Spaceability on Morrey Spaces

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ABSTRACT. In this paper, as a main result for Morrey spaces, we prove
that the set $\mathcal{M}_q^p(\mathbb{R}^n) \setminus \bigcup_{q < r \leq p} \mathcal{M}_r^p(\mathbb{R}^n)$ is spaceable in $\mathcal{M}_q^p(\mathbb{R}^n)$, where
$0 < q < p < \infty$.

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1. INTRODUCTION

Let $0 < q \leq p < \infty$. Define the *Morrey (quasi-)norm* $\| \cdot \|_{\mathcal{M}_q^p}$ by

$$\|f\|_{\mathcal{M}_q^p} \equiv \sup \left\{ |Q|^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q(Q)} \; : \; Q \text{ is a cube in } \mathbb{R}^n \right\}$$

for a measurable function f, where a cube is defined to be the set of the form
$\{a + y \, : \, a \in \mathbb{R}^n, y \in [0,t]^n\}$ for some $a \in \mathbb{R}^n$ and $t > 0$. The *Morrey space* $\mathcal{M}_q^p$
is the set of all measurable functions f for which $\|f\|_{\mathcal{M}_q^p}$ is finite. Morrey spaces
date back to 1938, when C.B. Morrey considered elliptic differential equations
and discussed continuity of the solutions using a lemma [7]. His lemma was
refined by Peetre [8]. Morrey's lemma gave rise to the theory of function spaces;

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see [8] and [10]. The function spaces dealt with are called Morrey spaces. Let $0 < q < p < \infty$. For each $r \in (q, p]$ the set $\mathcal{M}_r^p$ is a proper subset of $\mathcal{M}_q^p$ [4, 9]. In this paper, we show that the difference of these two sets is large enough. Precisely, by a techmical lemma we prove:

Theorem 1.1. *If $0 < q < p$, then $\mathcal{M}_q^p \setminus \bigcup_{q < r \leq p} \mathcal{M}_r^p$ is a spaceable subset of $\mathcal{M}_q^p$.*

Another different proof will also be given via [6, Theorem 3.3]. We recall that a subset S of a topological vector space X is called *spaceable* if $S \cup \{0\}$ contains a closed infinite-dimensional linear subspace of X. The concepts spaceability and lineability were introduced by the paper [1] and then have been studied on different kinds of function or sequence spaces (see [5, 6]). In particular, for spaceability of the difference of Lebesgue spaces (see [2, 3, 11]).

2. Main Result

Let $p > q > 0$ and $R > 1$ be fixed so that

$$(R + 1)^{-\frac{1}{p}} = 2^{\frac{1}{q}} (1 + R)^{-\frac{1}{q}}. \tag{2.1}$$

For a vector $\varepsilon \in \{0, 1\}^n$, we define an affine transformation T_ε by

$$T_\varepsilon(x) \equiv \frac{1}{R + 1} x + \frac{R}{R + 1} \varepsilon \quad (x \in \mathbb{R}^n).$$

Let $E_0 := [0, 1]^n$. Suppose that we have defined $E_0, E_1, E_2, \ldots, E_j$. Define

$$E_{j+1} := \bigcup_{\varepsilon \in \{0,1\}^n} T_\varepsilon(E_j)$$

and

$$E_{j,0} := [0, (1 + R)^{-j}]^n.$$

The following technical lemma is proved in [10, Proposition 4.1]. Here, for the sake of convenience of readers, we reproduce the proof.

Lemma 2.1. *Under above notations we have*

$$\|\chi_{E_j}\|_{\mathcal{M}_q^p} \sim (1 + R)^{-j \frac{n}{p}} = \|\chi_{E_{j,0}}\|_{\mathcal{M}_q^p} = \|\chi_{E_{j,0}}\|_p = \|\chi_{E_j}\|_q, \tag{2.2}$$

where the implicit constants in $\sim$ does not depend on j but can depend on p and q and $\| \cdot \|_{\mathcal{M}_q^p}$ is the Morrey norm.

Proof. A direct calculation shows that

$$\|\chi_{E_j}\|_{\mathcal{M}_q^p} \geq \|\chi_{E_{j,0}}\|_{\mathcal{M}_q^p} = (1 + R)^{-j \frac{n}{p}} = \|\chi_{E_{j,0}}\|_p = \|\chi_{E_j}\|_q,$$

Thus, we need to show

$$\|\chi_{E_j}\|_{\mathcal{M}_q^p} \lesssim \|\chi_{E_{j,0}}\|_p.$$

Let us calculate

$$\|\chi_{E_j}\|_{\mathcal{M}_q^p} \sim \sup_{S \in \mathcal{Q}} |S|^{\frac{1}{p} - \frac{1}{q}} |S \cap E_j|^{\frac{1}{q}},$$

where $\mathcal{Q}$ denotes the set of all cubes. Fix $j \in \mathbb{N}$. Let us temporally say that $Q \in \mathcal{Q}$ is wasteful, if there exists a cube $S \in \mathcal{Q}$ such that

$$|Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} < |S|^{\frac{1}{p}-\frac{1}{q}}|S \cap E_j|^{\frac{1}{q}}.$$

Thus by definition, if $\ell(Q) := |Q|^{\frac{1}{n}} > 1$, then

$$|Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} < |E_j|^{\frac{1}{q}} = |[0,1]^n|^{\frac{1}{p}-\frac{1}{q}}|[0,1]^n \cap E_j|^{\frac{1}{q}}.$$

In addition, if the side-length of a cube Q is less than $(R+1)^{-j}$, then it is wasteful. Indeed, then the equalities

$$\sup\left\{|Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} \, : \, Q \in \mathcal{Q}, |Q| \leq (R+1)^{-jn}\right\}$$
$$= \sup\{|Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} \, : \, Q \in \mathcal{Q}, Q \subset E_j\}$$
$$= \sup\left\{|Q|^{\frac{1}{p}} \, : \, Q \in \mathcal{Q}, Q \subset E_j\right\}$$
$$= |E_{j,0}|^{\frac{1}{p}}$$

hold. Here, we obtain the first equality by translating Q so that Q which is included in $E_{j,0}$. This calculation shows that

$$|Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} < |E_{j,0}|^{\frac{1}{p}}$$

for any such cube. Thus, if the cube Q is not wasteful, then $(R+1)^{-j} \leq \ell(Q) \leq 1$. So, there exists $k \in \{1, 2, \ldots, n\}$ such that $(R+1)^{-kn} \leq |Q| \leq (R+1)^{-(k-1)n}$. In this case, since any connected component P of E_k satisfies $(R+1)^{-kn} = |P| \leq |Q| \leq (R+1)^{-(k-1)n}$, $3Q$ contains a connected component of E_k. Hence, it follows that

$$\|\chi_{E_j}\|_{\mathcal{M}_q^p}$$
$$\sim \sup\{|Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} \, : \, Q \text{ contains a connected component of } E_j \, \}.$$

Let S be a cube which contains a connected component of E_j and is not wasteful. By symmetry, we may assume $S = I \times I \times \cdots \times I$ for some interval I. We define

$$S^* := \text{co}\left(\bigcup\{W \, : \, W \text{ is a connected component of } E_j \text{ intersecting } S\}\right),$$

where $\text{co}(A)$ stands for the smallest convex set containing a set A. Then a geometric observation shows that S^* engulfs k^n connected component of E_j for some $1 \leq k \leq 2^j$. Take an integer l such that $2^{l-1} \leq k \leq 2^l$. Then we have

$$|S^* \cap E_j| = k^n(1+R)^{-jn}, \quad |S^*| \sim (1+R)^{-jn+ln}.$$

Consequently, from (2.1) we have

$$
|S^*|^{\frac{1}{p}-\frac{1}{q}}|S^* \cap E_j|^{\frac{1}{q}} \sim 2^{\frac{ln}{q}}(1+R)^{-\frac{jn}{q}}(1+R)^{(-j+l)\left(\frac{n}{p}-\frac{n}{q}\right)}
$$

$$
= 2^{\frac{ln}{q}}(1+R)^{-j\frac{n}{p}+ln\left(\frac{1}{p}-\frac{1}{q}\right)}
$$

$$
= 2^{\frac{ln}{q}}(1+R)^{ln\left(\frac{1}{p}-\frac{1}{q}\right)}(1+R)^{-j\frac{n}{p}} = (1+R)^{-j\frac{n}{p}}.
$$

Therefore, (2.2) is obtained.   □

So, we can say that the Morrey norm $\|\cdot\|_{\mathcal{M}_q^p}$ reflects local regularity of the functions more precisely than the Lebesgue norm $\|\cdot\|_p$.

A chain of equalities in (2.2) is the motivation of choosing R in (2.1).

Remark 2.2. In Lemma 2.1, if one defines $F_j := \{x \in \mathbb{R}^n : (R+1)^{-j}x \in E_j\}$, then $\{F_j\}_{j=1}^\infty$ is an increasing sequence of sets and each F_j is made up of disjoint union of cubes of length 1, $\|\chi_{F_j}\|_{\mathcal{M}_q^p} \sim 1$, and each component of F_j is a cube of size 1.

Proposition 2.3. *Let* $0 < q < r \le p$. *Then* $\|\chi_{F_j}\|_{\mathcal{M}_r^p} \ge \left(\frac{1+R}{2}\right)^{\left(\frac{n}{q}-\frac{n}{r}\right)j}$, *and so that* $\lim\limits_{j\to\infty}\|\chi_{F_j}\|_{\mathcal{M}_r^p} = \infty$.

Proof. Simply use

$$
\|\chi_{F_j}\|_{\mathcal{M}_r^p} \ge (1+R)^{\frac{jn}{p}-\frac{jn}{r}}|F_j|^{\frac{1}{r}} = (1+R)^{\frac{jn}{q}-\frac{jn}{r}}|F_j|^{\frac{1}{r}-\frac{1}{q}} = \left(\frac{1+R}{2}\right)^{\left(\frac{n}{q}-\frac{n}{r}\right)j}.
$$

                 □

Now, we prove the main result of this paper.

Proof of Theorem 1.1. Under above notations, put

$$
F := \bigcup_{j=1}^\infty F_j. \tag{2.3}
$$

Then $\chi_F \in \mathcal{M}_q^p$. For each $j = 1, 2, 3, \ldots$ let us define

$$
g_j(x) := \chi_F((R+1)^{-j-1}x) - \chi_F((R+1)^{-j}x) \qquad (x \in \mathbb{R}^n),
$$

and

$$
h_j(x) := \chi_F(R^{-j-1}x) \quad (x \in \mathbb{R}^n).
$$

Then by Lemma 2.1 we have $\|g_j\|_{\mathcal{M}_q^p} \sim \|h_j\|_{\mathcal{M}_q^p} < \infty$, while $\|g_j\|_{\mathcal{M}_r^p} \sim \|h_j\|_{\mathcal{M}_r^p} = \infty$ for all $q < r \le p$. We set

$$
V := \left\{\sum_{j=1}^\infty \lambda_j g_j : \text{for all } j, \lambda_j \in \mathbb{C}\right\} \bigcap \mathcal{M}_q^p.
$$

Since the convergence with norm of $\mathcal{M}_q^p$ is stronger than the almost eveywhere convergence, V is closed in $\mathcal{M}_q^p$. Also, note that

$$V \cap \bigcup_{q < r \le p} \mathcal{M}_r^p = \{0\},$$

since the charactristic function χ_F belongs to the set $\mathcal{M}_q^p \setminus \bigcup_{q < r \le p} \mathcal{M}_r^p$. Therefore, $\mathcal{M}_q^p \setminus \bigcup_{q < r \le p} \mathcal{M}_r^p$ is spaceable in $\mathcal{M}_q^p$. $\qquad \square$

Here is an alternative proof of Thoerem 1.1 with $q \ge 1$ using a result by Kitson and Timoney [6]. We invoke this result from [6].

Theorem 2.4. [6, Theorem 3.3]. *Let Z_m ($m \in \mathbb{N}$) be Banach spaces and X a Fréchet space. Let $T_m : Z_m \to X$ be continuous linear operators and Y the linear span of $\bigcup_{m=1}^{\infty} T_m(Z_m)$. If Y is not closed in X, then the complement $X \setminus Y$ is spaceable.*

For a different proof of Theorem 1.1, simply apply Theorem 2.4 with

$$X = \mathcal{M}_q^p, \quad Z_m = \mathcal{M}_{q+\frac{p-q}{m}}^p, \quad T_m = \text{inclusion mapping from } Z_m \text{ to } \mathcal{M}_q^p.$$

We will check the following property of the Morrey space $\mathcal{M}_q^p$ to see that Theorem 2.4 is applicable.

Proposition 2.5. *Let $0 < q < p < \infty$. Then $\displaystyle\bigcup_{q < r \le p} \mathcal{M}_r^p$ is not closed in $\mathcal{M}_q^p$.*

We torelate the case $0 < q < 1$ here. Once this is shown, Theorem 1.1 will be proved with the help of the aforementioned theorem.

Proof. Let F_j be as in Remark 2.2. For each $k \in \mathbb{N}$ define

$$f_k := \sum_{j=1}^{k} \frac{1}{j^2 \|\chi_{F_j}\|_{\mathcal{M}_q^p}} \chi_{F_j}.$$

Then $(f_k)_{k=1}^{\infty}$ is a Cauchy sequence in $\mathcal{M}_q^p$. So, $(f_k)_{k=1}^{\infty}$ converges to a function f in $\mathcal{M}_q^p$ since $\mathcal{M}_q^p$ is a Banach space. Note that each $f_k \in L^p \subset \mathcal{M}_r^p$ for all $q < r \le p$. If f is a member in $\mathcal{M}_r^p$ for some $r \in (q, p]$, then

$$\left\{ \frac{1}{j^2 \|\chi_{F_j}\|_{\mathcal{M}_q^p}} \chi_{F_j} \right\}_{j=1}^{\infty}$$

would form a bounded set in $\mathcal{M}_r^p$ since $\mathcal{M}_r^p$ enjoys the lattice property. This is a contradiction to Proposition 2.3. $\qquad \square$

3. Appendix

The situation is different from the case of Lebesgue spaces.

Proposition 3.1. *Let $0 < q < p < \infty$. Then $\bigcup_{q < r \leq p} \mathcal{M}_r^p$ is not dense in $\mathcal{M}_q^p$.*

Proof. Let F be as in (2.3). We prove that $2\chi_F$ is not in the closure of $\bigcup_{q < r \leq p} \mathcal{M}_r^p$ by showing that for every $q < r \leq p$, $f \notin \mathcal{M}_r^p$ if $f \in \mathcal{M}_q^p$ satisfies $\|2\chi_F - f\|_{\mathcal{M}_q^p} < 1$. Indeed, if K is one of the connected components of F, then $\|f\|_{L^r(K)} \geq \|f\|_{L^q(K)} > c_q = (2^{\min(1,q)} - 1)^{\frac{1}{\min(1,q)}}$ since

$$1 > (\|2\chi_F - f\|_{\mathcal{M}_q^p})^{\min(1,q)}$$
$$\geq (\|2 - f\|_{L^q(K)})^{\min(1,q)}$$
$$\geq 2^{\min(1,q)} - (\|f\|_{L^q(K)})^{\min(1,q)}.$$

Thus,

$$\|f\|_{\mathcal{M}_r^p} \geq |[0, R^j]^n|^{\frac{1}{p} - \frac{1}{r}} \|f\|_{L^r([0,R^j]^n)} \geq c_q R^{\frac{jn}{p} - \frac{jn}{r}} |E_j|^{\frac{1}{r}} = c_q 2^{\frac{jn}{r} - \frac{jn}{q}} R^{\frac{jn}{q} - \frac{jn}{r}}$$

for all $j \in \mathbb{N}$. Hence, $\|f\|_{\mathcal{M}_r^p} = \infty$, or equivalently $f \notin \mathcal{M}_r^p$. $\qquad\square$

Acknowledgments

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