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Solis Graphs and Uniquely Metric Basis Graphs

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Abstract. A set $W \subset V(G)$ is called a resolving set, if for every two distinct vertices $u, v \in V(G)$ there exists $w \in W$ such that $d(u, w) \neq d(v, w)$, where $d(x, y)$ is the distance between the vertices x and y. A resolving set for G with minimum cardinality is called a metric basis. A graph with a unique metric basis is called a uniquely dimensional graph. In this paper, we establish a family of graph called Solis graph, and we prove that if G is a minimal edge unique base graph with the base of size two, then G belongs to the Solis graphs family. Finally, an algorithm is given for finding the metric dimension of a Solis graph.

Keywords: Metric dimension, Resolving set, Metric basis, Uniquely metric basis graphs, Solis graph.

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1. Introduction

Let $G = (V, E)$ be a connected simple graph. The distance $d(u, v)$ between two vertices u and v of a graph is the length of a shortest u-v path. For an

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ordered set $R = \{r_1, \ldots, r_k\}$ of vertices and a vertex v in a connected graph G, the ordered k-vector $\langle v|R\rangle = (d(v, r_1), \ldots, d(v, r_k))$ is called the representation of v with respect to R.
The set R is called a **resolving set** for G, if distinct vertices of G have distinct representations with respect to R. A resolving set of the minimum cardinality is called a **basis** for G and its cardinality is called **the metric dimension** of G and denoted by $dim(G)$. A graph with a unique metric basis is called a **uniquely dimensional graph**. According to these facts, it is desirable to recognize uniquely each vertex of the graph; see [5].

In 1975, when Slater [18] was working on tracking submarines and determining the route of planes and ships by sea telecommunications, he discovered the importance of the explorers' collection. He used the terms of the locating set and locating number. Next year, in 1976, Harary F. and Melter [9], independently of Slater, defined these concepts and used modern terminology.

Metric dimension explicitly obtained for other special graphs as, tree [11, 14], circuit graph [6], Johnson and Kneser graphs [1], complete k-partite graph [15], and wheel graph [3, 17]. In addition, the proper bounds for graphs can be seen in [5, 7, 4, 10].

The concept of a resolving set has various applications in different areas including network discovery and verification [2], problems of pattern recognition and image processing [12], robot navigation [11], mastermind game [4], combinatorial search and optimization [16].

As another application of the metric dimension, computer network modeling can be mentioned. Servers in a network represent vertices in a graph and edges represent connections between them. Each vertex in a graph is a possible location for an intruder and, in this sense, a more effective surveillance of each vertex of the graph to control such a possible intruder is worthwhile. According to these facts, it is desirable to recognize uniquely each vertex of the graph [8].

2. Primary

In this section, after mentioning a theorem [5], we introduce several definitions and new symbols, and we will show each uniquely dimensional graph must have at least one cycle. At the end of this section, all uniquely 2-dimensional graphs, with the number of vertices less than 11, are also displayed.

A **major vertex** of G is a vertex of degree at least 3. A leaf u of G is said to be a **terminal vertex** of a major vertex v of G if $d(u, v) < d(u, w)$ for every other major vertex w of G. The terminal degree $ter(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is an exterior major vertex of G if it has positive terminal degree. Let $\sigma(G)$ denote the sum

A network intruder is gaining access through unauthorized access to networking devices through physical, system and remote attempts.

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of the terminal degrees of the major vertices of G, and let $ex(G)$ denote the number of exterior major vertices of G; see[5].

Theorem 2.1. [5, 11, 14]. *Let G be a connected graph of order $n \geq 2$.*

(1) $dim(G) = 1$ *if and only if* $G = P_n$.
(2) $dim(G) = n - 1$ *if and only if* $G = K_n$.
(3) $dim(C_n) = 2$, *where* $n \geq 3$.
(4) *If T is a tree that is not a path, then* $dim(T) = \sigma(T) - ex(T)$.

Definition 2.2. The largest path leading to a leaf of the graph is defined a **flagellum**($\mathfrak{f}$). Also the last vertex of this path is called **pivot vertex** for $\mathfrak{f}$.

So the flagellum $\mathfrak{f}$ with length of m and pivot vertex V_i is shown as $\mathfrak{f} = \{V_i^0, V_i^1, V_i^2, \ldots, V_i^m\}$, $(V_i = V_i^0)$.
Radial distance for the two flagellum is defined as distance of two pivot vertices denoted as $Rd(\mathfrak{f}_1, \mathfrak{f}_2)$. The radial distance of 2 vertices is defined by the radial distance of the flagellums carrying it two. The two flagellums are neighbor if $Rd(\mathfrak{f}_1, \mathfrak{f}_2) = 1$. Also the two flagellums are same root if $Rd(\mathfrak{f}_1, \mathfrak{f}_2) = 0$.
$\mathfrak{f}$ is **internal flagellum** if it has two neighbor flagellums. $\mathfrak{f}$ is **boundary flagellum** if it has the maximum one neighbor flagellum. $\mathfrak{f}_1, \mathfrak{f}_2$ are **coborder flagellums** if the degree of all vertices on the shortest path between them is 2.

Proposition 2.3. *If $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_k$, for $k \geq 2$, are flagellums with same root, then at least $k-1$ vertices of $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_k$ belong to resolving set. Therefore, in the uniquely dimensional graphs, flagellums with the same root are not found.*

Proof. Let $\mathfrak{B}$ be a resolving set for G, and let $\mathfrak{f}_1 = \{V_0, V_1^1, V_1^2, \ldots, V_1^m\}, \mathfrak{f}_2 = \{V_0, V_2^1, V_2^2, \ldots, V_2^n\}$ have a common pivot vertex such that $B \cap \{\mathfrak{f}_1 \cup \mathfrak{f}_2\} = \emptyset$. So

$$\langle V_1^1 | \mathfrak{B} \rangle = \langle V_2^1 | \mathfrak{B} \rangle = \langle V_0 | \mathfrak{B} \rangle + \overrightarrow{\mathbf{1}}$$

($\overrightarrow{\mathbf{1}}$ is a vector of length $\mathfrak{B}$ and the element 1). Thus $\mathfrak{B}$ is not a resolving set for G. This is contradiction. □

Notation 1. *A uniquely dimensional graph G with $dim(G) = k$ is called a uniquely k-dimensional graph. We denote by **k-UBG** the collection of uniquely k-dimensional graphs.*

Notice that 1-UBG= $\{K_1\}$ is a trivial graph.

Definition 2.4. G is a **Core Graph** if $G \in k$-UBG and for all subgraphs of it as H, $H \notin k$-UBG. We use k-$\widehat{\mathbf{UBG}}$ to represent the core graphs.

Diagram of two-dimensional graphs.

Example 2.5. In Figure 2, two 2-UBG graphs are shown. Also, the subgraphs as their cores are drawn with bold lines. Try to find their uniquely basis.

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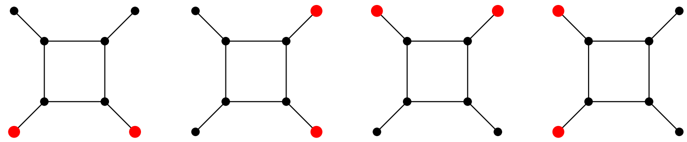

FIGURE 1. All the basis of a non uniquely dimensional graph.

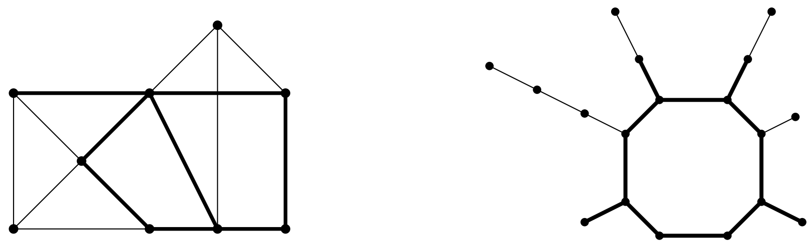

FIGURE 2. Two 2-UBG graphs and their cores.

2.1. **The minimum number of circuit in $G \in 2$-$\widehat{\mathbf{UBG}}$.** In this section we show that, if G is a 2-UBG, then G contains a cycle.

Lemma 2.6. *If $G \in k$-UBG, then the maximum number of flagellums with the same root is one.*

Proof. Let $G \in k$-UBG, and let $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_s$ be of the same root in G. According to Proposition 2.3, $s-1$ vertices of $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_s$ must be in basis. The number of choices is $\binom{s}{s-1} = s$. On the other hand, the graph G is a uniquely metric basis graph. Therefore, $s = 1$. □

Corollary 2.7. *If $G \in k$-UBG, $k \geq 2$, then G has at least one cycle.*

Proof. Let T be a tree that $T \in k$-UBG. According to Lemma 2.6, T does not have two flagellums with the same root. So T is a path. This is contradiction with the first part of Theorem 2.1. □

There are exactly 6 graphs with maximum 10 vertices and 2-$\widehat{\text{UBG}}$. We performed it by Mathematica program (see Figure 3).
We will see, for $n \geq 11$, there exists a family of unicyclic graph as G that $G \in 2$-$\widehat{\text{UBG}}$.

3. Solis Graphs

In this section, we first define the Solis graph. Finally, we for even Solic graphs prove that there are positions that the base vertices cannot be there.

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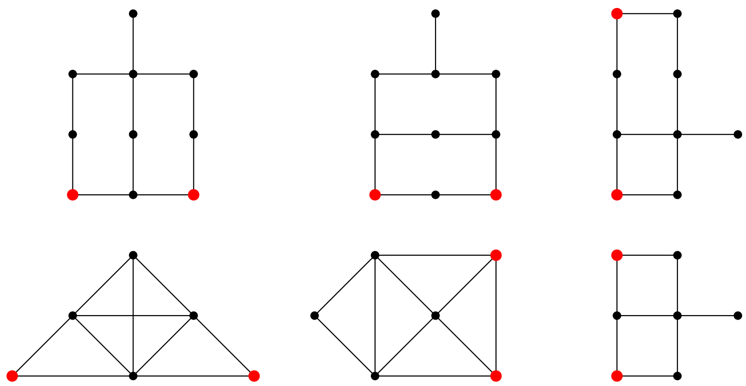

FIGURE 3. All graphs 2-$\widehat{\text{UBG}}$ maximum 10 vertices with their unique base.

Definition 3.1. Let $S = (x_1, x_2, \ldots, x_n)$ be a finite sequence of non-negative integers. The **Solis graph** generated with S denoted by $\langle S \rangle$ is the union of a cycle C_n arranged with vertices $V_1, V_2, \ldots, V_n$ and flagllums with the length of x_i on vertex V_i as $\mathfrak{f}_\mathrm{i}$.
So the Solis graphs generated with S and any rotation or reverse of S are isomorphic. Furthermore, order of Solis graph $\langle S \rangle$ is $|S|$.

Even Solis graph means that the order of Solis graph is even. A Solis graph that is not an even Solis graph is an **Odd Solis graph**.

EXAMPLE 3.2. The Solis graph generated by $(2,1,1,0,0,0,3)$, $(0,0,0,0)$ and $(1,0,2,0,0)$ are shown in Figure 4.
Note that, $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are neighbor flagellums. Also $\mathfrak{f}_3, \mathfrak{f}_7$ are boundary flagellums and $\mathfrak{f}_1, \mathfrak{f}_2$ are internal flagellums. Also, $\mathfrak{f}_3$ and $\mathfrak{f}_7$ are coborder flagellums. The generalized graph $\langle (0,0,0,0) \rangle$ is a cycle graph of order four, which is an even Solis graph. Also $\langle (1,0,2,0,0) \rangle \sim \langle (2,0,0,1,0) \rangle \sim \langle (0,0,2,0,1) \rangle \sim \langle (1,0,0,2,0) \rangle$ is an odd Solis graph.

In [14], following results are obtained for the metric dimension of the Solis graphs.

Theorem 3.3. [14]

- *If $\langle S \rangle$ is an odd Solis graph, then $dim\langle S \rangle = 2$,*
- *If $\langle S \rangle$ is an even Solis graph, then $2 \leq dim\langle S \rangle \leq 3$.*

Also in the proof of the first part of Theorem 3.3, the authors showed that each two vertex x and y with $d(x,y) = |S|$ is a basis for $\langle S \rangle$; So the following result is achieved.

The concept "Solis graph" has already been mentioned in [14] as "Unicyclic graph of type 1". In our definition, more precise details of such a graph are said.

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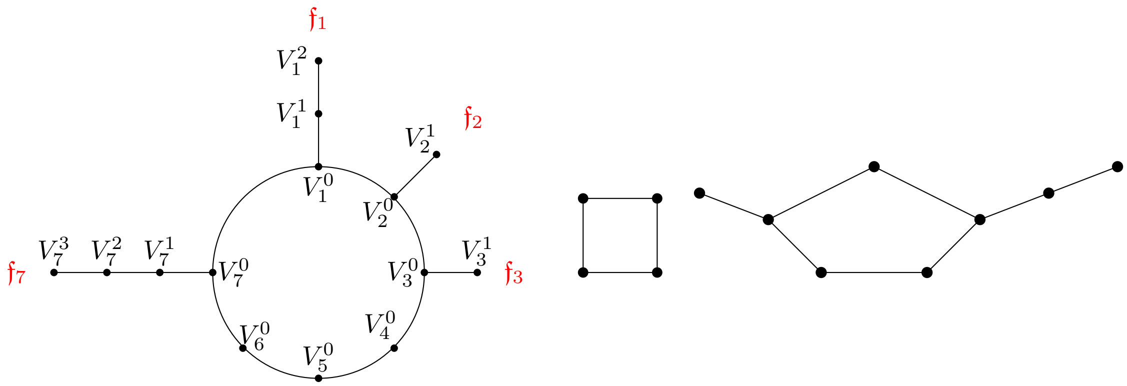

FIGURE 4. Solis graphs generated by $(2, 1, 1, 0, 0, 0, 3)$, $(0, 0, 0, 0)$, and $(0, 0, 2, 0, 1)$.

Corollary 3.4. *If $\langle S\rangle$ is an odd Solis graph, then $\langle S\rangle \notin 2$-UBG.*

According to Lemma 2.6 and Corollary 3.4, the following results are obtained.

Corollary 3.5. *Unicycle and unique two-dimensional graphs are even Solis graphs.*

Notation 2. *We define the symbol of $\mathbb{S}$, as the set of all Solis graphs that are 2-UBG, and $\widehat{\mathbb{S}}$ is the core of members of $\mathbb{S}$.*

Diagram of uniquely two-dimensional graphs.

Corollary 3.6. *Let $\langle S\rangle \in \mathbb{S}$. Then $\langle S\rangle$ contains an even cycle and $Girth\langle S\rangle \geq 6$.*

Proof. Figure 1 along with Corollary 3.5, shows that $Girth\langle S\rangle \geq 6$. □

Lemma 3.7. *Let $\langle S\rangle$ be an even Solis graph, and let $\mathfrak{B} = (a, b)$ be a resolving set for it. Then,*

I. *a, b are not on the same flagellum.*
II. *a, b are not pivot vertices.*
III. *a, b are not on the internal flagellums.*
IV. *a, b are not on the antipodal flagellums.*

Proof. I. Suppose that the flagellum $\mathfrak{f}_1$ contains a, b. In this case, the two vertices X, Y shown in Figure 5 are not resolved.

II. Let $\mathfrak{B} = (a, b)$ be a basis for G, and let a be a pivot vertex. The following table shows the different cases of position vertex b and all vertices not resolved (see Figure 6). In each case, you will see that two vertices of the three vertices X, Y, Z will not be resolved.

Or the position of the flagellum include b.

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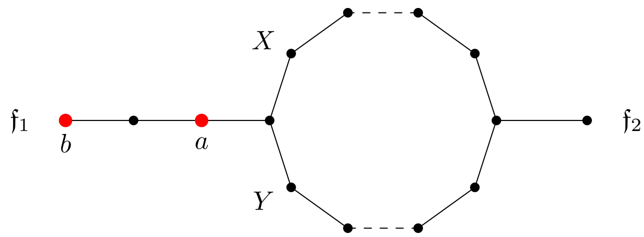

FIGURE 5. Basis vertices are on the same flagellum.

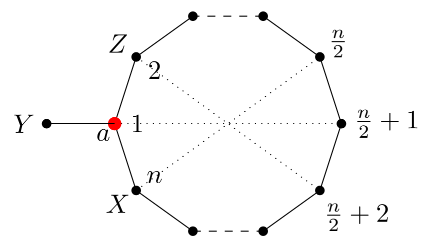

FIGURE 6. One of the base vertices is a pivot vertex.

Position of b	not resolved
$2, 3, \ldots, \frac{n}{2}$	X, Y
$\frac{n}{2}+1$	X, Z
$\frac{n}{2}+2$, $\frac{n}{2}+3, \ldots, n$	Y, Z
Y	X, Z

III. Let flagellum $\mathfrak{f}_1 = \{1, a\}$ have two neighbor flagellums. The following table shows the different cases of position vertex b and other vertices that are not resolved (see Figure 7). In each case, you will see that two of the four vertices Z, T, U, W will not be resolved.

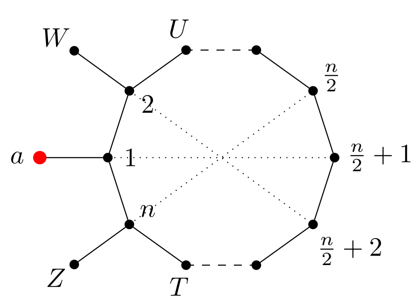

FIGURE 7. One of the base vertices is between two flagellum.

Or the position of the flagellum include b.

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Position of b	not resolved
$2, 3, \dots, \frac{n}{2} - 1$	T, Z
$\frac{n}{2}$	T, W
$\frac{n}{2} + 1$	Z, W
$\frac{n}{2} + 2, \frac{n}{2} + 3, \dots, n$	U, W

Now assume that the length of flagellum $\mathfrak{f}_1$ is m. In this case, if the vertex a is in each position of the flagellum $\mathfrak{f}_1$, then all the details of the above proof will be correct; Even for the pivot vertex, it is true.

IV. Let $\mathfrak{f}_1$ and $\mathfrak{f}_2$ be two antipodal flagellums, and let $a \in \mathfrak{f}_1,\ b \in \mathfrak{f}_2$. In the proof of the first part, choose vertex b from the flagellum $\mathfrak{f}_2$, and it is again seen that vertices X and Y are not resolved (see Figure 5).

□

As an immediate result, we have following corollary.

Corollary 3.8. *For all $\langle S\rangle \in \mathbb{S}$ of even order, if $Rd(\mathfrak{f}_1, \mathfrak{f}_2) = 2$, then no basis vertex can be found between the pivot vertices $\mathfrak{f}_1$ and $\mathfrak{f}_2$.*

Example 3.9. As an application of recent results, consider the graph of Figure 8. To review the basis unique $\langle(1, 0, 1, 1, 1, 1)\rangle$, it is sufficient that only vertices $\{7, 11\}$ are controlled. Because the vertices of the two sets $V_I = \{1, 3, 4, 5, 6\}$ and $V_{II} = \{2, 8, 9, 10\}$ are, respectively, pivot vertices and vertices of internal flagellums. According to Lemma 3.7, they do not have the chance to join the resolving set. Consider $\mathfrak{B} = (7, 11)$; so the address of the vertices of the graph are clearly unique.

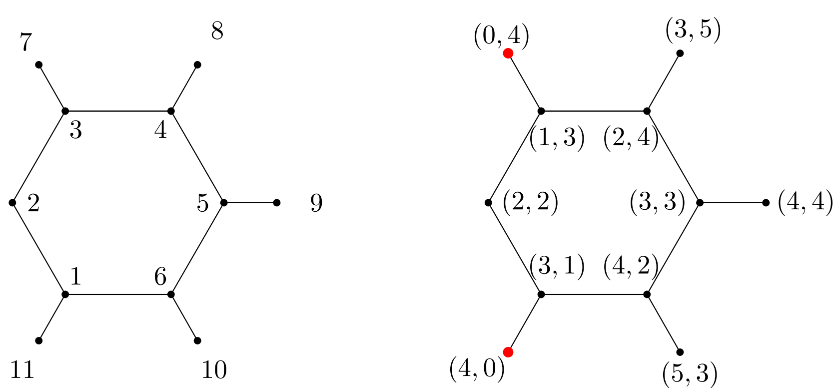

Figure 8. The Solis graph $\langle(1, 0, 1, 1, 1, 1)\rangle$ and vertices addresses obtained by the unique basis.

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4. Standard structure of the Solis graphs

In this section, the standard structure of Solis graphs is introduced. With this structure, the field of characterization of the desired graphs is provided.

Suppose that $n \geq 4$ is even and that $G = \langle (x_1, x_2, \ldots, x_n) \rangle$ is a Solis graph and has a resolving set $\mathfrak{B} = (a, b)$ with $Rd(a, b) = t$.

We know that basis vertices have the maximum distance of one from C_n. Because if the distance a of C_n is two, then $(N_G(a), b)$ will also be a basis; consequently it is a contradiction.

So, there are two vertices α, β of C_n such that satisfy in the following relationships:

$$\alpha = \underset{v \in V(C_n)}{\operatorname{Argmin}} \{d(v, a)\},$$

$$\beta = \underset{v \in V(C_n)}{\operatorname{Argmin}} \{d(v, b)\}.$$

We also define $\theta = (d(a, \alpha), d(b, \beta))$, and it can easily be concluded that $\theta \in \{(0,0), (0,1), (1,0), (1,1)\}$. In Example 14, you will see that there is a 2-UBG for each θ.

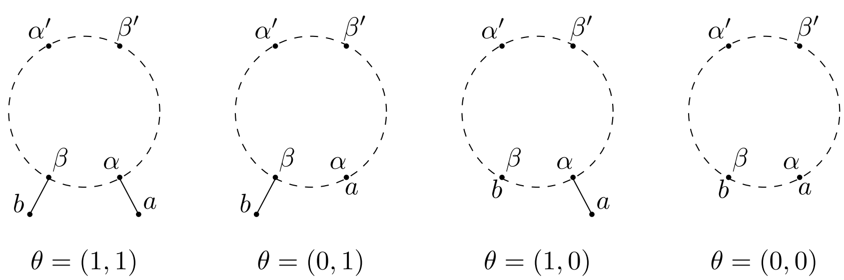

Figure 9. $\mathfrak{B} = (a, b)$ is a base and $\widetilde{\mathfrak{B}} = (\alpha, \beta)$ is a semibase for graph.

In addition, suppose that α' and β' are antipodal vertices of α and β, respectively, on the cycle C_n. It is clear that $d(\alpha, \beta) = t$, and since $t < \frac{n}{2}$, then $\alpha \neq \beta, \alpha' \neq \beta'$.

In this case, we will use the following format as a standard numbering of G.

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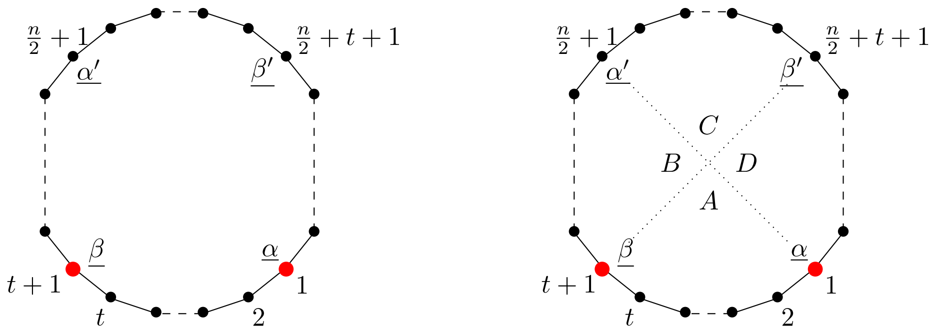

FIGURE 10. Vertices standard numbering and partitions of the Solis graphs.

We also call $\widetilde{\mathfrak{B}} = (\alpha, \beta)$ as a **semibase**. Now we can partition the vertices of C_n by $\alpha, \alpha', \beta, \beta'$ into five sets as the following:

$$\begin{aligned}
A(n,t) &= \{2, 3, \ldots, t\},\\
B(n,t) &= \{t+2, t+3, \ldots, \frac{n}{2}\},\\
C(n,t) &= \{\frac{n}{2}+2, \frac{n}{2}+3, \ldots, \frac{n}{2}+t\},\\
D(n,t) &= \{\frac{n}{2}+t+2, \frac{n}{2}+t+3, \ldots, n\},\\
E(n,t) &= \{1, t+1, \frac{n}{2}+1, \frac{n}{2}+t+1\}.
\end{aligned}$$

Finally, if V_i^j is jth vertex of ith flagellum, then it is clear that:

$$\langle V_i^j | \mathfrak{B} \rangle = (d(V_i^0, \alpha), d(V_i^0, \beta)) + (j, j) + \theta,$$

where

$i \in$	$A(n,t)$	$B(n,t)$	$C(n,t)$	$D(n,t)$
$d(V_i^0, \alpha)$	$i-1$	$i-1$	$n-i+1$	$n-i+1$
$d(V_i^0, \beta)$	$t-i+1$	$i-t-1$	$i-t-1$	$n-i+t+1$

Further, the next section shows that $x_i = 0$ for all $i \in B(n,t) \cup D(n,t)$. Therefore we can display a Solis graph as a ordered triple (T, T', Z), such that,

$$\begin{aligned}
T &= \{x_1, x_2, \ldots, x_{t+1}\},\\
T' &= \{x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_{\frac{n}{2}+t+1}\},\\
Z &= |B(n,t)| = |D(n,t)| = \frac{n}{2} - t - 1.
\end{aligned}$$

Note that all members of $\{x_i | i \in A(n,t) \cup C(n,t)\}$ are present in T or T'. However, by $S = [T, T', Z]$ we can display the graph in a unique way. $[T, T', Z]$ is called **canonical form**, that is more appropriate to storage a graph and computer search.

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5. Main Results

In this section, we obtain the necessary and sufficient condition for the Solis graphs dimension to be two and the necessary condition for 2-UBG Solis graphs. So we show that for all $n \geq 6$, there exists a Solis graph $\langle S\rangle \in \mathbb{S}$ with girth of n. Also an algorithm is presented, which can obtain metric dimensional of a Solis graph.

Theorem 5.1. *Suppose that $\langle S\rangle = \langle(x_1, x_2, \ldots, x_n)\rangle, x_i \in \mathbb{N}_0$ is an even Solis graph.*

In this case $dim\langle S\rangle = 2$ with $\widetilde{\mathfrak{B}} = (V_1, V_{t+1}), t < \frac{n}{2}$, as the semibase of its if and only if:

(1) $\forall i \in A(n,t) \Longrightarrow x_i \leq \frac{n}{2} - t - 1$,
(2) $\forall i \in B(n,t) \Longrightarrow x_i = 0$,
(3) $\forall i \in D(n,t) \Longrightarrow x_i = 0$.

Proof. • $(\Longrightarrow)$

(1) Suppose that we have $x_i \geq \frac{n}{2} - t$ for all $i \in A(n,t)$. In this case, we show that $\langle V_i^{\frac{n}{2}-t}|\mathfrak{B}\rangle = \langle V^0_{\frac{n}{2}+t-i+2}|\mathfrak{B}\rangle$.

$$\begin{aligned}\langle V_i^{\frac{n}{2}-t}|\mathfrak{B}\rangle &= (d(V_i^0,\alpha), d(V_i^0,\beta)) + (\frac{n}{2}-t, \frac{n}{2}-t) + \theta\\ &= (i-1, t-i+1) + (\frac{n}{2}-t, \frac{n}{2}-t) + \theta\\ &= (i+\frac{n}{2}-t-1, \frac{n}{2}-i+1) + \theta.\end{aligned}$$

On the other hand, notice that if $i \in A(n,t)$, then $\frac{n}{2}+t-i+2 \in C(n,t)$. So

$$\begin{aligned}\langle V^0_{\frac{n}{2}+t-i+2}|\mathfrak{B}\rangle &= (d(V^0_{\frac{n}{2}+t-i+2},\alpha), d(V^0_{\frac{n}{2}+t-i+2},\beta)) + \theta\\ &= (n-(\frac{n}{2}+t-i+2)+1, (\frac{n}{2}+t-i+2)-t-1) + (0,0) + \theta\\ &= (i+\frac{n}{2}-t-1, \frac{n}{2}-i+1) + \theta.\end{aligned}$$

Therefore $\mathfrak{B}$ is not a basis; this is a contradiction.

(2) Suppose that there exists $i \in B(n,t)$ such that $x_i \neq 0$ (see Figure 11).

$$\begin{aligned}\langle V_i^1|\mathfrak{B}\rangle &= (d(V_i^0,\alpha), d(V_i^0,\beta)) + (1,1) + \theta\\ &= (i-1, i-t-1) + (1,1) + \theta\\ &= (i, i-t) + \theta\\ &= (d(V_{i+1}^0,\alpha), d(V_{i+1}^0,\beta)) + \theta\\ &= \langle V_{i+1}^0|\mathfrak{B}\rangle.\end{aligned}$$

Therefore $\mathfrak{B}$ is not a basis; this is a contradiction.

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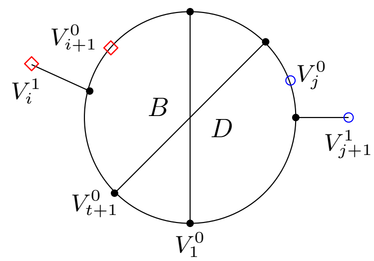

FIGURE 11. The existence of a flagellum in $B(n,t)$ or $D(n,t)$.

(3) Suppose that there exists $j \in D(n,t)$ such that $x_j \neq 0$ (see Figure 11).

$$\begin{aligned}
\langle V_{j+1}^1|\mathfrak{B}\rangle &= (d(V_{j+1}^0,\alpha), d(V_{j+1}^0,\beta)) + (1,1) + \theta \\
&= (n-(j+1)+1, n-(j+1)+t+1) + (1,1) + \theta \\
&= (n-(j+1)+2, n-(j+1)+t+2) + \theta \\
&= (n-j+1, n-j+t+1) + \theta \\
&= (d(V_j^0,\alpha), d(V_j^0,\beta)) + \theta \\
&= \langle V_j^0|\mathfrak{B}\rangle.
\end{aligned}$$

Therefore $\mathfrak{B}$ is not a basis; this is a contradiction.

- $(\Longleftarrow)$ Suppose that there is a t such that
  - for all $i \in A(n,t) \Longrightarrow x_i \leq \frac{n}{2} - t - 1$,
  - for all $i \in B(n,t) \Longrightarrow x_i = 0$,
  - for all $i \in D(n,t) \Longrightarrow x_i = 0$.

  It is sufficient to prove that $\mathfrak{B} = (V_1, V_{t+1})$ is a valid base.
  For this purpose, suppose that $x = V_i^j$ and $y = V_{i'}^{j'}$ are two arbitrary vertices of S and that $\langle x|\mathfrak{B}\rangle = \langle y|\mathfrak{B}\rangle$. In each case we show that $x = y$.
  - $x, y \in A(n,t)$.

$$(i-1, t-i+1) + (j,j) + \theta = (i'-1, t-i'+1) + (j',j') + \theta \Rightarrow \begin{cases} i = i' \\ j = j' \end{cases}$$

Then $x = y$.

- $x \in A(n,t), y \in B(n,t)$.

$$(i-1, t-i+1) + (j,j) + \theta = (i'-1, i'-t-1) + (j',j') + \theta \Rightarrow i = t+1$$

The above case is not possible.

- $x \in A(n,t), y \in C(n,t)$. Because $i' \leq \frac{n}{2} - t - 1$.

$$(i-1, t-i+1) + (j,j) + \theta = (n-i'-1, i'-t-1) + (j',j') + \theta \Rightarrow \begin{cases} i = -i' + \frac{n}{2} + t + 2 \\ j = j' + \frac{n}{2} - t \end{cases}$$

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The above case is not possible.

– $x \in A(n,t), y \in D(n,t)$.

$$(i-1, t-i+1) + (j,j) + \theta = (n-i'+1, n-i'+t+1) + (j',j') + \theta \Rightarrow i = 1$$

The above case is not possible.

– $x, y \in B(n,t)$. According to assumption $j = j' = 0$,

$$(i-1, i-t-1) + (j,j) + \theta = (i'-1, i'-t-1) + (j',j') + \theta \Rightarrow i = i'$$

Then $x = y$.

– $x \in B(n,t), y \in C(n,t)$.

$$(i-1, i-t-1) + (j,j) + \theta = (n-i'+1, i'-t-1) + (j',j') + \theta \Rightarrow i = \frac{n}{2} + 1$$

The above case is not possible.

– $x \in B(n,t), y \in D(n,t)$.

$$(i-1, i-t-1) + (j,j) + \theta = (n-i'+1, n-i'+t+1) + (j',j') + \theta \Rightarrow 2 = 0$$

The above case is not possible.

– $x, y \in C(n,t)$.

$$(n-i+1, i-t-1) + (j,j) + \theta = (n-i'+1, i'-t-1) + (j',j') + \theta \Rightarrow \begin{cases} i = i' \\ j = j' \end{cases}$$

Then $x = y$.

– $x \in C(n,t), y \in D(n,t)$.

$$(n-i+1, i-t-1) + (j,j) + \theta = (n-i'+1, n-i'+t+1) + (j',j') + \theta \Rightarrow i = \frac{n}{2} + t + 1$$

The above case is not possible.

– $x, y \in D(n,t)$. According to assumption $j = j' = 0$,

$$(n-i+1, n-i+t+1) + (j,j) + \theta = (n-i'+1, n-i'+t+1) + (j',j') + \theta \Rightarrow i = i'$$

Then $x = y$.

□

The following theorem gives a necessary condition for $S \in \mathbb{S}$.

Theorem 5.2. *Suppose that* $\langle S \rangle = \langle (x_1, x_2, \ldots, x_n) \rangle \in \mathbb{S}$ *such that* $\widetilde{\mathfrak{B}} = (V_1, V_{t+1})$ *is a semibase of it. Then exists* $i \in C(n,t)$ *such that* $x_i \geq \frac{n}{2} - t$.

Proof. Suppose $x_i \leq \frac{n}{2} - t - 1$ for all $i \in C(n,t)$. It can be shown that $\widetilde{\mathfrak{B}'} = (V_{\frac{n}{2}+1}, V_{\frac{n}{2}+t+1})$ is another semibase of S. To do this, we perform the rotation of size $\frac{n}{2}$ on the vertices name of C_n. $\widetilde{\mathfrak{B}'} = (V_{\frac{n}{2}+1}, V_{\frac{n}{2}+t+1})$ satisfy conditions of Theorem 5.1.

So $\widetilde{\mathfrak{B}'}$ is a semibase of S. This is a contradiction because $\langle S \rangle \in \mathbb{S}$. □

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Lemma 5.3. *For all $\langle S\rangle \in \mathbb{S}$, the maximum number of consecutive zeroes in $S = (x_1, , x_2, \dots, x_n)$ is $\frac{n}{2} - 2$. In this case, two vertices before and after this zeroes sequence are semibases.*

Proof. Suppose that the maximum number of consecutive zeroes in $S = (x_1, x_2, \dots, x_n)$ is $\frac{n}{2} - 1$. Without loss of generality, let the last $\frac{n}{2} - 1$ components of it be zero,

$$S = (x_1, x_2, \dots, x_{\frac{n}{2}+1}, 0, 0, \dots, 0).$$

It is clear that if

$$S_1 = (x_1, 0, 0, \dots, 0, x_{\frac{n}{2}+1}, x_{\frac{n}{2}}, \dots, x_2)$$

and

$$S_2 = (x_{\frac{n}{2}+1}, 0, 0, \dots, 0, x_1, x_2, \dots, x_{\frac{n}{2}})$$

, then $\langle S\rangle \sim \langle S_1\rangle \sim \langle S_2\rangle$. Now, we see that two sets $\widetilde{\mathfrak{B}}_1 = (V_1, V_{\frac{n}{2}+2})$ and $\widetilde{\mathfrak{B}}_2 = (V_{\frac{n}{2}+1}, V_n)$ along with S_1, S_2 satisfy conditions of Theorem 5.1, and they are semibases for $\langle S\rangle$. it is a contradiction.

To end the proof, consider the Solis graph that has $\frac{n}{2} - 2$ consecutive zeroes, as $S = (x_1, 0, 0, \dots, 0, x_{\frac{n}{2}}, x_{\frac{n}{2}+1}, \dots, x_n)$ and $x_1, x_{\frac{n}{2}} \neq 0$. It is clear that $d(V_1, V_{\frac{n}{2}}) = \frac{n}{2} - 1$ and $B(n, \frac{n}{2} - 1) \cup D(n, \frac{n}{2} - 1) = \emptyset$. According to Theorem 5.1, $\widetilde{\mathfrak{B}} = (V_1, V_{\frac{n}{2}})$ is a semibase for S. □

Corollary 5.4. *Suppose that $\langle S\rangle$ is a Solis graph and $\mathfrak{f}_i, \mathfrak{f}_j$ are coborder flagellums that not contain basis vertices, Then the number of zeroes between them is the maximum of $\frac{n}{2} - 3$.*

Proof. Let $\mathfrak{f}_i, \mathfrak{f}_j \in S \in \mathbb{S}$ be coborder falgellums, and let $Rd(\mathfrak{f}_i, \mathfrak{f}_j) = \frac{n}{2} - 1$. Therefore the number of zeroes between them is $\frac{n}{2} - 2$, and according to Lemma 5.3, $\mathfrak{f}_i, \mathfrak{f}_j$ contain basis vertices. This is a contradiction. So $Rd(\mathfrak{f}_i, \mathfrak{f}_j) \leq \frac{n}{2} - 2$ and the number of zeroes between them are maximum of $\frac{n}{2} - 3$. □

Theorem 5.5. *If $\langle S\rangle = \langle (x_1, , x_2, \dots, x_n)\rangle \in \mathbb{S}$ and $A_0 = \{x_i \in S | x_i \neq 0\}$, then $|A_0| \geq 4$.*

Proof. Let $|A_0| = 3$. Without loss of generality, the indexes $1 = a < b < c \leq n$ exist such that

$$\begin{cases} x_i \neq 0, & i \in \{a, b, c\}, \\ x_i = 0, & i \notin \{a, b, c\}. \end{cases}$$

For all a and b, there are two logical values for c (see Figure 12).

- $b < c \leq a' = 1 + \frac{n}{2}$ or $b + \frac{n}{2} = b' \leq c < a$.
  In this case, the number of consecutive zeroes is greater than $\frac{n}{2} - 2$. Which is not possible by Lemma 5.3.

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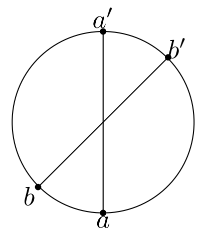

FIGURE 12. Partitions of Solis graph by two vertices.

- $a' < c < b'$.
  In this case, three sets $\{a,b\},\{a,c\}$, and $\{b,c\}$ satisfy all the necessary conditions of Theorem 5.1. Therefore $\{a,b\},\{a,c\}$, and $\{b,c\}$ are semibases. This is a contradiction.

□

Theorem 5.6. *For all $\langle S\rangle \in \mathbb{S}$, the radial distance of the basis vertices a and b is at least two. Therefore,*

$$2 \leq Rd(a,b) \leq \frac{n}{2} - 1.$$

Proof. Radial distance cannot be zero. Because if $Rd(a,b)=0$, then a and b are on the same flagellum, which contradicts with Part I of Lemma 3.7. Let $Rd(a,b)=1$. For $t=1$, we have

$$\begin{aligned}
&A(n,1) = \{\},\\
&B(n,1) = \{3,4,\ldots,\frac{n}{2}\},\\
&C(n,1) = \{\},\\
&D(n,1) = \{\frac{n}{2}+3,\frac{n}{2}+4,\ldots,n\},\\
&E(n,1) = \{1,2,\frac{n}{2}+1,\frac{n}{2}+2\}.
\end{aligned}$$

We also know $x_i = 0$, for $i \in A(n,1)\cup B(n,1)$, and according to Theorem 5.5, four remaining vertices must be nonzero. So $x_1, x_2, x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2} \neq 0$ (see Figure 13).

Now $\widetilde{\mathfrak{B}}_1 = \{V_1,V_2\}$ and $\widetilde{\mathfrak{B}}_2 = \{V_{\frac{n}{2}+1},V_{\frac{n}{2}+2}\}$ satisfy conditions of Theorem 5.1 and they can resolve graph vertices. it is a contradiction. Therefore $Rd(a,b)\geq 2$. Also according to part IV of Lemma 3.7, we have $Rd(a,b)\leq \frac{n}{2}-1$. □

EXAMPLE 5.7. With a program written by Mathematica software, several examples of Solis graphs 2-$\widehat{\text{UBG}}$ are shown in Figure 14.

Radial distance of their basis are $2,3,6,8$ and the distance of the vertex basis of the cycle are $(0,0),(1,1),(1,0)$.

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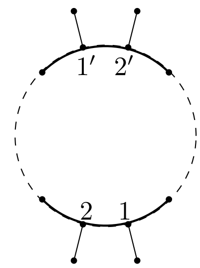

FIGURE 13. Solis graph with $Rd(a, b) = 1$.

Note that the four numbers that have been underlined, are semibase and antipodal's $(\alpha, \beta, \alpha', \beta')$.

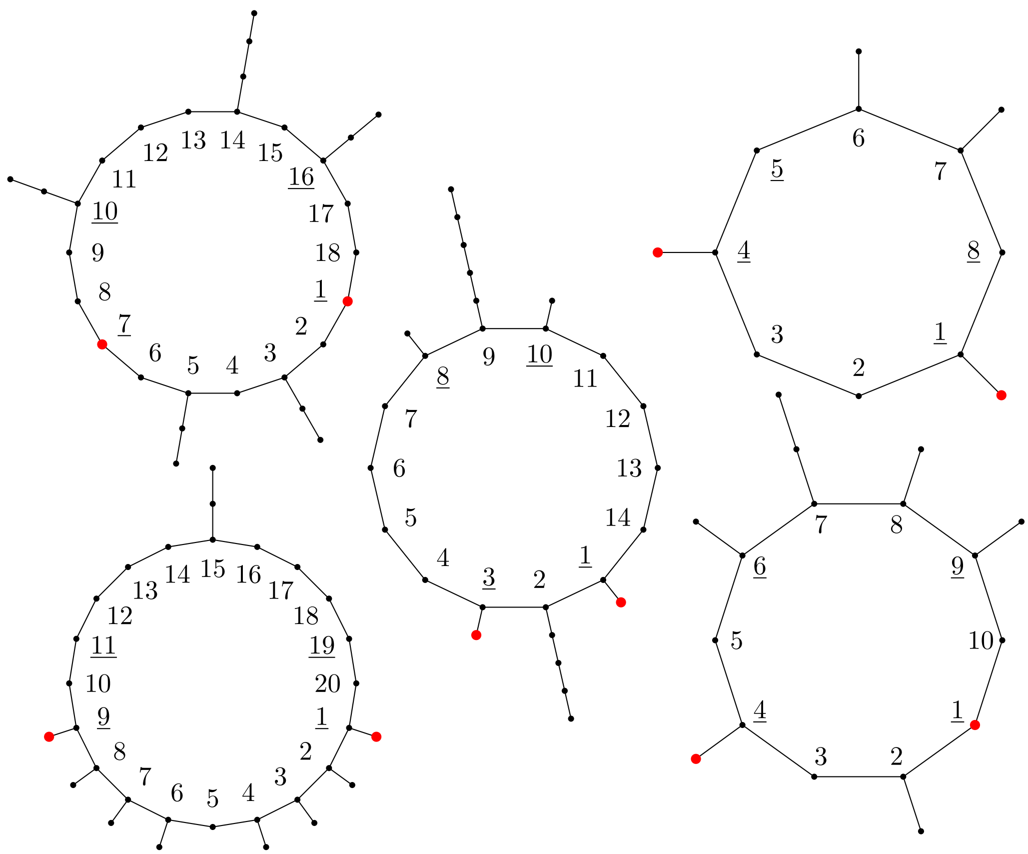

FIGURE 14. Some core of Solis graphs that are 2-UBG.

5.1. **Characterization of $\mathbb{S}$ with minimum radial distance.** In this section, we find all $S \in \mathbb{S}$, that their distance basis is at least.

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Let $S = (x_1, x_2, \ldots, x_n)$ be a core of Solis graph with $\mathfrak{B} = (a, b)$ as its unique base. According to Theorem 5.6, the minimum $Rd(a, b)$ is two. So $\widetilde{\mathfrak{B}} = (V_1, V_3)$ is a semibase of it and $t = 1$.

$$\begin{aligned}
A(n,1) &= \{x_2\},\\
B(n,1) &= \{x_4, x_5, \ldots, x_{\frac{n}{2}}\},\\
C(n,1) &= \{x_{\frac{n}{2}+2}\},\\
D(n,1) &= \{x_{\frac{n}{2}+4}, x_{\frac{n}{2}+5}, \ldots, x_n\},\\
E(n,1) &= \{x_1, x_3, x_{\frac{n}{2}+1}, x_{\frac{n}{2}+3}\}.
\end{aligned}$$

Correspond to Theorems 5.1 and 5.2,

$$x_i = \begin{cases} \frac{n}{2} - 3, & i = 2,\\ 0, & i = 4, 5, \ldots, \frac{n}{2},\\ \frac{n}{2} - 2, & i = \frac{n}{2} + 2,\\ 0, & i = \frac{n}{2} + 4, \frac{n}{2} + 5, \ldots, n. \end{cases}$$

Now values of four remaining elements $i \in E(n,1) = \{x_1, x_3, x_{\frac{n}{2}+1}, x_{\frac{n}{2}+3}\}$ must be one. Otherwise, without loss of generality suppose that $x_1 = 0$. in this case, the number of zeroes between V_2 and $V_{\frac{n}{2}+3}$ will be greater than the limit specified in Corollary 5.4.

Now, we have characterized the optimal graph of this section as below.

$$S = (1, \frac{n}{2} - 3, 1, \overbrace{0, 0, \ldots, 0}^{\frac{n}{2}-3}, 1, \frac{n}{2} - 2, 1, \overbrace{0, 0, \ldots, 0}^{\frac{n}{2}-3});$$

in the other words, if $\mathbb{S}_y^n = \{S \in \mathbb{S} | Rd(a, b) = y, |S| = n\}$, then canonical form of the above graphs is as follows:

$$\mathbb{S}_2^n = [(1, \frac{n}{2} - 3, 1), (1, \frac{n}{2} - 2, 1), \frac{n}{2} - 3].$$

Theorem 5.8. *For all even $n \geq 6$, there is a 2-UBG with $Girth(G) = n$.*

Proof. Suppose that the integer and even number $n \geq 6$ is given.

Consider $\mathbb{S}_2^n = [(1, \frac{n}{2} - 3, 1), (1, \frac{n}{2} - 2, 1), \frac{n}{2} - 3]$. Based on what is described in Section 5.1, this graph is 2-UBG and $Girth\langle S \rangle = 2 \times 3 + 2(\frac{n}{2} - 3) = n$. □

In fact, all graphs in $\mathbb{S}_2^n$ are core graph except $\mathbb{S}_2^8$, which are checked with a computer program. More precisely that,

$$\forall S \in \mathbb{S}_2^n, \quad Core(S) = \begin{cases} [(1, 0, 1), (1, 2, 1), 1], & n = 8,\\ S, & else. \end{cases}$$

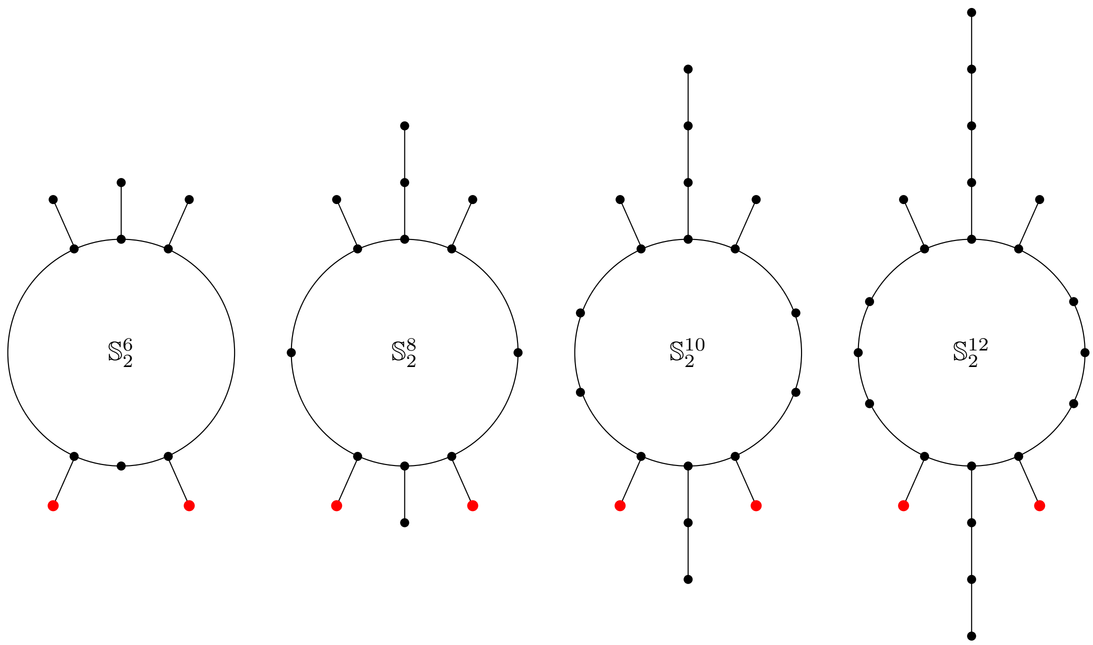

FIGURE 15. Several uniquely 2-dimensional graph.

5.2. **Summary of obtained properties.** Let $S = (x_1, x_2, \ldots, x_n)$ be a sequence of integer and non-negative numbers. For the Solis graph $\langle S \rangle$ with $\widetilde{\mathfrak{B}} = (V_1, V_{t+1})$ as the unique semibase of it, we obtained the following properties.

(1) n is even.
(2) $n \geq 6$.
(3) At least four elements of S are nonzero.
(4) The maximum number of consecutive zeroes in S is $\frac{n}{2} - 2$.
(5) basis vertices are not pivot vertices.
(6) basis vertices are not on the inner flagellum.
(7) $2 \leq t \leq \frac{n}{2} - 1$.
(8) $x_i = 0$ for all $i \in B(n,t) \cup D(n,t)$.
(9) $x_i \leq \frac{n}{2} - t - 1$ for all $i \in A(n,t)$.
(10) There exists $i \in C(n,t)$ such that $x_i \geq \frac{n}{2} - t$.

Remark 5.9. If $dim\langle S \rangle = 2$ but it is not unique metric dimensional, then items $5, 6, 8, 9$ are correct.

5.3. **An algorithm to find the metric dimension of a Solis graph.** In this Algorithm, the symbol $\circlearrowleft$ is a modular function, that starts from one.

$$\circlearrowleft m = x \iff 1 \leq x \leq n, \quad x = m - kn \quad \text{for some } k \in \mathbb{Z}$$

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(1) Input $S = (x_1, x_2, \dots, x_n)$ as integer and non-negative numbers.
(2) If n is an odd number, then $dim\langle S\rangle = 2$; go to 15.
(3) If the maximum number of consecutive zeroes in S is at least $\frac{n}{2} - 1$, then
$dim\langle S\rangle = 2$; go to 15.
(4) If the number of nonzero elements S is less than 4, then $dim\langle S\rangle = 2$; go to 15.
(5) Rotate the elements of S such that $x_1 \neq 0$.
(6) Find the boundary flagellums of S. Let,

$$T_1 = \{1 \leq i \leq n | x_{\circlearrowleft(i-1)} x_{\circlearrowleft(i+1)} = 0\}.$$

(7) For all $j \in T_1$, let

$$L_j = \begin{cases} n + 1 - Max\{i | x_i \neq 0\}, & j = 1, \\ j - Max\{i \leq j - 1 | x_i \neq 0\}, & else. \end{cases}$$

(8) For all $j \in T_1$, let

$$U_j = \begin{cases} Min\{L_j, L_{j+\frac{n}{2}}\}, & j \leq \frac{n}{2}, \\ Min\{L_j, L_{j-\frac{n}{2}}\}, & else. \end{cases}$$

(9) For all $j \in T_1$, let

$$T_2^j = \{i \in T_1 | \circlearrowleft (\circlearrowleft (\frac{n}{2} + j) - U_j) \leq i \leq \circlearrowleft (\frac{n}{2} + j - 1)\}.$$

(10) For all $j \in T_1$ that $j \leq \frac{n}{2} + 1$, let

$$T_{31}^j = \{i \in T_2^j | i \leq j + \frac{n}{2} - 1 - Max\{x_k\}_{k=j+1}^{i-1}\}.$$

(11) For all $j \in T_1$ that $j \geq \frac{n}{2} + 2$, let

$$T_{32}^j = \{i \in T_2^j | i \leq j - \frac{n}{2} - 1 - Max\{x_{\circlearrowleft k}|\}_{k=j+1}^{n+i-1}\}.$$

(12) For all $j \in T_1$, let

$$Index(j) = \begin{cases} T_{31}^j, & 1 \leq j \leq \frac{n}{2} + 1, \\ T_{32}^j, & \frac{n}{2} + 2 \leq j \leq n. \end{cases}$$

(13) **The basis identification:** Now all the vertices that locate on the ith flagellum $i \in T_1$ with all the vertices that locate on the jth flagellum $j \in Index(j)$ are the resolving set for $\langle S\rangle$.
(14) **Metric dimension identification:** If there exists j such that $Index(j) \neq \emptyset$, then $dim\langle S\rangle = 2$; Else $dim\langle S\rangle = 3$.
(15) End.

By the above algorithm, we can find the number of the 2-member basis of even Solis graph.

Let $\langle S\rangle$ have resolving sets as $\mathfrak{B}_1 = (V_i^j, x)$. In the process of adding a vertex and connecting to V_i^j, two logical states occur. In the Solis graph, the

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basis are on cycle or on of the flagellums. So they are either degree one or two. Also, we know that the pivot vertices cannot be the base.

- When $V_i^j \in C_n$, it does not affect the number of basis (the new vertex is new base).
- When $V_i^j \notin C_n$, one adds to the number of basis (the new vertex and the vertex attached to it are base).

Suppose that

$$\gamma(i) = \begin{cases} 1, & x_i = 0, \\ x_i, & x_i > 0. \end{cases}$$

Also, let $\tau(S)$ be the number of 2-member basis of even Solis graph. To find $\tau(S)$, if, for all i, $x_i = 0$, then the selection of both vertices is a base unless they face each other. Therefore,

$$\tau(S) = \binom{n}{2} - \frac{n}{2} = \frac{n^2}{2} - n.$$

Else if there exists i such that $x_i \neq 0$, remove step 2,3,4 of the algorithm and use it on $S = (x_1, x_2, \ldots, x_n)$, then let

$$\tau(S) = \sum_{j \in T_1} \sum_{k \in Index(j)} \gamma(j)\gamma(k).$$

Theorem 5.10. *Suppose that $\langle S \rangle$ is a Solis graph. Therefore,*

$$\begin{cases} \text{if } \tau(S) = 0, & \text{then } \langle S \rangle \text{ is a three dimensional graph,} \\ \text{if } \tau(S) = 1, & \text{then } \langle S \rangle \text{ is 2-UBG,} \\ \text{if } \tau(S) \geq 2, & \text{then } \langle S \rangle \text{ is a two dimensional graph.} \end{cases}$$

Comparing the direct search algorithm and algorithm 5.3, consider the particular family graphs as,

$$G(n) = \langle (2, 1, 1, 2, 1, 4, \overbrace{0, \ldots, 0}^{n-6}) \rangle,\ n \geq 6.$$

It is clear that $Girth(G(n)) = n$. Using one computer, we obtain the number of 2-member basis of the graph $G(n)$, applying two algorithms. While the numbers obtained were exactly the same, the time to do the calculations was very different. This suggests that the algorithm 5.3 is faster than direct search algorithm (see Figure 16).

Review all 2-member subsets of vertices, to find the number of 2-member basis.
Set $\tau = 0$ and $Y = \{\{u, v\} | u, v \in V(G)\}$. Select one member in Y. If they resolve the graph vertices, increase τ. Otherwise, select the other member. Repeat this algorithm for all members of Y.

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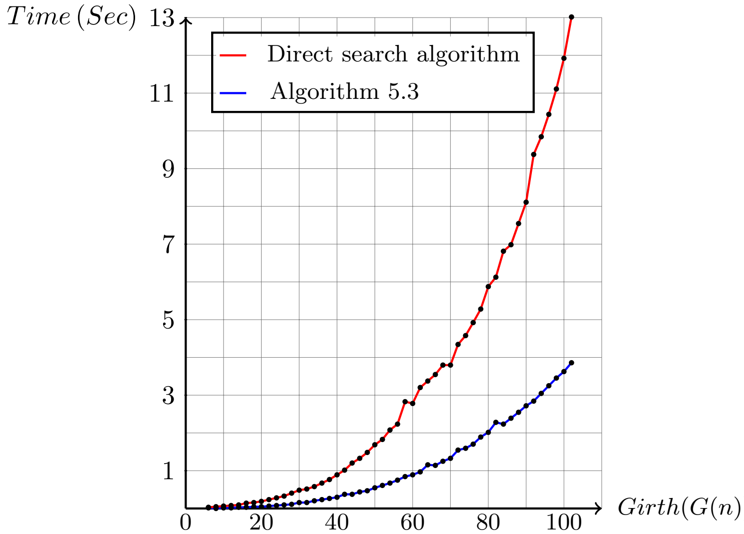

FIGURE 16. The duration of the calculation of the direct search algorithm and algorithm 5.3.

EXAMPLE 5.11. In Figure 17, a 5-UBG graph and its core are shown. In addition, they have subgraph in the shape of the Solis graph.

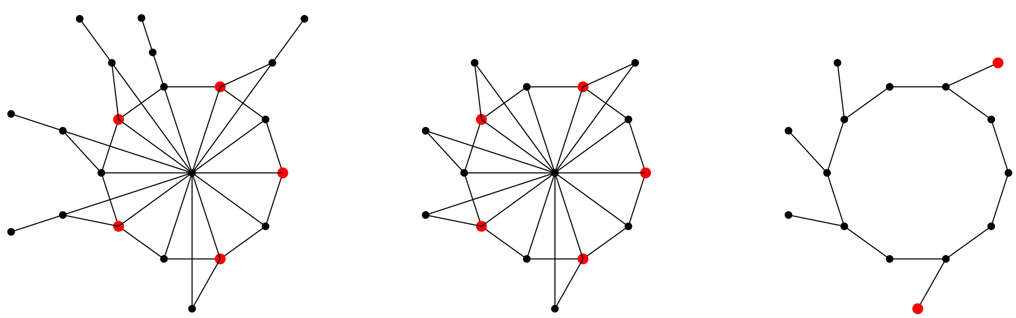

FIGURE 17. A 5-UBG and its core with a subgraph of them that is a Solis graph.

Problem. *Characterize a Solis graph with* $Rd(a, b) \geq 3$.

Problem. *Find the minimum number of cycle in a* K*-UBG.*

Conjecture. *Any graph* $G \in K$*-UBG contains a subgraph* $G' \in (K-1)$*-UBG.*

Conjecture. *All core of two uniquely dimensional graphs are either in* $\widehat{\mathbb{S}}$ *or as one of the graphs in Figure 3.*

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