

n -submodules

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ABSTRACT. Let R be a commutative ring with identity. A proper submodule N of an R -module M is an n -submodule if $rm \in N$ ($r \in R, m \in M$) with $r \notin \sqrt{\text{Ann}_R(M)}$, then $m \in N$. A number of results concerning n -submodules are given. For example, we give other characterizations of n -submodules. Also various properties of n -submodules are considered.

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1. INTRODUCTION

Throughout this article, R denotes a commutative ring with identity and all modules are unitary. Also \mathbb{N} , \mathbb{Z} , and \mathbb{Q} will denote, respectively, the natural numbers, the ring of integers, and the field of rational numbers. If N is an R -submodule of M , annihilator of R -module $\frac{M}{N}$ is defined to be $\text{Ann}_R(\frac{M}{N}) = (N :_R M) = \{r \in R : rM \subseteq N\}$. Also the annihilator of M , denoted by $\text{Ann}_R(M)$, is $(0 :_R M)$. Suppose that I is an ideal of R . We denote the radical of I by $\sqrt{I} = \{a \in R : a^n \in I \text{ for some } n \in \mathbb{N}\}$.

A proper submodule N of M is called prime (*primary*) if $rx \in N$, for $r \in R$ and $x \in M$, implies that either $x \in N$ or $r \in (N :_R M)$ ($r^n \in (N :_R M)$), for

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some $n \in \mathbb{N}$ (see [1], [6], [9], [11]).

An R -module M is said to be a multiplication module, if for each submodule N of M , there is an ideal I of R , such that $N = IM$. Equivalently, M is a multiplication module if and only if $N = (N :_R M)M$, for each submodule N of M [2], [3].

The concepts of n -ideals and n -submodules were introduced in [12]. A proper ideal I of R is said to be an n -ideal if the condition $ab \in I$ with $a \notin \sqrt{0} = \{a \in R : a^n = 0 \text{ for some } n \in \mathbb{N}\}$ implies $b \in I$, for every $a, b \in R$. Also a proper submodule N of M is called an n -submodule if for $a \in R$, $x \in M$, $ax \in N$ with $a \notin \sqrt{\text{Ann}_R(M)}$, then $x \in N$.

In Section 2, we investigate some properties of n -submodules analogous with n -ideals and also obtain some basic results. Among many results in this article, it is shown in Theorem 2.2, that a proper submodule N of M is an n -submodule if and only if $N = (N :_M a)$ for every $a \notin \sqrt{\text{Ann}_R(M)}$. In Theorem 2.22, we show that every n -submodule is a primary submodule. Furthermore, in Theorem 2.27, we characterize torsion-free modules in terms of n -submodules.

2. n -SUBMODULES

Recall that a proper submodule N of a module M over a commutative ring R is said to be an n -submodule, if for $a \in R$, $x \in M$, $ax \in N$ with $a \notin \sqrt{\text{Ann}_R(M)}$, then $x \in N$.

EXAMPLE 2.1. (i) Suppose that R is a ring that has only one prime ideal. Then every proper submodule of R -module R is an n -submodule.

(ii) \mathbb{Z}_6 as \mathbb{Z} -module has not any n -submodule.

Theorem 2.2. *Let M be an R -module and N be a proper submodule of M . Then the following statements are equivalent:*

- (i) N is an n -submodule of M ;
- (ii) $N = (N :_M a)$, for every $a \notin \sqrt{\text{Ann}_R(M)}$;
- (iii) For any ideal I of R and submodule K of M , $IK \subseteq N$ with $I \not\subseteq \sqrt{\text{Ann}_R(M)}$ implies $K \subseteq N$.

Proof. (i) \Rightarrow (ii) Let N be an n -submodule of M . For every $a \in R$, the inclusion $N \subseteq (N :_M a)$ always holds. Let $a \notin \sqrt{\text{Ann}_R(M)}$ and $x \in (N :_M a)$. Then we have $ax \in N$. Since N is an n -submodule, we conclude that $x \in N$ and thus $N = (N :_M a)$.

(ii) \Rightarrow (iii) Suppose that $IK \subseteq N$ where $I \not\subseteq \sqrt{\text{Ann}_R(M)}$, for ideal I of R and submodule K of M . Since $I \not\subseteq \sqrt{\text{Ann}_R(M)}$, there exists $a \in I$ such that $a \notin \sqrt{\text{Ann}_R(M)}$. Then we have $aK \subseteq N$, and so $K \subseteq (N :_M a) = N$ by (ii).

(iii) \Rightarrow (i) Let $ax \in N$ with $a \notin \sqrt{\text{Ann}_R(M)}$ for $a \in R$ and $x \in M$. It is sufficient to take $I := Ra$ and $K := Rx$ to prove the result. \square

Proposition 2.3. i) If N is an n -submodule of M , then $(N :_R M) \subseteq \sqrt{\text{Ann}_R(M)}$.
 ii) Let $\{N_i\}_{i \in I}$ be a nonempty set of n -submodules of an R -module M . Then $\bigcap_{i \in I} N_i$ is an n -submodule.
 iii) Let $\{N_i\}_{i \in I}$ be a chain of n -submodules of a finitely generated R -module M . Then $\bigcup_{i \in I} N_i$ is an n -submodule of M .

Proof. i) Assume that N is an n -submodule; but $(N :_R M) \not\subseteq \sqrt{\text{Ann}_R(M)}$. Then there exists $r \in (N :_R M)$ such that $r \notin \sqrt{\text{Ann}_R(M)}$. Thus $rM \subseteq N$ and since N is an n -submodule, we conclude that $N = M$, a contradiction. Hence $(N :_R M) \subseteq \sqrt{\text{Ann}_R(M)}$.

ii) Let $rx \in \bigcap_{i \in I} N_i$ with $r \notin \sqrt{\text{Ann}_R(M)}$, for $r \in R$ and $x \in M$. Then $rx \in N_i$, for every $i \in I$. Since for every $i \in I$, N_i is an n -submodule of M , we get $x \in N_i$ and so $x \in \bigcap_{i \in I} N_i$.

iii) Let $rx \in \bigcup_{i \in I} N_i$ where $r \notin \sqrt{\text{Ann}_R(M)}$ for $r \in R$ and $x \in M$. Then $rx \in N_k$ for some $k \in \mathbb{N}$. Since N_k is an n -submodule, we conclude that $x \in N_k \subseteq \bigcup_{i \in I} N_i$ and so $\bigcup_{i \in I} N_i$ is an n -submodule. \square

Proposition 2.4. Let I be an ideal of R such that $I \not\subseteq \sqrt{\text{Ann}_R(M)}$. Then the followings hold:

- (i) If K_1 and K_2 are n -submodules of M with $IK_1 = IK_2$, then $K_1 = K_2$.
- (ii) If IK is an n -submodule of M , then $IK = K$.

Proof. (i) Since K_1 is an n -submodule and $IK_2 \subseteq K_1$, by Theorem 2.2, we get that $K_2 \subseteq K_1$. Likewise, $K_1 \subseteq K_2$.

(ii) Since IK is an n -submodule and $IK \subseteq IK$, we conclude that $K \subseteq IK$, so this completes the proof. \square

The next lemma provides a useful characterization of modules that have n -submodule.

Lemma 2.5. Let M be a torsion-free R -module. Then zero submodule is an n -submodule of M .

Proof. Let $ax = 0$ with $a \notin \sqrt{\text{Ann}_R(M)}$, for $a \in R$ and $x \in M$. Since M is torsion-free, $x = 0$. Thus zero submodule of M is an n -submodule. \square

Lemma 2.6. If M is a torsion-free multiplication R -module, then zero submodule is the only n -submodule of M .

Proof. Suppose that N is an n -submodule of M . Then by Proposition 2.3(i), we have $(N :_R M) \subseteq \sqrt{\text{Ann}_R(M)} = 0$ and so $(N :_R M) = 0$. As M is multiplication, then $N = 0$. So by Lemma 2.5, the zero submodule is the only n -submodule. \square

Proposition 2.7. *Let M be an R -module and I be an ideal of R . If N is an n -submodule of M such that $I \not\subseteq (N :_R M)$, then $(N :_M I)$ is an n -submodule of M .*

Proof. Let $ax \in (N :_M I)$ with $a \notin \sqrt{\text{Ann}_R(M)}$, for $a \in R$ and $x \in M$. So $aIx \subseteq N$ and as N is an n -submodule, $Ix \subseteq N$. Hence $x \in (N :_M I)$. \square

Proposition 2.8. *Let N be a proper submodule of M . Then N is an n -submodule if and only if for every $x \in M$, $(N :_R x) = R$ or $(N :_R x) \subseteq \sqrt{\text{Ann}_R(M)}$.*

Proof. Assume that N is an n -submodule. If $(N :_R x) \not\subseteq \sqrt{\text{Ann}_R(M)}$, then there exists $r \in (N :_R x) - \sqrt{\text{Ann}_R(M)}$. So $rx \in N$ where $r \notin \sqrt{\text{Ann}_R(M)}$. Since N is an n -submodule, $x \in N$. Hence $(N :_R x) = R$. Conversely, let $rx \in N$ where $r \notin \sqrt{\text{Ann}_R(M)}$, for $r \in R$ and $x \in M$. So $r \in (N :_R x) - \sqrt{\text{Ann}_R(M)}$. By assumption, we have $(N :_R x) = R$ and therefore $x \in N$. \square

Corollary 2.9. *Let N be a proper submodule of M . Then N is an n -submodule if and only if for every $x \in M - N$, $(N :_R x) \subseteq \sqrt{\text{Ann}_R(M)}$.*

Recall that, $r \in R$ is said to be a zero divisor of an R -module M , if there exists a non-zero element $x \in M$ such that $rx = 0$.

Theorem 2.10. *Let M be an R -module and N be a submodule of M . Then N is an n -submodule if and only if every zero divisor of an R -module $\frac{M}{N}$ is in $\sqrt{\text{Ann}_R(M)}$.*

Proof. Let N be an n -submodule and r be a zero divisor of $\frac{M}{N}$. Then there exists $x \in M - N$ such that $rx \in N$. Since N is an n -submodule, we have $r \in \sqrt{\text{Ann}_R(M)}$. For the converse, assume that $rx \in N$ where $x \notin N$, for $r \in R$ and $x \in M$. Then r is a zero divisor of $\frac{M}{N}$ and so $r \in \sqrt{\text{Ann}_R(M)}$. \square

Theorem 2.11. *Every maximal n -submodule is a prime submodule.*

Proof. Let N be a maximal n -submodule of M and $ax \in N$ where $a \notin (N :_R M)$, for $a \in R$ and $x \in M$. By Proposition 2.7, $(N :_M a)$ is an n -submodule. Thus $x \in (N :_M a) = N$, by maximality of N . So N is a prime submodule. \square

Theorem 2.12. *Let M be a finitely generated R -module. If M has an n -submodule, then M has a prime submodule.*

Proof. Suppose that N is an n -submodule and $\Omega = \{L : L \text{ is an } n\text{-submodule of } M; N \subseteq L\}$. By Zorn's Lemma, Ω has a maximal element $K \in \Omega$. Then by Theorem 2.11, K is a prime submodule of M . \square

In ring theory (and so in module theory), the concepts prime ideal and n -ideal are not the same in general. (see Example 3.2 in [12]). In the following, we try to find some relations between them.

Proposition 2.13. *For a prime submodule N of M , N is an n -submodule if and only if $(N :_R M) = \sqrt{\text{Ann}_R(M)}$.*

Proof. Suppose that N is a prime submodule of M . It is clear that $\sqrt{\text{Ann}_R(M)} \subseteq (N :_R M)$. If N is an n -submodule, then by Proposition 2.3(i), we have $(N :_R M) \subseteq \sqrt{\text{Ann}_R(M)}$ and so $(N :_R M) = \sqrt{\text{Ann}_R(M)}$. For the converse, assume that $(N :_R M) = \sqrt{\text{Ann}_R(M)}$. Now we show that N is an n -submodule. Let $ax \in N$ and $a \notin \sqrt{\text{Ann}_R(M)}$, for $a \in R$ and $x \in M$. Since N is a prime submodule and $a \notin (N :_R M)$, we get $x \in N$ and so N is an n -submodule. \square

Recall from [11], the intersection of all prime submodules contains N , denoted $\text{rad}(N)$, is called the radical of N . If there is no prime submodule containing N , $\text{rad}(N) = M$.

Proposition 2.14. *Let M be a finitely generated R -module. Then $\text{rad}(0)$ is an n -submodule if and only if $\text{rad}(0)$ is a prime submodule.*

Proof. Since M is finitely generated, by Theorem 4.4 in [8], $(\text{rad}(0) :_R M) = \sqrt{\text{Ann}_R(M)}$. Suppose that $\text{rad}(0)$ is an n -submodule. Let $ax \in \text{rad}(0)$ with $a \notin (\text{rad}(0) :_R M)$, for $a \in R$ and $x \in M$. So $a \notin \sqrt{\text{Ann}_R(M)}$ and since $\text{rad}(0)$ is an n -submodule, we have $x \in \text{rad}(0)$. Thus $\text{rad}(0)$ is a prime submodule. Now assume that $\text{rad}(0)$ is a prime submodule. By Proposition 2.13, $\text{rad}(0)$ is an n -submodule. \square

Lemma 2.15. *Let N be an n -submodule of an R -module M such that $(N :_R M) = \sqrt{\text{Ann}_R(M)}$. Then N is a prime submodule.*

Proof. It is clear. \square

Proposition 2.16. *If zero submodule of an R -module M is an n -submodule, then $\sqrt{\text{Ann}_R(M)}$ is a prime ideal of R .*

Proof. Let $ab \in \sqrt{\text{Ann}_R(M)}$ for $a, b \in R$. So there exists $n \in \mathbb{N}$ such that $a^n b^n M = 0$. If $a \notin \sqrt{\text{Ann}_R(M)}$, then since the zero submodule is a n -submodule, we get $b^n M = 0$; i.e. $b \in \sqrt{\text{Ann}_R(M)}$. \square

Remember that if N is a prime submodule of an R -module M , then $(N :_R M)$ is a prime ideal of R . Now, we give a similar result for n -submodules.

Lemma 2.17. *If M is a faithful R -module and N is an n -submodule of M , then $(N :_R M)$ is an n -ideal of R .*

Proof. Assume that $ab \in (N :_R M)$ with $a \notin \sqrt{0}$, for $a, b \in R$. Since $\text{Ann}_R(M) = 0$ and N is an n -submodule, then $b \in (N :_R M)$. \square

Corollary 2.18. *Let M be a faithful R -module and R has no n -ideal. Then M has no n -submodule.*

Lemma 2.19. *Let M be a multiplication R -module and N be a submodule of M such that $(N :_R M)$ is an n -ideal of R . Then N is an n -submodule.*

Proof. Let $IK \subseteq N$ with $I \not\subseteq \sqrt{\text{Ann}_R(M)}$, where I is an ideal of R and K is a submodule of M . Since M is multiplication and $(N :_R M)$ is an n -ideal, $I(K :_R M) \subseteq (N :_R M)$ and so $(K :_R M) \subseteq (N :_R M)$, by Theorem 2.7 in [12]. Thus $K \subseteq N$ and by Theorem 2.2, N is an n -submodule. \square

Corollary 2.20. *Let M be a cyclic R -module and N be a submodule of M such that $(N :_R M)$ is an n -ideal of R . Then N is an n -submodule of M .*

Recall that a proper submodule N of M is said to be an r -submodule, if for $a \in R$, $m \in M$ and whenever $am \in N$ with $\text{ann}_M(a) = 0$, then $m \in N$ [5].

Proposition 2.21. *Every n -submodule is an r -submodule.*

Proof. Let N be an n -submodule of M . Now, we will show that N is an r -submodule. Let $am \in N$ with $\text{ann}_M(a) = 0$, for some $a \in R$, $m \in M$. Assume that $a \in \sqrt{\text{Ann}_R(M)}$. Then there exists $n \in \mathbb{N}$ such that $a^n M = 0$. Choose the smallest positive integer n such that $a^n M = 0$. Then we have $a^{n-1} M \neq 0$. Since $a(a^{n-1} M) = a^n M = 0$, we have $a^{n-1} M \subseteq \text{ann}_M(a) = 0$ and so $a^{n-1} M = 0$ which is a contradiction. So that $a \notin \sqrt{\text{Ann}_R(M)}$. As N is an n -submodule and $am \in N$, we get $m \in N$. Hence, N is an r -submodule of M . \square

Theorem 2.22. *Let N be a submodule of M such that $(N :_R M) \subseteq \sqrt{\text{Ann}_R(M)}$. Then the following statements are equivalent:*

- (i) N is an n -submodule;
- (ii) N is a primary submodule of M .

Proof. (i) \Rightarrow (ii) Let $ax \in N$ with $a \notin \sqrt{(N :_R M)}$, for $a \in R$ and $x \in M$. As N is an n -submodule, we have $x \in N$. Thus N is a primary submodule.
(ii) \Rightarrow (i) Let $ax \in N$ with $a \notin \sqrt{\text{Ann}_R(M)}$, for $a \in R$ and $x \in M$. As $\sqrt{(N :_R M)} = \sqrt{\text{Ann}_R(M)}$, we have $a \notin \sqrt{\text{Ann}_R(M)}$. Since N is a primary submodule, we get $x \in N$. Therefore N is an n -submodule. \square

By the proof of previous theorem, every n -submodule is a primary submodule. So it is straightforward to get that if N is an n -submodule of R -module M , then $(N :_R M)$ is a primary ideal of R . Recall if $(N :_R M)$ is a maximal ideal of ring R , then N is a primary submodule of M . So we have:

Corollary 2.23. *Let $\text{Ann}_R(M)$ be a maximal ideal of R . Then every proper submodule of M is an n -submodule.*

By using the fact that every irreducible submodule of a Noetherian module is a primary submodule (Proposition 1-17 in [4]), we can get the following corollary:

Corollary 2.24. *Let M be a Noetherian R -module and N be an irreducible submodule of M such that $(N :_R M) \subseteq \sqrt{\text{Ann}_R(M)}$. Then N is an n -submodule of M .*

Proposition 2.25. *If N is a primary R -submodule of M such that $(N :_R M)$ is maximal in the set of all n -ideals, then N is an n -submodule of M .*

Proof. Let $ax \in N$ with $a \notin \sqrt{\text{Ann}_R(M)}$, for $a \in R$ and $x \in M$. By Theorem 2.11 [12], $\sqrt{0} = \sqrt{(N :_R M)}$. Since N is a primary submodule and $a \notin \sqrt{(N :_R M)}$, $x \in N$. \square

Lemma 2.26. *If N is an n -submodule and L is a primary submodule of an R -module M such that $(L :_R M) \subseteq \text{Ann}_R(M)$, then $N \cap L$ is an n -submodule of M .*

Proof. Let $rx \in N \cap L$ where $r \notin \sqrt{\text{Ann}_R(M)}$, for $r \in R$, $x \in M$. Then $r \notin \sqrt{(L :_R M)}$. Since L is primary, $x \in L$. Also, since N is an n -submodule, $x \in N$. Thus $x \in N \cap L$. \square

Recall that a proper ideal I of R is called semiprime, if whenever $a^n \in I$ for $a \in R$ and $n \in \mathbb{N}$, then $a \in I$ [10]. Now, in the following theorem we give a characterization for torsion free modules in terms of n -submodules.

Theorem 2.27. *Let M be an R -module. Then the following statements are equivalent:*

- (i) M is a torsionfree R -module;
- (ii) M is faithful, zero submodule is an n -submodule of M and zero ideal is a semiprime ideal of R .

Proof. (i) \Rightarrow (ii) It follows from Lemma 2.5.

(ii) \Rightarrow (i) Let $rx = 0$ and $r \neq 0$, for $r \in R$, $x \in M$. Since (0) is a semiprime ideal of R , $\sqrt{0} = 0$. As M is faithful, it follows that $r \notin \sqrt{\text{Ann}_R(M)} = \sqrt{0} = 0$. Since the zero submodule is an n -submodule, $x = 0$. Therefore, M is a torsion-free module. \square

Theorem 2.28. *Let $f : M \rightarrow M'$ be an R -homomorphism. Then the followings hold:*

- (i) If f is an epimorphism and N is an n -submodule of M containing $\ker(f)$, then $f(N)$ is an n -submodule of M' .
- (ii) If f is a monomorphism and L' is an n -submodule of M' , then $f^{-1}(L') = M$ or $f^{-1}(L')$ is an n -submodule of M .

Proof. (i) Let $rx' \in f(N)$ where $r \notin \sqrt{\text{Ann}_R(M')}$, for $r \in R$, $x' \in M'$. Since f is epimorphism, there exists $x \in M$ such that $x' = f(x)$. Then $rx' = rf(x) = f(rx) \in f(N)$. As $\ker(f) \subseteq N$, we conclude that $rx \in N$. Also, note that $r \notin \sqrt{\text{Ann}_R(M)}$. Since N is an n -submodule of M , we get the result that $x \in N$ and so $x' = f(x) \in f(N)$.

(ii) Let $f^{-1}(L') \neq M$ and $rx \in f^{-1}(L')$ where $r \notin \sqrt{\text{Ann}_R(M)}$, for $r \in R$, $x \in M$. Then $f(rx) = rf(x) \in L'$. Since f is a monomorphism and $r \notin \sqrt{\text{Ann}_R(M)}$, we get $r \notin \sqrt{\text{Ann}_R(M')}$. Since L' is an n -submodule of M' , $f(x) \in L'$ and so $x \in f^{-1}(L')$. Consequently, $f^{-1}(L')$ is an n -submodule of M . \square

Corollary 2.29. *Let M be an R -module and $L \subseteq N$ be two submodules of M . Then the followings hold:*

- (i) *If N is an n -submodule of M , then $\frac{N}{L}$ is an n -submodule of $\frac{M}{L}$.*
- (ii) *If $\frac{N}{L}$ is an n -submodule of $\frac{M}{L}$ and $(L :_R M) \subseteq \sqrt{\text{Ann}_R(M)}$, then N is an n -submodule of M .*
- (iii) *If $\frac{N}{L}$ is an n -submodule of $\frac{M}{L}$ and L is an n -submodule of M , then N is an n -submodule of M .*

Proof. (i) Assume that N is an n -submodule of M and $L \subseteq N$. Let $\pi : M \rightarrow \frac{M}{L}$ be the natural homomorphism. Note that $\ker(\pi) = L \subseteq N$, and so by Theorem 2.28(i), $\frac{N}{L}$ is an n -submodule of $\frac{M}{L}$.

(ii) Let $rx \in N$ where $r \notin \sqrt{\text{Ann}_R(M)}$ for $r \in R$, $x \in M$. Then we have $(r + I)(x + L) = rx + L \in \frac{N}{L}$ and $r + I \notin \sqrt{\text{Ann}_{\frac{R}{I}}(\frac{M}{L})}$, where $I = (L :_R M)$. Since $\frac{N}{L}$ is an n -submodule of $\frac{M}{L}$, we conclude that $x + L \in \frac{N}{L}$ and so $x \in N$. Consequently, N is an n -submodule of M .

(iii) It follows from (ii) and Proposition 2.3(i). \square

Corollary 2.30. *Let M be an R -module and N be a submodule of M . If L is an n -submodule of M such that $N \not\subseteq L$, then $L \cap N$ is an n -submodule of N .*

Proof. Consider the injection $i : N \rightarrow M$. Note that $i^{-1}(L) = L \cap N$, so by Theorem 2.28(ii), $L \cap N$ is an n -submodule of N . \square

Let M be an R -module and S be a multiplicative closed subset of R . Consider the natural homomorphism π from M to M_S as $\pi(m) = \frac{m}{1}$, for any $m \in M$. Then for each submodule L of M_S , we define L^c as an inverse image of L under this natural homomorphism.

Proposition 2.31. *Let M be an R -module and S a multiplicative closed subset of R .*

- (i) *If N is an n -submodule of M , then $N_S = M_S$ or N_S is an n -submodule of M_S .*
- (ii) *If M is finitely generated, L is an n -submodule of M_S and $S \cap (\text{Ann}_R(M) :_R a) = \emptyset$ for every $a \notin \text{Ann}_R(M)$, then $L^c = M$ or L^c is an n -submodule of M .*

Proof. (i) Let $N_S \neq M_S$ and $\frac{a}{s} \frac{m}{t} \in N_S$ where $\frac{a}{s} \notin \sqrt{\text{Ann}_{R_S}(M_S)}$, for $a \in R$, $s, t \in S$, $m \in M$. Then we have $uam \in N$, for some $u \in S$. It is clear that $a \notin \sqrt{\text{Ann}_R(M)}$. Since N is an n -submodule of M , we conclude that $um \in N$ and so $\frac{m}{t} = \frac{um}{ut} \in N_S$. Therefore N_S is an n -submodule of M_S .

(ii) Let $L^c \neq M$ and $am \in L^c$ where $a \notin \sqrt{\text{Ann}_R(M)}$ for $a \in R$, $m \in M$. Then we have $\frac{a}{1} \frac{m}{1} \in L$. Now we show that $\frac{a}{1} \notin \sqrt{\text{Ann}_{R_S}(M_S)}$. Suppose $\frac{a}{1} \in \sqrt{\text{Ann}_{R_S}(M_S)}$. There exists a positive integer k such that $(\frac{a}{1})^k M_S = 0$. Then we get $ua^k M = 0$ for some $u \in S$, as M is finitely generated. Since $a \notin \sqrt{\text{Ann}_R(M)}$, $a^k M \neq 0$ and so $u \in (\text{Ann}_R(M) :_R a^k) \cap S$, which is a contradiction. Thus we have $\frac{a}{1} \notin \sqrt{\text{Ann}_{R_S}(M_S)}$. As L is an n -submodule of M_S , we conclude that $\frac{m}{1} \in L$ and so $m \in L^c$. \square

Lemma 2.32. *Let M be a finitely generated R -module such that for every multiplicative closed set $S \subseteq R$, the kernel of $\varphi : M \rightarrow M_S$ is either (0) or M . Then (0) is an n -submodule of M .*

Proof. Let $rx = 0$ where $r \in R - \sqrt{\text{Ann}_R(M)}$ and $x \in M$. So $r^n \neq 0$, for every $n \in \mathbb{N}$. We put $S = \{r^n : n \in \mathbb{N} \cup \{0\}\}$. Clearly S is a multiplicative closed set in R . If $\ker(\varphi) = 0$, then as $\varphi(x) = \frac{x}{1} = \frac{rx}{r} = 0$ we have $x = 0$. Let $\ker(\varphi) = M$. Since M is finitely generated, we can write $M = Rx_1 + Rx_2 + \dots + Rx_t$, for some $x_1, x_2, \dots, x_t \in M$. Then $\varphi(x_i) = \frac{x_i}{1} = 0$ for any $1 \leq i \leq t$. Thus for any i , there exists $l_i \in \mathbb{N}$ such that $r^{l_i} x_i = 0$. Put $j := \max\{l_1, l_2, \dots, l_t\}$. Thus we have $r^j M = 0$ and so $r \in \sqrt{\text{Ann}_R(M)}$, which is a contradiction. \square

We recall that a nonempty subset S of R where $R - \sqrt{0} \subseteq S$ is said to be an n -multiplicatively closed subset of R , if $xy \in S$ for all $x \in R - \sqrt{0}$ and all $y \in S$ (see [12]).

Theorem 2.33. *Let M be a finitely generated R -module and N be a proper submodule of M such that $(N :_R M) \cap S = \emptyset$, where S is an n -multiplicatively closed set in R . Then there exists an n -submodule L of M containing N such that $(L :_R M) \cap S = \emptyset$.*

Proof. Consider that set $\Omega = \{L : L \text{ is a submodule of } M; (L :_R M) \cap S = \emptyset\}$. Since $N \in \Omega$, we have $\Omega \neq \emptyset$. Since M is finitely generated, by using Zorn's lemma, we get a maximal element K of Ω . Now we show that K is an n -submodule of M . Suppose that $rx \in K$, for some $r \notin \sqrt{\text{Ann}_R(M)}$ and $x \notin K$. Thus we get $x \in (K :_M r)$ and $K \subset (K :_M r)$. By maximality of K , we have $((K :_M r) :_R M) \cap S \neq \emptyset$ and thus there exists $t \in S$ such that $tM \subseteq (K :_M r)$. Also $rt \in S$, because $r \in R - \sqrt{0}$ and $t \in S$ and S is an n -multiplicatively closed subset of R . We get $(K :_R M) \cap S \neq \emptyset$, which is a contradiction. Hence K is an n -submodule of M . \square

Proposition 2.34. *Suppose that $N \subseteq \bigcup_{i=1}^n N_i$, where N, N_i ($1 \leq i \leq n$), are R -submodules of M . If there exists N_j such that $N \not\subseteq \bigcup_{i \neq j} N_i$, N_j is an n -submodule and $(\bigcap_{i \neq j} N_i :_R M) \not\subseteq \sqrt{\text{Ann}_R(M)}$, then $N \subseteq N_j$.*

Proof. We may assume that $j = 1$. Since $N \not\subseteq \bigcup_{i \geq 2} N_i$, there exists $x \in N - \bigcup_{i=2}^n N_i$. Thus we have $x \in N_1$. Let $y \in N \cap (\bigcap_{i=2}^n N_i)$. Since $x \notin N_k$ and $y \in N_k$ for every $2 \leq k \leq n$, we have $x + y \notin N_k$. Thus $x + y \in N - \bigcup_{i=2}^n N_i$

and so $x + y \in N_1$. As $x + y \in N_1$ and $x \in N_1$, it follows that $y \in N_1$ and so $N \cap (\bigcap_{i=2}^n N_i) \subseteq N_1$. Also we have $(\bigcap_{i=2}^n N_i :_R M)N \subseteq N \cap (\bigcap_{i=2}^n N_i)$. Now since $(\bigcap_{i=2}^n N_i :_R M)N \subseteq N_1$, $(\bigcap_{i=2}^n N_i :_R M) \not\subseteq \sqrt{\text{Ann}_R(M)}$ and N_1 is an n -submodule of M , we have $N \subseteq N_1$. \square

Following Lemma 1.1 in [9], a submodule K of an R -module M is prime if and only if $p = (K :_R M)$ is a prime ideal of R and the $\frac{R}{p}$ -module $\frac{M}{K}$ is torsion-free. Now, we give a similar result for n -submodules.

Theorem 2.35. *Let N be an R -submodule of M such that $I = \sqrt{\text{Ann}_R(M)} \subseteq (N :_R M)$. Then N is an n -submodule of M if and only if $\frac{M}{N}$ is a torsion-free $\frac{R}{I}$ -module.*

Proof. Let N be an n -submodule and $(r + I)(x + N) = 0_{\frac{M}{N}}$, for $r \in R$ and $x \in M$. Then we have $rx \in N$. If $r \in I$, then $r + I = 0$. Otherwise, since N is an n -submodule, we conclude that $x \in N$ and so $x + N = 0$. For the converse, assume that $\frac{M}{N}$ is a torsion-free $\frac{R}{I}$ -module and $rx \in N$, for $x \in M$ and $r \in R - \sqrt{\text{Ann}_R(M)}$. Then $(r + I)(x + N) = rx + N = N = 0_{\frac{M}{N}}$. Now as $\frac{M}{N}$ is a torsion-free $\frac{R}{I}$ -module and $r \notin I$, we have $x \in N$. So N is an n -submodule of M . \square

Lemma 2.36. *Let $\{L_i\}_{i \in I}$ be a family of R -submodules of $\{M_i\}_{i \in I}$. If $\Pi_{i \in I} L_i$ is an n -submodule of $\Pi_{i \in I} M_i$, then for every $i \in I$, L_i is an n -submodule of M_i .*

Proof. Let $\Pi_{i \in I} L_i$ be an n -submodule of $\Pi_{i \in I} M_i$ and i be an arbitrary in I . We will prove L_i is an n -submodule of M_i . Suppose that $rx \in L_i$ where $r \notin \sqrt{\text{Ann}_R(M_i)}$, for $r \in R$ and $x \in M_i$. Put $x_i := x$ and $x_j := 0$ for all $j \neq i$. Then we have $r(x_j)_{j \in I} \in \Pi_{j \in I} L_j$ and $r \notin \sqrt{\text{Ann}_R(\Pi_{j \in I} M_j)}$. Since $\Pi_{j \in I} L_j$ is an n -submodule of $\Pi_{j \in I} M_j$, so $(x_j)_{j \in I} \in \Pi_{j \in I} L_j$. Hence $x_i \in L_i$. \square

Corollary 2.37. *Let M_1 and M_2 be R -module and $M = M_1 \times M_2$. Then the following are satisfied:*

- (i) *If $L_1 \times M_2$ is an n -submodule of M , then L_1 is an n -submodule of M_1 .*
- (ii) *If $M_1 \times L_2$ is an n -submodule of M , then L_2 is an n -submodule of M_2 .*

Theorem 2.38. *Let N be a proper R -submodule of M . Then N is an n -submodule of M if and only if for each $a \in R - \sqrt{\text{Ann}_R(M)}$, the homothety $\lambda_a : \frac{M}{N} \rightarrow \frac{M}{N}$ is an injective.*

Proof. Suppose that N is an n -submodule and $\lambda_a(x + N) = 0_{\frac{M}{N}}$ for $a \in R - \sqrt{\text{Ann}_R(M)}$, $x \in M$. Then $ax \in N$ and since N is an n -submodule, so $x \in N$ and $x + N = 0$. Hence λ_a is injective. Conversely, suppose that $rx \in N$ where $r \notin \sqrt{\text{Ann}_R(M)}$, for $r \in R$, $x \in M$. It follows that $\lambda_r(x + N) = 0$. Since λ_r is injective, $x + N = 0$ and so $x \in N$. \square

In [7], I.G. Macdonald introduced the notion of secondary modules. A nonzero R -module M is said to be secondary, if for each $a \in R$ the endomorphism of M given by multiplication by a is either surjective or nilpotent.

Proposition 2.39. *If M is a secondary R -module such that every ascending chain of cyclic submodules of it stops, then every proper submodule of M is an n -submodule.*

Proof. Let N be a proper submodule of M and $rx \in N$, for $r \in R$ and $x \in M$. Assume that φ_r is the homothety $M \rightarrow M$ for $r \in R$. If φ_r is nilpotent, then there exists $n \in \mathbb{N}$ such that $(\varphi_r)^n = 0$. It follows that $r^n \in \text{Ann}_R(M)$ and so $r \in \sqrt{\text{Ann}_R(M)}$. If φ_r is surjective, then we have

$$x = rx_1$$

$$x_1 = rx_2$$

$$x_2 = rx_3$$

...

$$x_n = rx_{n+1}$$

...

for some $x_i \in M$. Then $\langle x \rangle \subseteq \langle x_1 \rangle \subseteq \langle x_2 \rangle \subseteq \dots \subseteq \langle x_n \rangle \subseteq \dots$. Since M is complete, there exists $n \in \mathbb{N}$ such that $\langle x_n \rangle = \langle x_i \rangle$, for every $i \geq n$. Hence there exists $s \in R$ such that $x_{n+1} = sx_n$. It follows that $x_n = rsx_n$. So $(1 - rs)x = 0$ and we have $x = rsx$. As $rx \in N$, so $x \in N$. \square

Corollary 2.40. *Let M be a Noetherian secondary module. Then every proper submodule is an n -submodule.*

Proposition 2.41. *If N is an n - R -submodule of M , then $N[x]$ an n -submodule of $M[x]$.*

Proof. Let r be a zero divisor of an R -module $\frac{M[x]}{N[x]}$. Since $\frac{M[x]}{N[x]} \cong \frac{M}{N}[x]$, then there exists $f(x) = a_0 + a_1x + \dots + a_tx^t \in M[x]$ such that $0 \notin \overline{f(x)} \in \frac{M}{N}[x]$ and $r\overline{f(x)} = \bar{0}$. Hence $ra_i \in N$, for $1 \leq i \leq t$. If for every i , $a_i \in N$, then $\overline{f(x)} = \bar{0}$, which is a contradiction. Thus there exists $1 \leq i \leq t$ such that $a_i \notin N$ with $ra_i \in N$. On the other hand, as N is an n -submodule, so $r \in \sqrt{\text{Ann}_R(M)}$. Since $M \subseteq M[x]$, so $r \in \sqrt{\text{Ann}_R(M[x])}$. Then by Theorem 2.10, $N[x]$ is an n -submodule of $M[x]$. \square

3. EXAMPLES

EXAMPLE 3.1. Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $R = \mathbb{Z}$. Then every proper submodule of M is an n-submodule. It is clear that every proper submodule of M is prime and the colon ideal of M into submodules are equal $2\mathbb{Z}$. Now according to Proposition 2.13, every proper submodule of M is an n-submodule.

Now we have an example which shows that there exists an R -module that does not have an n-submodule.

EXAMPLE 3.2. Let p be any prime number. Let $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ and $R = \mathbb{Z}$. Then every proper submodule of M is not an n-submodule. Let N be an n-submodule of M . By Proposition 2.3(i), $(N :_R M) \subseteq \sqrt{\text{Ann}_R(M)} = \sqrt{0} = 0$. It follows that $(N :_R M) = \sqrt{\text{Ann}_R(M)}$. Then by Lemma 2.15, N is a prime submodule. On the other hand, pM is the only prime submodule of M . So $N = pM$ and $(N :_R M) = (pM :_R M) = p\mathbb{Z}$, which is a contradiction.

Remark 3.3. (i) By Theorem 2.22, every n-submodule of a module is a primary submodule. However, the converse is not true in general. Since for example: if $R = \mathbb{Z}$, $M = \mathbb{Z}$ and $N = 4\mathbb{Z}$, then N is a primary submodule of M , however it is not n-submodule, as $2 \in N$, but $2 \notin \sqrt{\text{Ann}_R(M)}$ and $2 \notin N$.

(ii) It is well known that if N is a prime submodule of M , then $(N :_R M)$ is a prime ideal of R . Contrary to what happens for a prime submodules, if N is an n-submodule, the ideal $(N :_R M)$ is not in general an n-ideal of R . For example: Let $M = \mathbb{Z}_4$, $R = \mathbb{Z}$. Take $N = (\bar{0})$. Certainly N is an n-submodule of M , but $(N :_R M) = 4\mathbb{Z}$ is not an n-ideal of R .

The following example shows that the converse of Lemma 2.5, is not necessarily true.

EXAMPLE 3.4. Consider the \mathbb{Z} -module \mathbb{Z}_4 and $N = (\bar{0})$. Clearly N is an n-submodule, but M is not a torsion-free module.

In the next example, we show that zero submodule is not always the only n-submodule of torsion-free modules.

EXAMPLE 3.5. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ and $R = \mathbb{Z}$. consider the submodule $N = 0 \oplus \mathbb{Z}$. Let $a(m, n) = (am, an) \in N$ with $a \notin \sqrt{\text{Ann}_R(M)} = 0$ for some $a, m, n, \in \mathbb{Z}$. Then we have $am = 0$ and so $m = 0$. This implies that $(m, n) = (0, n) \in N$. Thus N is a nonzero n-submodule of M .

The next example shows that the sum of two n-submodule is not an n-submodule in general.

EXAMPLE 3.6. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ and $R = \mathbb{Z}$. Consider the submodules $N = 0 \oplus \mathbb{Z}$ and $K = \mathbb{Z} \oplus 0$. One can easily see that K and N are n-submodules. Since $N + K = M$, $N + K$ is not an n-submodule of M .

Proposition 3.7. \mathbb{Q} as \mathbb{Z} -module has only one n-submodule.

Proof. By Lemma 2.5, zero submodule is an n-submodule of \mathbb{Q} . Let N be an n-submodule. It follows that $(N :_{\mathbb{Z}} \mathbb{Q}) = 0$. Then by Lemma 2.15, N is an prime submodule of \mathbb{Q} , which is zero. \square

Now we give an example to show that in Theorem 2.27, it is necessary that zero submodule be an n-submodule.

EXAMPLE 3.8. Let M be the \mathbb{Z} -module $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$. M is faithful and zero ideal is a semiprime ideal. By Example 3.2, zero submodule is not n-submodule of M and M is not torsion-free.

In the following examples we show that the condition $\ker(f) \subseteq N$ in Theorem 2.28(i) and the condition monomorphism in Theorem 2.28(ii), are necessary.

EXAMPLE 3.9. Consider the \mathbb{Z} -epimorphism

$$\psi : \mathbb{Z} \longrightarrow \mathbb{Z}_6; \quad a \longmapsto \bar{a}$$

Clearly $\psi(0) = \bar{0}$ and $\ker(\psi) = 6\mathbb{Z} \not\subseteq (0)$. By Example 2.1(ii), $(\bar{0})$ is not n-submodule of \mathbb{Z}_6 .

EXAMPLE 3.10. Consider the zero homomorphism

$$g : \mathbb{Q} \longrightarrow \mathbb{Z};$$

clearly $\ker(g) = \mathbb{Q}$. So g is not monomorphism. By Proposition 3.7, $g^{-1}(0)$ is not an n-submodule.

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