \leq q^J q^\rho$ and $\mathfrak{P}^{-j+r} \subset \mathfrak{P}^{-J-\rho}$.

Hence, $||P_j\psi||_{H^s(\mathbb{K})} \to 0$ as $j \to -\infty$. Therefore there exist j such that

$$||P_j\psi||_{H^s(\mathbb{K})} < \epsilon.$$

Hence,

$$||f||_{H^s(\mathbb{K})} < 2\epsilon.$$

Since ϵ was arbitrary we get $f = 0$ a.e..

This shows that $\cap_{j\in\mathbb{Z}} V_j = \{0\}$. □

Now, we prove the theorem 4.1.

[DOI: 10.61186/ijmsi.19.1.161]

Proof. Suppose $\phi^{(j)}$ is a scaling function for an MRA. Then $\{q^{j/2}\phi^{(j)}(\mathfrak{p}^{-j}x - u(k) : k \in \mathbb{Z}\}$ forms orthonormal system in $H^s(\mathbb{K})$ which is equivalent to equation (4.1). It follows from theorem 3.2. Since $\{V_j : j \in \mathbb{Z}\}$ is an MRA for $H^s(\mathbb{K})$, we have $\bigcup_{j\in\mathbb{Z}} V_j = H^s(\mathbb{K})$ and hence equality (4.3) holds. Thus on the other hand equality (4.2) follows from lemma 4.2.
We now prove the converse part of the theorem. Let us assume that (4.1) , (4.2) and (4.3) are satisfied. The orthogonality of system $\left\{q^{j/2}\phi^{(j)}(\mathfrak{p}^{-j}. - u(k)) : k \in \mathbb{Z}\right\}$ is equivalent to (4.1). Due to this fact, we get condition (4) of the definition of MRA.
Consider the closed space of $H^s(\mathbb{K})$ as that

$$V_j = \overline{\left\{q^{j/2}\phi^{(j)}(\mathfrak{p}^{-j}. - u(k)) : k \in \mathbb{N}\right\}} \tag{4.27}$$

for all $j \in \mathbb{Z}$. We can easily show that for each $j \in \mathbb{Z}$,

$$V_j = \left\{f \in H^s(\mathbb{K}) : \hat{f}(\mathfrak{p}^{-j}\xi) = h_j(\xi)\hat{\phi}^{(j)}(\xi) \text{ for some integral periodic } h_j \in L^2(\mathfrak{D})\right\}.$$

Now we prove the inclusion $V_j \subset V_{j+1}$. Thus , using (4.3), we get

$$\hat{f}(\mathfrak{p}^{-j-1}\xi) = h_j(\mathfrak{p}^{-j}\xi)m_0^{(j+1)}(\xi)\hat{\phi}^{(j+1)}(\xi).$$

It is clear that the function $h_j(\mathfrak{p}^{-j}.)m_0^{(j+1)}(.)$ is integral periodic and belongs to $L^2(\mathfrak{D})$.Thus the inclusion $V_j \subset V_{j+1}$ follows from (4.19) .
The property (2) and (3) in the definition of MRA follows from lemmas 4.2 and 4.3 respectively. The proof of theorem 4.1 is completed. □

Acknowledgments

The authors are thankful to the referee for his/her thorough review and highly appreciate the comments and suggestions for various improvements in the original manuscript.The research of the second author is supported by University Grants Commission (UGC), grant number: F.No. 16-6(DEC. 2017)/2018(NET/CSIR), New Delhi, India.

References

1. C. K. Chui, *An Introduction to Wavelets, Wavelet Analysis and Its Applications*, Academic Press, Boston, MA, 1992.
2. E. Hernández, G. Weiss, *A First Course on Wavelets*, CRC Press, Boca Raton,FL, 1996.
3. Ashish Pathak, Guru P. Singh, Wavelet in Sobolve Space over Local Fields of Positive Characteristics, *International Journal of Wavelets, Multiresolution and Information Processing*, **16**(4), (2018), 1850027(1-16).
4. S. Mallat, Multiresolution Approximations and Wavelet Orthonormal Bases of $L^2(\mathbb{R})$, *Transactions of the American Mathematical Society*, **315** (1), (1989), 69-87.
5. Y. Meyer, *Wavelets and Operators*, Cambridge University Press, Cambridge, 1992.
6. I. Novikov, V. Protasov, M. Skopina, *Wavelet theory*, Translations of Mathematical Monographs, American Mathematical Society, **239**, 2011.
7. S. Dahlke, Multiresolution Analysis and Wavelets on Locally Compact Abelian Groups, *In: P. J. Laurent, A. Lemehaute and L. L. Schumaker, Eds., Wavelets, Images and Surface Fitting, A. K. Peters Wellesley, Wellesley*, (1994), 141-156.

[DOI: 10.61186/ijmsi.19.1.161]

8. J. J. Benedetto, R. L. Benedetto, A Wavelet Theory for Local Fields and Related Groups, *Journal of Geometric Analysis*, **14**, (2004), 423-456.
9. R. L. Benedetto, Examples of Wavelets for Local Fields, in Wavelets, Frames and Operator Theory, *Contemporary Mathematics, American Mathematical Society,Providence, RI*, **345**, (2004), 27-47.
10. S. Albeverio, S. Kozyrev, Multidimensional Basis of p-adic Wavelets and Representation Theory, *P-Adic Numbers Ultrametric Analysis, and Applications*, **1**, (2009), 181-189.
11. A. Y. Khrennikov, V. M. Shelkovich, M. Skopina, p-adic Refinable Functions and MRA-based Wavelets, *Journal of Approximation Theory*, **161**(1), (2009), 226-238.
12. S. Kozyrev, Wavelet Theory as p-adic Spectral Analysis, *Izv. Ross. Akad. Nauk Ser. Mat.*, **66**(2), (2002), 149-158 (in Russian); *translation in Izv. Math.*, **66**(2), (2002), 367-376.
13. H. Jiang, D. Li, N. Jin, Multiresolution Analysis on Local Fields, *Journal of Mathematical Analysis and Applications*, **294**(2), (2004), 523-532.
14. D. Ramakrishnan, R. J. Valenza, *Fourier Analysis on Number Fields*, Graduate Texts in Mathematics, Springer-Verlag, New York, **186**, 1999.
15. M. H. Taibleson, *Fourier Analysis on Local Fields*, Mathematical Notes, Princeton University Press, Princeton, NJ, **15**, 1975.
16. Biswaranjan Behera, Qaiser Jahan, Multiresolution Analysis on Local Fields and Characterization of Scaling Functions, *Advances in Pure and Applied Mathematics*, **3**, (2012), 181-202.
17. H. M. Srivastava, W. Z. Lone, F. A. Shah, A. I. Zayed, Discrete Quadratic-Phase Fourier Transform: Theory and Convolution Structures, *Entropy, Multidisciplinary Digital Publishing Institute(MDPI)*, **24**, (2022), 1340.
18. H. M. Srivastava, Firdous A. Shah, Aajaz A. Teali, On Quantum Representation of the Linear Canonical Wavelet Transform, *Universe, Multidisciplinary Digital Publishing Institute(MDPI)*, **8**(9), (2022), 477.
19. H. M. Srivastava, Firdous A. Shah, Waseem Z. Lone, Fractional Nonuniform Multiresolution Analysis in $L^2(\mathbb{R})$, *Mathematical Methods in the Applied Sciences*, **44**(11), (2021), 9351-9372.
20. H. M. Srivastava, F. A. Shah, W. Z. Lone, Quadratic-phase Wave-packet Transform in $L^2(\mathbb{R})$, *Symmetry, Multidisciplinary Digital Publishing Institute(MDPI)*, **14**(10), (2022), 1-16.

[DOI: 10.61186/ijmsi.19.1.161]