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A Note on Absolute Central Automorphisms of Finite p-Groups

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ABSTRACT. Let G be a finite group. The automorphism σ of a group G is said to be an absolute central automorphism, if for all $x \in G$, $x^{-1}x^{\sigma} \in L(G)$, where L(G) be the absolute centre of G. In this paper, we study some properties of absolute central automorphisms of a given finite p-group.

Keywords: Absolute centre, Absolute central automorphisms, Finite p-groups.

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1. Introduction

Let G be a finite group and N a characteristic subgroup of G. Suppose σ is an automorphism of G. If $(Ng)^{\sigma} = Ng$ for all g in G or equivalently σ induces the identity automorphism on G/N, we shall say σ centralizes G/N. We let $\operatorname{Aut}^N(G)$ denote the group of all automorphisms of G centralizing G/N. Clearly $\sigma \in \operatorname{Aut}^N(G)$ if and only if $x^{-1}x^{\sigma} \in N$ for all $x \in G$. Now let M be a normal subgroup of G. Let us denote by $C_{\operatorname{Aut}^N(G)}(M)$ the group of all automorphisms of $\operatorname{Aut}^N(G)$ centralizing M. Various authors have studied the groups $\operatorname{Aut}^Z(G)$, the central automorphisms of G, where G stands for the commutator subgroup of G, and $\operatorname{Aut}^{\Phi}(G)$, where G denote the Frattini subgroup of G, the intersection of all maximal subgroups of G, see for example [14, 17, 19, 20]. For any

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element $g \in G$ and $\sigma \in \text{Aut}(G)$, the element $[g, \sigma] = g^{-1}g^{\sigma}$ is called the autocommutator of g and σ . Also inductively, for all $\sigma_1, \sigma_2, ..., \sigma_n \in \text{Aut}(G)$, define $[g, \sigma_1, \sigma_2, ..., \sigma_n] = [[g, \sigma_1, \sigma_2, ..., \sigma_{n-1}], \sigma_n]$. Hegarty [7], generalized the concept of centre into absolute centre L(G) of a group G as

$$L(G) = \{g \in G \mid [g, \sigma] = 1, \forall \sigma \in Aut(G)\}.$$

One can easily check that the absolute centre is a characteristic subgroup contained in the centre of G. Also he introduced the concept of the absolute central automorphism. An automorphism σ of G is called an absolute central automorphism if σ centralizes G/L(G). We denote the set of all absolute central automorphisms of G by $\operatorname{Aut}^L(G)$. Singh and Gumber [18], Kaboutari Farimani [9], also Shabani-Attar [17] have given some necessary and sufficient conditions for a finite non-abelian p-group such that all absolute central automorphisms are inner. In this paper, we will characterize the finite non-abelian p-groups G such that $\operatorname{Aut}^L(G) = \operatorname{Aut}^{G'}(G)$. Then, we determine the finite non-abelian p-groups G with cyclic Frattini subgroup for which $\operatorname{Aut}^L(G) = \operatorname{Aut}^\Phi(G)$. Finally, we classify all finite p-groups G of order $p^n(3 \leq n \leq 5)$, such that $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$.

Throughout this paper all groups are assumed to be finite and p always denotes a prime number. Most of our notation is standard, and can be found in [5], for example. In particular, a p-group G is said to be extraspecial if $G' = Z(G) = \Phi(G)$ is of order p. Let $L_1(G) = L(G)$ and for $n \geq 2$, define $L_n(G)$ inductively as

$$L_n(G) = \{ g \in G \mid [g, \sigma_1, \sigma_2, ..., \sigma_n] = 1, \forall \sigma_1, \sigma_2, ..., \sigma_n \in Aut(G) \}.$$

A group G is called autonilpotent of class at most n if $L_n(G) = G$, for some $n \in \mathbb{N}$. If σ is an automorphism of G and X is an element of G, we write X^{σ} for the image of X under σ and o(X) for the order of X. For a finite group G, $\exp(G)$, d(G) and $\operatorname{cl}(G)$, denote the exponent of G, minimal number of generators of G and the nilpotency class of G, respectively. Recall that a group G is called a central product of its subgroups $G_1, ..., G_n$ if $G = G_1 \cdots G_n$ and $[G_i, G_j] = 1$, for all $1 \leq i < j \leq n$. In this situation, we shall write $G = G_1 * \cdots * G_n$. For $s \geq 1$, we use the notation G^{*s} for the iterated central product defined by $G^{*s} = G * G^{*(s-1)}$ with $G^{*1} = G$, where G is a finite p-group. We also make the convention $G^{*0} = 1$. Finally, we use X^n for the direct product of n-copies of a group X, C_n for the cyclic group of order n where $n \geq 1$, as usual, D_8 for the dihedral group, Q_8 for the quaternion group, of order 8, respectively and $M_p(n,m)$ and $M_p(n,m,1)$ for the minimal non-abelian p-groups of order p^{n+m} and p^{n+m+1} defined respectively by

$$\langle a, b \mid a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-1}} \rangle,$$

where $n \ge 2$, $m \ge 1$ and

$$\langle a, b, c \mid a^{p^n} = b^{p^m} = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle,$$

where $n \ge m \ge 1$ and if p = 2, then m + n > 2.

2. Preliminary results

In this section we give some results which will be used in the rest of the paper.

Let G and H be any two groups. We denote by $\operatorname{Hom}(G,H)$ the set of all homomorphisms from G into H. Clearly, if H is an abelian group, then $\operatorname{Hom}(G,H)$ forms an abelian group under the following operation (fg)(x)=f(x)g(x), for all $f,g\in\operatorname{Hom}(G,H)$ and $x\in G$.

The following lemma is a well-known.

Lemma 2.1. Let A, B and C be finite abelian groups. Then

- (i) $\operatorname{Hom}(A \times B, C) \cong \operatorname{Hom}(A, C) \times \operatorname{Hom}(B, C)$;
- (ii) $\operatorname{Hom}(A, B \times C) \cong \operatorname{Hom}(A, B) \times \operatorname{Hom}(A, C)$;
- (iii) $\operatorname{Hom}(C_m, C_n) \cong C_e$, where e is the greatest common divisor of m and n.

We have the following theorem due to Müller [14].

Theorem 2.2. [14, Theorem] If G is a finite p-group which is neither elementary abelian nor extraspecial, then $Aut^{\Phi}(G)/\text{Inn}(G)$ is a non-trivial normal p-subgroup of the group of outer automorphisms of G.

The following preliminary lemma is well-known result [19, Lemma 2.2].

Lemma 2.3. Let G be a group and M, N be normal subgroups of G with $N \leq M$ and $C_N(M) \leq Z(G)$. Then $C_{\operatorname{Aut}^N(G)}(M) \cong \operatorname{Hom}(G/M, C_N(M))$.

Corollary 2.4. If G is a finite group, then

$$C_{\operatorname{Aut}^L(G)}(Z(G)) \cong \operatorname{Hom}(G/Z(G), L(G)),$$

where L = L(G).

Moghaddam and Safa [12], proved that for a finite group G,

$$\operatorname{Aut}^{L}(G) \cong \operatorname{Hom}(G/L(G), L(G)).$$

The following theorem states a useful result for finite p-groups.

Theorem 2.5. Let G be a finite p-group different from C_2 . Then $\operatorname{Aut}^L(G) \cong \operatorname{Hom}(G, L(G))$.

Proof. Let $\theta \in \operatorname{Aut}^L(G)$. We define the map $f_\theta : G \to L(G)$ by $f_\theta(g) = g^{-1}g^\theta$. It is easy to see that f_θ is a homomorphism, and $\theta \mapsto f_\theta$ is an injective map from $\operatorname{Aut}^L(G)$ to $\operatorname{Hom}(G, L(G))$. Conversely, assume that $f \in \operatorname{Hom}(G, L(G))$. Then we define $\theta = \theta_f : G \to G$ by $g^\theta = gf(g)$. Since by [11, Corollary 3.7], $g^{-1}g^\theta \in L(G) \leq \Phi(G)$, for every element $g \in G$, we may write G as the product of the image of θ and the Frattini subgroup of G and so the image of θ must be G itself. Hence θ is an automorphism of G. Now $\theta = \theta_f \in \operatorname{Aut}^L(G)$ and $f_{\theta_f} = f$. Finally, suppose that $\alpha, \beta \in \operatorname{Aut}^L(G)$. Then for any $x \in G$,

$$f_{\alpha\beta}(x) = x^{-1}x^{\alpha\beta} = x^{-1}(xx^{-1}x^{\alpha})^{\beta} = x^{-1}x^{\beta}x^{-1}x^{\alpha} = x^{-1}x^{\alpha}x^{-1}x^{\beta},$$

since $x^{-1}x^{\alpha} \in L(G)$. Thus $f_{\alpha\beta}(x) = f_{\alpha}(x)f_{\beta}(x)$ and so $\theta \mapsto f_{\theta}$ is a homomorphism, which completes the proof.

We next give a necessary and sufficient condition on a finite p-group G for the group $\operatorname{Aut}^L(G)$ to be elementary abelian.

Corollary 2.6. Let G be a finite p-group. Then $\operatorname{Aut}^L(G)$ is elementary abelian if and only if $\exp(G/G') = p$ or $\exp(L(G)) = p$.

Proof. It is straightforward by Lemma 2.1 and Theorem 2.5. \Box

3. Main results

For a finite abelian p-group G, |L(G)| = 1, 2 by [11, Lemma 4.4] and so $|\operatorname{Aut}^L(G)| = 1$ or $\operatorname{Aut}^L(G) \cong C_2^d$, with d = d(G). Thus we may assume that G is a non-abelian p-group. In this section, first we characterize the finite non-abelian p-groups G such that $\operatorname{Aut}^L(G) = \operatorname{Aut}^{G'}(G)$. Then, we determine the finite non-abelian p-groups G with cyclic Frattini subgroup for which $\operatorname{Aut}^L(G) = \operatorname{Aut}^\Phi(G)$.

In [9], Kaboutari Farimani proved the following two results giving some information of absolute central automorphisms of a finite p-group.

Lemma 3.1. Let G be a finite non-abelian p-group. Then $C_{\operatorname{Aut}^L(G)}(Z(G)) = \operatorname{Inn}(G)$ if and only if G/L(G) is abelian and L(G) is cyclic.

Theorem 3.2. Let G be a finite non-abelian p-group. Then $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$ if and only if G/L(G) is abelian, L(G) is cyclic and $Z(G) = L(G)G^{p^n}$ where $\exp(L(G)) = p^n$.

Note that the Theorem 3.2 yields the following corollary that is the Corollary 1 of Singh and Gumber [18].

Let G be a finite non-abelian p-group such that $G' \leq L(G)$. Let $G/Z(G) = C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \cdots \times C_{p^{\alpha_r}}$, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r \geq 1$. Also let $G/L(G) = C_{p^{\beta_1}} \times C_{p^{\beta_2}} \times \cdots \times C_{p^{\beta_s}}$, where $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_s \geq 1$ and $L(G) = C_{p^{\gamma_1}} \times C_{p^{\gamma_1}} \times C_{p^{\gamma_1}} \times C_{p^{\gamma_2}} \times \cdots \times C_{p^{\gamma_s}}$

 $C_{p^{\gamma_2}} \times \cdots \times C_{p^{\gamma_t}}$, where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_t \geq 1$. Since G/Z(G) is a quotient group of G/L(G) by [2, Section 25], $r \leq s$ and $\alpha_i \leq \beta_i$ for all $1 \leq i \leq r$.

By the above notation, we prove the following corollary:

Corollary 3.3. [18, Corollary 1] Let G be a finite non-abelian p-group. Then $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$ if and only if $G' \leq L(G)$, L(G) is cyclic and either L(G) = Z(G) or d(G/L(G)) = d(G/Z(G)), $\alpha_i = \gamma_1$ for $1 \leq i \leq k$ and $\alpha_i = \beta_i$ for $k+1 \leq i \leq r$, where k is the largest integer such that $\beta_k > \gamma_1$.

Proof. First assume that $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$. Hence by Theorem 3.2, $G' \leq L(G)$ and L(G) is cyclic. If $\exp(G/L(G)) \leq \exp(L(G))$, then

$$G/Z(G) \cong \operatorname{Aut}^{L}(G) \cong \operatorname{Hom}(G/L(G), L(G)) \cong G/L(G),$$

because L(G) is cyclic and by [12, Proposition 1]. Therefore L(G) = Z(G). Next, let $\exp(G/L(G)) > \exp(L(G))$ and k is the largest integer such that $\beta_k > \gamma_1$. Since L(G) and G/L(G) are abelian,

$$d(G/Z(G))=d(\operatorname{Hom}(G/L(G),L(G)))=d(G/L(G))d(L(G))=d(G/L(G)).$$

Now we have $\operatorname{Hom}(G/L(G),L(G)) \cong C_{p^{\gamma_1}} \times C_{p^{\gamma_1}} \times \cdots \times C_{p^{\gamma_1}} \times C_{p^{\beta_{k+1}}} \times \cdots \times C_{p^{\beta_s}}$ and $\operatorname{Hom}(G/L(G),L(G)) \cong G/Z(G) = C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \cdots \times C_{p^{\alpha_r}}$. Hence $\alpha_1 = \alpha_2 = \cdots = \alpha_k = \gamma_1$ and $\alpha_i = \beta_i$ for $k+1 \leq i \leq r$, as required.

Conversely if L(G) = Z(G), then $\exp(G/Z(G)) = \exp(G')|\exp(Z(G))$, since $G' \leq L(G)$ and by [13, Lemma 0.4]. Now

$$\operatorname{Hom}(G/L(G), L(G)) = \operatorname{Hom}(G/Z(G), Z(G)) \cong G/Z(G),$$

because Z(G) is cyclic and so $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$. Next assume that L(G) < Z(G), s = d(G/L(G)) = d(G/Z(G)) = r, $\alpha_i = \gamma_1$ for $1 \le i \le k$ and $\alpha_i = \beta_i$ for $k+1 \le i \le r$, where k is the largest integer such that $\beta_k > \gamma_1$. We claim that $Z(G) = L(G)G^{p^{\gamma_1}}$. Since $\exp(G/Z(G)) = \exp(L(G))$, we have $L(G) \le L(G)G^{p^{\gamma_1}} \le Z(G)$. It follows that G/Z(G) is a quotient group of $G/L(G)G^{p^{\gamma_1}}$. Now let $G/L(G)G^{p^{\gamma_1}} = C_{p^{\gamma_1}} \times C_{p^{\delta_2}} \times \cdots \times C_{p^{\delta_r}}$, where $\delta_1 = \gamma_1 \ge \delta_2 \ge \cdots \ge \delta_r \ge 1$, since $d(G/L(G)) = d(G/L(G)G^{p^{\gamma_1}})$ and $\exp(G/L(G)G^{p^{\gamma_1}}) = p^{\gamma_1}$. Therefore $\gamma_1 = \alpha_i \le \delta_i \le \gamma_1$ for $1 \le i \le k$, whence we have $\delta_i = \gamma_1 = \alpha_i$ for $1 \le i \le k$. As $\beta_i = \alpha_i \le \delta_i \le \beta_i$ for $k+1 \le i \le r$, it follows that $\delta_i = \alpha_i = \beta_i$ for $k+1 \le i \le r$. Hence $G/Z(G) = G/L(G)G^{p^{\gamma_1}}$ and consequently $Z(G) = L(G)G^{p^{\gamma_1}}$. Therefore by Theorem 3.2, $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$. This completes the proof.

As an application of Theorem 3.2, we get another proof of the main result of [15].

Theorem 3.4. [15, Theorem 3.2] Let G be a non-abelian autonilpotent finite p-group of class 2. Then $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$ if and only if L(G) = Z(G) and L(G) is cyclic.

Proof. Suppose that $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$. Hence L(G) is cyclic and $Z(G) = L(G)G^{p^n}$, where $\exp(L(G)) = p^n$. Now by [15, Proposition 2.13], $\exp(G/L(G))$ divides $\exp(L(G))$ and so $Z(G) = L(G)G^{p^n} = L(G)$. Conversely, assume that L(G) = Z(G) and L(G) is cyclic. Since G be a non-abelian autonilpotent p-group of class 2, $\operatorname{Aut}^L(G) = \operatorname{Aut}(G)$, by [15, Lemma 2.11]. Therefore $\operatorname{Inn}(G) \leq \operatorname{Aut}^L(G)$, $G' \leq L(G)$ and G/L(G) is abelian. Obviously, $Z(G) = L(G) = L(G)G^{p^n}$, where $\exp(L(G)) = p^n$, and so $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$, by Theorem 3.2, as required.

Corollary 3.5. Let G be an extraspecial p-group.

- (i) If p > 2, then L(G) and $Aut^{L}(G)$ is trivial.
- (ii) If p = 2, then $L(G) \cong C_2$ and $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$.

Proof. Let G be an extraspecial p-group. First assume that p > 2. By [10, Theorem 3], L(G) is trivial and so $\operatorname{Aut}^{L}(G) = 1$.

To prove (ii), since |G'|=2, and G' is a characteristic subgroup of G, we have $G' \leq L(G) \leq Z(G)$. Thus $G' = L(G) = Z(G) = \Phi(G)$ is cyclic of order 2. Now by Theorem 3.2, $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$.

Let G be a finite non-abelian p-group such that G/L(G) is abelian. Then G is of class 2 and $\operatorname{Aut}^{G'}(G) \leq \operatorname{Aut}^{L}(G)$. Let $G/G' = C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}}$, where $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$. Also let $L(G) = C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_l}}$, where $b_1 \geq b_2 \geq \cdots \geq b_l \geq 1$ and $G' = C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_n}}$, where $e_1 \geq e_2 \geq \cdots \geq e_n \geq 1$. Since $G' \leq L(G)$, by [2, Section 25] we have $n \leq l$ and $e_j \leq b_j$ for all $1 \leq j \leq n$. By the above notation, we prove the following theorem:

Theorem 3.6. Let G be a finite non-abelian p-group. Then $\operatorname{Aut}^L(G) = \operatorname{Aut}^{G'}(G)$ if and only if G' = L(G) or G' < L(G), d(G') = d(L(G)) and $a_1 = e_t$, where t is the largest integer between 1 and n such that $b_t > e_t$.

Proof. Suppose that $\operatorname{Aut}^L(G) = \operatorname{Aut}^{G'}(G)$ and $G' \neq L(G)$. By Theorem 2.5 and Lemma 2.3, we have $|\operatorname{Hom}(G/G', L(G))| = |\operatorname{Hom}(G/G', G')|$. First, we claim that d(G') = d(L(G)). Suppose, for a contradiction, that d(G') = n < l = d(L(G)). Since $b_j \geq e_j$ for all j such that $1 \leq j \leq n$, by Lemma 2.1,

$$\begin{split} |\mathrm{Aut}^{G'}(G)| &= |\mathrm{Hom}(G/G',G')| = |\mathrm{Hom}(G/G',C_{p^{e_1}}\times C_{p^{e_2}}\times \cdots \times C_{p^{e_n}})| \\ &\leq |\mathrm{Hom}(G/G',C_{p^{b_1}}\times C_{p^{b_2}}\times \cdots \times C_{p^{b_n}})| < |\mathrm{Hom}(G/G',C_{p^{b_1}}\times C_{p^{b_2}}\times \cdots \times C_{p^{b_n}})| \\ &\times |\mathrm{Hom}(G/G',C_{p^{b_{n+1}}}\times \cdots \times C_{p^{b_l}})| = |\mathrm{Hom}(G/G',C_{p^{b_1}}\times C_{p^{b_2}}\times \cdots \times C_{p^{b_l}})| \\ &= |\mathrm{Hom}(G/G',L(G))| = |\mathrm{Aut}^L(G)|, \end{split}$$

which is a contradiction. So n=l, as required. Next, since $|\mathrm{Aut}^L(G)|=|\mathrm{Aut}^{G'}(G)|$, we have

$$\prod_{1 \le i \le k, 1 \le j \le l} p^{\min\{a_i, b_j\}} = \prod_{1 \le i \le k, 1 \le j \le l} p^{\min\{a_i, e_j\}}.$$

Since $b_j \geq e_j$ for all j such that $1 \leq j \leq l$, we have $\min\{a_i, b_j\} \geq \min\{a_i, e_j\}$, where $1 \leq i \leq k, 1 \leq j \leq l$. Thus $\min\{a_i, b_j\} = \min\{a_i, e_j\}$, for all $1 \leq i \leq k, 1 \leq j \leq l$. Next, since G' < L(G), there exists some $1 \leq j \leq l$ such that $e_j < b_j$. Let t be the largest integer between 1 and n such that $e_t < b_t$. We show that $a_1 \leq e_t$. Suppose, on the contrary, that $a_1 > e_t$. Then by the above equality, we must have $\min\{a_1, b_t\} = \min\{a_1, e_t\} = e_t$, which is impossible. Hence $a_1 \leq e_t$. Let $\exp(G/Z(G)) = p^f$, where $f \in \mathbb{N}$. Since $\operatorname{cl}(G) = 2$, by [13, Lemma 0.4], $f = e_1$. But $a_1 \leq e_t \leq e_{t-1} \leq \cdots \leq e_1 = f \leq a_1$. Whence $a_1 = e_t$.

Conversely, if G' = L(G), then $\operatorname{Aut}^{G'}(G) = \operatorname{Aut}^{L}(G)$. Assume that G' < L(G), d(G') = n = d(L(G)) = l and $a_1 = e_t$, where t is the largest integer between 1 and n such that $b_t > e_t$. Now by Lemma 2.3,

$$|\mathrm{Aut}^{G'}(G)|=|\mathrm{Hom}(G/G',G')|=\prod_{1\leq i\leq k, 1\leq j\leq l}p^{\min\{a_i,e_j\}},$$

and by Theorem 2.5,

$$|\mathrm{Aut}^L(G)| = |\mathrm{Hom}(G/G', L(G))| = \prod_{1 \leq i \leq k, 1 \leq j \leq l} p^{\min\{a_i, b_j\}}.$$

Since $a_1 = e_t$, we have $1 \le a_k \le \cdots \le a_2 \le a_1 = e_t \le e_{t-1} \le \cdots \le e_2 \le e_1$. Thus $b_j \ge e_j \ge a_i$ for all $1 \le i \le k$ and $1 \le j \le t$, which shows that $\min\{a_i, e_j\} = a_i = \min\{a_i, b_j\}$ for $1 \le i \le k$ and $1 \le j \le t$. Since $e_j = b_j$ for all $j \ge t+1$, we have $\min\{a_i, e_j\} = \min\{a_i, b_j\}$ for all $1 \le i \le k$ and $t+1 \le j \le l$. Thus $\min\{a_i, e_j\} = \min\{a_i, b_j\}$ for all $1 \le i \le k$ and $1 \le j \le l$. Therefore $|\operatorname{Aut}^{G'}(G)| = |\operatorname{Aut}^{L}(G)|$. Since G' < L(G) we have $\operatorname{Aut}^{G'}(G) = \operatorname{Aut}^{L}(G)$, which completes the proof.

In [11], Meng and Guo proved that for a finite group G, if C_2 is not a direct factor of G, then $L(G) \leq \Phi(G)$. We end this section by characterizing the finite non-abelian p-groups G with cyclic Frattini subgroup for which $\operatorname{Aut}^L(G) = \operatorname{Aut}^\Phi(G)$.

First, we give some basic results about the finite non-abelian p-groups G with cyclic Frattini subgroup.

Let n > 1. Following [1], we denote by $D_{2^{n+3}}^+$ and $Q_{2^{n+3}}^+$ the 2-groups of order 2^{n+3} defined by the following presentations.

$$D_{2^{n+3}}^+ = \langle a,b,c \mid a^{2^{n+1}} = b^2 = c^2 = 1, a^b = a^{-1+2^n}, a^c = a^{1+2^n}, [b,c] = 1 \rangle,$$

 $\begin{aligned} Q_{2^{n+3}}^+ &= \langle a,b,c \mid a^{2^{n+1}} = b^2 = 1, a^b = a^{-1+2^n}, a^c = a^{1+2^n}, a^{2^n} = c^2, [b,c] = 1 \rangle. \\ \text{Note that if G is either $D_{2^{n+3}}^+$ or $Q_{2^{n+3}}^+$, then $\operatorname{cl}(G) = n+1$.} \end{aligned}$

In [1], Berger, Kovács and Newman proved the following result.

Theorem 3.7. [1, Theorem 2] If G is a finite p-group with $Z(\Phi(G))$ cyclic, then

$$G = E \times (G_0 * G_1 * \cdots * G_s),$$

where E is an elementary abelian, $G_1, ..., G_s$ are non-abelian of order p^3 , of exponent p for p odd and dihedral for p = 2, while $G_0 > 1$ if E > 1, $|G_0| > 2$ if s > 0, and G_0 is one of the following types: cyclic, non-abelian with a cyclic maximal subgroup, $D_{2^{n+2}} * \mathbb{Z}_4, S_{2^{n+2}} * \mathbb{Z}_4, D_{2^{n+3}}^+, Q_{2^{n+3}}^+, D_{2^{n+3}}^+ * \mathbb{Z}_4$, all with n > 1. Conversely, every such group has cyclic Frattini subgroup.

Theorem 3.8. [20, Theorem 2.3] Let G be a finite non-abelian p-group with cyclic Frattini subgroup $\Phi(G)$.

- (i) If p > 2, or p = 2 and cl(G) = 2, then $\Phi(G) \le Z(G)$.
- (ii) If cl(G) > 2, then $G' = \Phi(G)$.

Lemma 3.9. [20, Lemma 2.4] Let G be a finite group with $\Phi(G) \leq Z(G)$. Then there is a bijection from $\operatorname{Hom}(G/G', \Phi(G))$ onto $\operatorname{Aut}^{\Phi}(G)$ associating to every homomorphism $f: G \to \Phi(G)$ the automorphism $x \mapsto xf(x)$ of G. In particular, if G is a p-group and $\exp(\Phi(G)) = p$, then $\operatorname{Aut}^{\Phi}(G) \cong \operatorname{Hom}(G/G', \Phi(G))$.

In the following theorem, we will make use Theorem 3.7, which is the structural theorem for p-groups with cyclic Frattini subgroup.

Theorem 3.10. Let G be a finite non-abelian p-group with cyclic Frattini subgroup. Then $\operatorname{Aut}^L(G) = \operatorname{Aut}^\Phi(G)$ if and only if G is one of the following types: $C_2^m \times D_8^{*(s+1)}$ or $C_2^m \times (D_8^{*s} * Q_8)$, where $s, m \geq 0$.

Proof. Let $\operatorname{Aut}^L(G) = \operatorname{Aut}^\Phi(G)$. Hence $\operatorname{Aut}^\Phi(G)$ is abelian, G is of class 2 and by Theorem 3.8, $\Phi(G) \leq Z(G)$. It follows that $\exp(G') = \exp(G/Z(G)) = p$ and so |G'| = p. Assume that $|\Phi(G) : G'| = p^a$. Then $\Phi(G) \cong C_{p^{a+1}}$ and we observe that $\exp(G/G') \leq p^{a+1} = |\Phi(G)|$. Together with Lemma 3.9, we have $|\operatorname{Aut}^\Phi(G)| = |\operatorname{Hom}(G,\Phi(G))| = |G|/p$. Next, we note that $G' \cap L(G) \neq 1$; otherwise, $G' \cap L(G) = 1$ and $G' \times L(G)$ would be a subgroup of $\Phi(G)$. Hence either G' = 1 or L(G) = 1, a contradiction. Whence $G' \leq L(G)$. Now we are able to show that $G' = L(G) \cong C_p$. To do this, first assume that $L(G) \neq \Phi(G)$. By similar argument that was applied for Theorem 3.6, we have $\exp(G/G') \leq \exp(L(G))$, which implies that $\exp(G/L(G)) \leq \exp(G/G') \leq \exp(L(G)) = |L(G)|$. If $L(G) = \Phi(G)$, then $\exp(G/L(G)) = \exp(G/\Phi(G)) \leq \exp(L(G)) = |L(G)|$. Thus $|\operatorname{Aut}^L(G)| = |G/L(G)| = |\operatorname{Aut}^\Phi(G)| = |G/G'|$, by [12, Proposition 1] and so $G' = L(G) \cong C_p$. Now, we will make use of the notation of Theorem 3.7.

Since cl(G) = 2, by Theorem 3.7 and [5, Theorems 5.4.3 and 5.4.4], G_0 is one of the groups $M_p(n, 1)$, where $n \ge 3$, if p = 2; D_8 or Q_8 .

We claim that $G' = G'_0$ and $\Phi(G) = \Phi(G_0)$. To see this, since $G'_0 \cap G'_i \neq 1$ for $1 \leq i \leq s$ and $|G'_i| = p$, we have $G'_i \leq G'_0$ and so $G' = G'_0$. Also $\Phi(G) = G'G^p = G'_0E^pG_0^pG_1^p \cdots G_s^p = G'_0G_0^p = \Phi(G_0)$. To continue the proof, we may consider two cases:

Case I. E = 1.

Let $G = G_0 * T$, where T be one of the groups $M_p(1,1,1)^{*s}$, while p > 2 or D_8^{*s} , where all $s \ge 0$. Note that if s = 0, then $G = G_0$ and $Z(G) = Z(G_0) = \Phi(G_0) = \Phi(G)$; otherwise, since $1 \ne G_0 \cap T = Z(T) \le Z(G_0)$, then $Z(G) = Z(G_0)$, because |Z(T)| = p, which implies that $\Phi(G) = \Phi(G_0) = Z(G_0) = Z(G)$. We claim that G is an extraspecial p-group. To see this, since $G' = L(G) \cong C_p$, by Theorem 3.2, $\operatorname{Aut}^{\Phi}(G) = \operatorname{Aut}^{L}(G) = \operatorname{Inn}(G)$. This shows that G is an extraspecial p-group, by Theorem 2.2. If F = 0, then by Corollary 3.5, F = 0, which is impossible. Whence F = 0, then by F = 0, which is impossible. Whence F = 0 is isomorphic either to F = 0, since F = 0, a contradiction. Therefore F = 0 is isomorphic either to F = 0, and F = 0, and F = 0, and F = 0, where F = 0, are F = 0, for some F = 0. Case II. F = 0.

In this case $G_0 > 1$ and $G = E \times (G_0 * T)$, where T be one of the groups lying in Case I.

We claim that $\operatorname{Aut}^{\Phi(G_0*T)}(G_0*T) = \operatorname{Aut}^{L(G_0*T)}(G_0*T)$. Choose a non-trivial element σ of $\operatorname{Aut}^{\Phi(G_0*T)}(G_0*T)$. Then the map $\overline{\sigma}$ defined by $(ef)^{\overline{\sigma}} = ef^{\sigma}$, for all $e \in E$, $f \in G_0*T$ denotes an automorphism of $\operatorname{Aut}^{\Phi}(G) = \operatorname{Aut}^{L}(G)$. Since $G' \cap L(G_0*T) \neq 1$, then $L(G) \leq L(G_0*T)$ and so σ is in $\operatorname{Aut}^{L(G_0*T)}(G_0*T)$. This shows that $\operatorname{Aut}^{\Phi(G_0*T)}(G_0*T) = \operatorname{Aut}^{L(G_0*T)}(G_0*T)$, as required. Next, by a similar argument as mentioned for the previous case, G_0 be one of the groups: D_8 or Q_8 . Therefore G has one of the following types: $C_2^m \times D_8^{*(s+1)}$ or $C_2^m \times (D_8^{*s}*Q_8)$, where $s \geq 0, m > 0$.

Conversely, assume that G be of the groups in Theorem 3.10. Hence $G' = L(G) \cong C_2$. Now the proof is complete, since $|\operatorname{Aut}^L(G)| = |\operatorname{Aut}^\Phi(G)| = |G|/2$.

4. Classify all finite *p*-groups G of order $p^n (3 \le n \le 5)$, such that $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$

Let G be a non-abelian group of order p^3 . Then by Corollary 3.5, $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$ if and only if p = 2. In the following corollaries, we use Theorems 4.7 and 5.1 of [11] and classify all finite p-groups G of order $p^n (4 \le n \le 5)$, such that $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$. First we recall the following concept, which was introduced by Hall in [6].

Definition 4.1. Two finite groups G and H are said to be isoclinic if there exist isomorphisms $\phi: G/Z(G) \to H/Z(H)$ and $\theta: G' \to H'$ such that, if $(x_1Z(G))^{\phi} = y_1Z(H)$ and $(x_2Z(G))^{\phi} = y_2Z(H)$, then $[x_1, x_2]^{\theta} = [y_1, y_2]$. Notice that isoclinism is an equivalence relation among finite groups and the equivalence classes are called isoclinism families.

Corollary 4.2. Let G be a non-abelian group of order p^4 . Then $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$ if and only if p = 2 and G is one of the following types: $M_2(3,1)$ or $M_2(2,1,1)$.

Proof. Assume that $|G| = p^4$ and $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$. We claim that $|Z(G)| = p^2$. Suppose for a contradiction, that |Z(G)| = p. We observe that $G' \leq Z(G) \cong C_p$, by Theorem 3.2 and so G is an extraspecial p-group, a contradiction since the order of G is not of the form p^{2n+1} , for some natural number n. Therefore $G/Z(G) \cong C_p^2$, and hence |G'| = p. We consider two cases:

Case I. p an odd prime. It is straightforward to see that the map $\sigma: G \to G$ by $x^{\sigma} = x^{1+p}$, is an automorphism of G. Hence for any element x of L(G), $x = x^{\sigma} = x^{1+p}$, and so $x^p = 1$. Thus $\exp(L(G)) = p$ and so $G' = L(G) \cong C_p$, by Theorem 3.2. If $G/L(G) \cong C_{p^3}$, then by [3, Theorem 2.2], G is cyclic, a contradiction. Next, we assume that $G/L(G) \cong C_{p^2} \times C_p$. Then G is an abelian group by [11, Theorem 5.1], which is impossible. Finally, if $G/L(G) \cong C_p^3$, then $L(G) = \Phi(G)$ and so $\operatorname{Aut}^{\Phi}(G) = \operatorname{Inn}(G)$. Therefore by Theorem 2.2, G is an extraspecial p-group, a contradiction.

Case II. p=2. Since |G'|=2, and G' be a characteristic subgroup of G, we have $G' \leq L(G) \leq Z(G)$. Thus |L(G)|=2 or 4. If |L(G)|=4, then L(G)=Z(G) and $G/L(G)\cong C_2^2$. Hence by [11, Theorems 5.1 and 4.7], $G\cong M_2(2,2)$, and $L(G)\cong C_2^2$, which is a contradiction by Theorem 3.2. Next we assume that |L(G)|=2. So G'=L(G) and |G/L(G)|=8. By a similar argument, G is isomorphic to one of the following groups: $M_2(3,1)$ or $M_2(2,1,1)$. The converse follows at once from Theorem 3.2.

Corollary 4.3. Let G be a non-abelian group of order p^5 . Then $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$ if and only if p = 2 and G is one of the following types: $M_2(3,2)$, $M_2(4,1)$, $M_2(2,2,1)$, D_8^{*2} or $D_8 * Q_8$.

Proof. Let G be a finite group such that $|G| = p^5$ and $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$. We consider two cases:

Case I. p > 2. These groups lying in the isoclinism families (5), (4) or (2) of [8, 4.5] and we show that $\operatorname{Aut}^{L}(G) \neq \operatorname{Inn}(G)$.

First, let G denote one of the groups in the isoclinism family (5). Hence |Z(G)| = p and $G' = Z(G) = \Phi(G) \cong C_p$, by Theorem 3.2. So G is an extraspecial p-group and by Corollary 3.5, |L(G)| = 1, a contradiction.

Next, let G be one of the groups in the isoclinism family (4). Then $G' \cong C_p^2$, which is a contradiction, since G' is cyclic.

Finally, let G denote one of the groups in the isoclinism family (2). Then $G/Z(G)\cong C_p^2$ and so d(G/L(G))>1. We observe that $G'=L(G)\cong C_p$ and $Z(G)=\Phi(G)$, by using Theorems 2.2, 3.2, [3, Theorem 2.2] and [11, Theorem 5.1]. So d(G)=2 and by [16], G is a minimal non-abelian p-group. If $G/L(G)\cong C_{p^3}\times C_p$, then G is an abelian group, by [11, Theorem 5.1], a contradiction. If $G/L(G)\cong C_{p^2}$, then by [16], $G\cong M_p(3,2)$ or $G\cong M_p(2,2,1)$. Thus L(G)=1, by [11, Theorem 4.7], a contradiction. Finally, assume that $G/L(G)\cong C_{p^2}\times C_p^2$ or $G/L(G)\cong C_p^4$. In this cases, $\operatorname{Aut}^L(G)\neq \operatorname{Inn}(G)$, by Theorem 2.5.

Case II. p=2. We can see that |L(G)|=2,4, by [3, Theorem 2.2] and [11, Theorem 5.1]. First, we assume that |L(G)|=4. Since G is a non-cyclic group, by [3, Theorem 2.2], d(G/L(G))>1. It follows that G/L(G) is one of the groups C_2^3 or $C_4\times C_2$. Now in the first case, $L(G)=\Phi(G)$ and so G is an extraspecial 2-group by Theorem 2.2. Hence $G'=L(G)\cong C_2$, a contradiction. Therefore $G/L(G)\cong C_4\times C_2$ and by [11, Theorems 5.1 and 4.7], G is one of the groups: $M_2(2,3)$ or $M_2(3,1,1)$, and $L(G)\cong C_2^2$, a contradiction by Theorem 3.2. Now we may suppose that |L(G)|=2. So $G'=L(G)\cong C_2$. We discuss the following cases.

If $G/L(G) \cong C_2^4$, then $L(G) = \Phi(G)$ and so $\operatorname{Aut}^\Phi(G) = \operatorname{Inn}(G)$. Therefore by Theorem 2.2, G is an extraspecial 2-group. Thus G is one of the groups D_8^{*2} or $D_8 * Q_8$, by [21]. Next, suppose that $G/L(G) \cong C_4 \times C_2^2$. Hence $G/L(G) = \langle \bar{a}, \bar{b}, \bar{c} \rangle$, where $\bar{a} = aL(G), \bar{b} = bL(G), \bar{c} = cL(G)$ and $o(\bar{a}) = 4$, $o(\bar{b}) = o(\bar{c}) = 2$. Therefore $G = \langle a, b, c, L(G) \rangle = \langle a, b, c \rangle$, by [11, Corollary 3.7]. Since $\langle a^2 \rangle \times G' \leq Z(G)$, we have either $Z(G) \cong C_4 \times C_2$ or C_2^2 . If $Z(G) \cong C_4 \times C_2$, then $\operatorname{Aut}^L(G) \neq \operatorname{Inn}(G)$, by Theorem 2.5. Therefore $Z(G) \cong C_2^2$. Now by using GAP [4], we find that there are no such groups. Next, if $G/L(G) \cong C_8 \times C_2$, then $G \cong M_2(4,1)$, by [11, Theorem 5.1]. Finally, suppose that $G/L(G) \cong C_4^2$. Then d(G) = 2, by [11, Corollary 3.7] and $G' = L(G) \cong C_2$. Hence by [16], G is a minimal non-abelian 2-group. Thus G is isomorphic to the group $M_2(3,2)$ or $M_2(2,2,1)$. The converse follows at once from Theorem 3.2.

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