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On the WZ Factorization of the Real and Integer Matrices

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Abstract. The *QIF* (Quadrant Interlocking Factorization) method of Evans and Hatzopoulos solves linear equation systems using *WZ* factorization. The WZ factorization can be faster than the *LU* factorization because, it performs the simultaneous evaluation of two columns or two rows. Here, we present a method for computing the real and integer *WZ* and *ZW* factorizations by using the null space generators of some special nested submatrices of a matrix A.

Keywords: Linear systems, Quadrant interlocking factorization, *WZ* factorization, *ZW* factorization, Null space generator.

2000 Mathematics subject classification: 15A03, 15A21, 15A23, 34C20, 49M27.

1. Introduction

Linear systems arise frequently in scientific and engineering computing. Various serial and parallel algorithms have been introduced for their serial solution [9, 4]. The *QIF* (Quadrant Interlocking Factorization) algorithm, introduced by Evans and Hatzopoulos, is a numerical method for finding a solution for systems of the type $Ax = b$, where A is a nonsingular matrix of dimensions

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$n \times n$, x is an unknown column vector, and b is the independent term vector provided. The *QIF* method is based on the WZ factorization of the coefficient matrix A. The main advantage of this factorization is that it presents a complexity order less than of the LU decomposition due to the fact that it performs the simultaneous evaluation of two columns or two rows. A detailed description of this algorithm for real and complex matrices can be found in [4, 5, 10, 11]. Golpar-Raboky and Mahdavi-Amiri presented new algorithms for computing the real and integer WZ and ZW matrix factorizations using ABS algorithms and the extended rank reduction process [6, 7, 8, 16]. Recently, some authors have considered simultaneous matrix decompositions [12, 13, 14].

The WZ factorization is used for solving Markovian linear systems [2] and network modeling [3], preconditioning of sparse matrices [18] and eigenvalue problems [15].

Let $\mathbb{R}$ and $\mathbb{R}^{m\times n}$ stand for the real number, and the set of all $m \times n$ matrices over $\mathbb{R}$ and A^T denotes the transpose of A. Let $A = (a_1, \cdots, a_m)^T \in \mathbb{R}^{m\times n}$. Assume that $a_{k_1}^T, \cdots, a_{k_i}^T$ be the rows of A and $H_1 \in \mathbb{R}^{n\times n}$ be an arbitrary nonsingular matrix. For $j = 1, \cdots, i$ update H_j by

$$H_{j+1} = H_j - \frac{H_j a_{k_j} w_j^T H_j}{w_j^T H_j a_{k_j}}, \tag{1.1}$$

where $w_j \in \mathbb{R}^n$ such that $w_j^T H_j a_{k_j} \neq 0$. Then, we have

$$a_{k_i}^T H_{j+1}^T = 0,\ i = 1, \cdots, j, \tag{1.2}$$

and the linear combination of the columns of H_{i+1}^T generates the null space of $\{a_{k_1}, \cdots a_{k_i}\}$ (see [1]).

Matrices H_i are generalizations of (oblique) projection matrices. They probably first appeared in a book by Wedderburn [19]. They have been named ***Abbafians*** since the First International Conference on ABS methods(Luoyang, China, 1991) and this name will be used here.

Notation: Let $A \in \mathbb{R}^{n\times n}$. Here and subsequently $J_n = \{j_1, \cdots, j_n\}$ denotes a permutation of $\mathcal{I}_n = \{1, 2, \cdots, n\}$ and, for $k = 1, \cdots, n$, $J_k = \{j_1, \cdots, j_k\}$ denotes a subset of J_n. Let

$$A_{J_k} = (a_{i,j}),\ i, j \in J_k. \tag{1.3}$$

denotes a submatrix of A, and

$$J_1 \subset J_2 \subset \cdots \subset J_n, \tag{1.4}$$

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and $\{A_{J_k}\}_{k=1}^{n}$ be a sequence of nested submatrices of A. The following theorem describes a necessary and sufficient condition for nonsingularity $A_{J_k},\ k = 1, \cdots, n$.

Theorem 1.1. *(Nested submatrices) Let $A \in \mathbb{R}^{n\times n}$ and $H_1 = I$. Then the nested submatrices $A_{J_i},\ i = 1, \cdots, n$, are nonsingular if and only if $e_{j_i}^T H_i a_{j_i} \neq 0,\ i = 1, \cdots, n$.*

Proof. Follow the lines of the proof for Theorem 6.5 in [1] by replacing i to j_i. □

From (1.1) and Theorem 1.1 we have the following result.

Theorem 1.2. *Let $A \in \mathbb{R}^{n\times n}$, $H_1 = I$, and for $i = 1, \cdots, n$, $e_{j_i}^T A H_i e_{j_i} \neq 0$. Then,*

$$H_{i+1} = H_i - \frac{H_i a_{j_i} e_{j_i}^T H_i}{e_{j_i}^T H_i a_{j_i}}, \tag{1.5}$$

is well defined.

The parameter choices in Theorem 1.2, induce a structure in the matrix H_i, described by the following theorem.

Theorem 1.3. *Let the conditions of Theorem 1.2 be satisfied and H_{i+1} defined by (1.5). Then, the following properties hold:*

(a) The jth row of H_{i+1} is zero, for $j \in J_i$.

(b) The jth column of H_{i+1} is equal to the jth column of H_1, for $j \notin J_i$.

Proof. See Theorem 6.3 in [1]. □

In this paper we present new algorithms for computing the *WZ* and *ZW* factorizations using null space of special submatrices of the matrix A.

The structure of this paper is organized as follows. In Section 2, we discuss our proposed algorithm for the *WZ* factorization of a matrix A by using null space of special submatrices of A. In Section 3, we propose a new algorithm for computing the WZ and ZW factorizations. In Section 4, we report a numerical experiment. We conclude in Section 5.

2. The *WZ* factorization

The *WZ* factorization is a parallel method for solving dense linear systems of the form

$$Ax = b, \tag{2.1}$$

where A is a square $n \times n$ matrix, and b is an n-vector.

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Definition 2.1. *Let s be a real number and denote by* $\lfloor s \rfloor$ ($\lceil s \rceil$)*, the greatest (least) integer less (greater) than or equal to s.*

Definition 2.2. *We say that a matrix A is factorized in the form WZ if*

$$A = WZ, \tag{2.2}$$

where the matrices W and Z have the following structures:

$$W = \begin{pmatrix} * & 0 & \cdots & 0 & * \\ * & * & 0 & * & * \\ * & * & * & * & * \\ * & * & 0 & * & * \\ * & 0 & \cdots & 0 & * \end{pmatrix}, Z = \begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & 0 \\ \vdots & 0 & * & 0 & \vdots \\ 0 & * & * & * & 0 \\ * & * & * & * & * \end{pmatrix} \tag{2.3}$$

where stars stand for possible nonzero entries.

The matrices W and Z have two zero opposite quadrants. Then, we refer to W and Z as the interlocking quadrant factors of A. The factorization is unique if W has $1's$ on the main diagonal and $0's$ on the cross diagonal entries(see [17]).

Now, we give a characterization for the existence of the WZ factorization of A.

Theorem 2.1. *Let* $A \in \mathbb{R}^{n\times n}$ *be a nonsingular matrix. A has quadrant interlocking factorization QIF, A=WZ, if and only if for every k,* $1 \le k \le s$*, where* $s = \lfloor n/2 \rfloor$ *if n is even and* $s = \lceil n/2 \rceil$ *if n is odd, the* $2k \times 2k$ *submatrix*

$$\Delta_k = \begin{pmatrix} a_{1,1} & \cdots & a_{1,k} & a_{1,n-k+1} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{k,1} & \cdots & a_{k,k} & a_{k,n-k+1} & \cdots & a_{k,n} \\ a_{n-k+1,1} & \cdots & a_{n-k+1,k} & a_{n-k+1,n-k+1} & \cdots & a_{n-k+1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{n,1} & \cdots & a_{n,k} & a_{n,n-k+1} & \cdots & a_{n,n} \end{pmatrix} \tag{2.4}$$

of A is invertible. Moreover, the factorization is unique.

Proof. See Theorem 2 in [17]. □

If $A \in \mathbb{R}^{n\times n}$ is nonsingular, then the WZ factorization with pivoting can always be carried out. Whenever Δ_k is nonsingular, it is always possible to interchange the rows $k \le i \le (n-k+1)$. These row interchanges can be viewed in a matrix form as premultiplication by a permutation matrix. Thus, we have the following result.

Theorem 2.2. *If* $A \in \mathbb{R}^{n\times n}$ *is nonsingular, then the with pivoting WZ factorization can always be carried out, that is, a row permutation matrix P and the factors W and Z exist so that,* $PA = WZ$.

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Proof. See [17]. □

Let $A \in \mathbb{R}^{n\times n}$ and there exists a WZ factorization without pivoting of A. Let, n be an even number. Here, we present a new algorithm for computing the WZ factorization of A using null space of the sequence submatrices

$$\Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_{n/2}. \tag{2.5}$$

For $k = 1, \cdots, s$, where $s = \dfrac{n}{2}$ consider Δ_k defined by (2.4). Let, the rows of H_k generate the null space of Δ_k expect the kth and the $(k{+}1)$th rows. Let $e_j \in \mathbb{R}^{2k}$ be the jth unit vector, then we have,

$$e_i^T \Delta_k {H_k}^T = 0,\ i \neq k, k+1, \tag{2.6}$$

and

$$(e_k^T, e_{k+1}^T)\Delta_k {H_k}^T \neq 0. \tag{2.7}$$

Therefore, there exist $1 \leq j_1, j_2 \leq 2k$ such that,

$$\alpha_1 = e_{j_1}^T H_k \Delta_k^T e_k \neq 0, \alpha_2 = e_{j_2}^T H_k \Delta_k^T e_{k+1} \neq 0. \tag{2.8}$$

Let $\mathcal{T}_k = (t_1, \cdots, t_{2k}) = {H_k}^T e_{j_1}/\alpha_1 \in \mathbb{R}^{2k}$ and $\mathcal{Y}_k = (y_1, \cdots, y_{2k}) = {H_k}^T e_{j_2}/\alpha_2 \in \mathbb{R}^{2k}$. Then, we have

$$\Delta_k \mathcal{T}_k = (\underbrace{0...0}_{k-1}, 1, \underbrace{0, ..., 0}_{k})^T,\ \Delta_k \mathcal{Y}_k = (\underbrace{0...0}_{k}, 1, \underbrace{0, ..., 0}_{k-1})^T. \tag{2.9}$$

Now, let

$$\bar{z}_k = (t_1, \cdots, t_k, \underbrace{0, \cdots, 0}_{n-2k}, t_{k+1}, \cdots, t_{2k})^T, \tag{2.10}$$

and

$$\bar{z}_{n-k+1} = (y_1, \cdots, y_k, \underbrace{0, \cdots, 0}_{n-2k}, y_{k+1}, \cdots, y_{2k})^T, \tag{2.11}$$

then, we have

$$w_k = A\bar{z}_k = (\underbrace{0, \cdots, 0}_{k-1}, 1, w_{k+1,k}, \cdots, w_{n-k,k}, \underbrace{0, \cdots, 0}_{k})^T \tag{2.12}$$

and

$$w_{n-k+1} = A\bar{z}_{n-k+1} = (\underbrace{0, \cdots, 0}_{k}, w_{k+1,n-k+1}, \cdots, w_{n-k,n-k+1}, 1, \underbrace{0, \cdots, 0}_{k-1})^T. \tag{2.13}$$

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$$\tilde{Z} = (\bar{z}_1, \cdots, \bar{z}_n),\ W = (w_1, \cdots, w_n), \tag{2.14}$$

then, we have,

$$A\bar{Z} = W \Rightarrow A = WZ,\ Z = \bar{Z}^{-1}.$$

Here, we are ready to present the WZ algorithm. Without loss of generality we assume that A is an even order matrix.

Algorithm 1. *WZ* algorithm

(1) Let $A^{(0)} = A$, $k = 1$, $s = n/2$.

(2) Compute P_k, $A^{(k)} = P_k A^{(k-1)}$ where, P_k is a permutation matrix and Δ_k is nonsingular.

(3) Compute H_k^k, so that the rows of H_k^k present the null space of the rows of Δ_k except the kth and $(k+1)$th rows.

(4) Determine $1 \leq j_1, j_2 \leq 2k$ such that,

$$\alpha_1 = e_{j_1}^T H_k \Lambda_k^T e_k \neq 0, \alpha_2 = e_{j_2}^T H_k \Lambda_k^T e_{k+1} \neq 0. \tag{2.15}$$

(5) Compute,

$$\mathcal{T}_k = (t_1, \cdots, t_{2k}) = {H_k}^T e_{j_1}/\alpha_1 \text{ and } \mathcal{Y}_k = (y_1, \cdots, y_{2k}) = {H_k}^T e_{j_2}/\alpha_2.$$

(6) Compute,

$$\bar{z}_k = (t_1, \cdots, t_k, \underbrace{0, \cdots, 0}_{n-2k}, t_{k+1}, \cdots, t_{2k})^T, \tag{2.16}$$

and

$$\bar{z}_{n-k+1} = (y_1, \cdots, y_k, \underbrace{0, \cdots, 0}_{n-2k}, y_{k+1}, \cdots, y_{2k})^T, \tag{2.17}$$

(7) **If** $k < s$ **then** k=k+1 and **go to** (2).

(8) Compute

$$PA = WZ,$$

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where, $P = P_s \cdots P_1,\ \bar{Z} = (\bar{z}_1, \cdots, \bar{z}_n),\ W = PA\bar{Z}$ and $Z = \bar{Z}^{-1}$.

(9) **Stop**.

The integer WZ factorization of an integer matrix, can be calculated as the real case if it exists. Here, we present the conditions for existence of the integer WZ factorization of an integer matrix.

Definition 2.3. *$A \in \mathbb{Z}^{n\times n}$ is a unimodular matrix if and only if $|det(A)| = 1$.*

If A is unimodular, then A^{-1} is also unimodular.

Definition 2.4. *We say that a matrix A is factorized in an integer WZ form if*

$$A = WZ, \tag{2.18}$$

where the matrices W and Z are matrices with integer entries defined by (2.3).

According to Theorem 2.1, we have the following result.

Theorem 2.3. *Let $A \in \mathbb{Z}^{n\times n}$ and the submatrices Δ_k defined by (2.4) be unimodular, then A has an integer WZ factorization.*

For computing an integer WZ factorization (if there exits), in the kth step H_k generates the integer null space of Δ_k expect the kth and the $(k+1)$th rows. Furthermore, in (2.8) we choose two integer vectors j_1 and j_2 such that

$$\alpha_1 = e_{j_1}^T H_k \Delta_k^T e_k = gcd(H_k \Delta_k^T e_k), \alpha_2 = e_{j_2}^T H_k \Delta_k^T e_{k+1} = gcd(H_k \Delta_k^T e_{k+1}), \tag{2.19}$$

where, $gcd(x)$ is the greatest common divisor of entries of x.

Definition 2.5. *A matrix $A \in \mathbb{Z}^{n\times n}$ is called totally unimodular if each square submatrix of A has determinant equal to 0, $+1$, or -1. In particular, each entry of a totally unimodular matrix is 0, $+1$, or -1.*

Corollary 2.1. *Every totally unimodular symmetric positive definite matrix has an integer WZ factorization.*

3. The ZW factorization

Definition 3.1. *We say that a matrix A is factorized in the form ZW if*

$$A = ZW, \tag{3.1}$$

where the matrices W and Z are defined as (2.3)

where the empty bullets stand for zero and the other bullets stand for possible nonzero entries.

The factorization is unique if Z has $1's$ on the main diagonal and $0's$ on the

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cross diagonal.

Without loss of generality, suppose that n be an even number and $s = \frac{n}{2}$. Here, we present a new algorithm for computing the ZW factorization of A using null space of the sequence of submatrices

$$\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_{n/2} \tag{3.2}$$

where

$$\Lambda_k = \begin{pmatrix} a_{s-k+1,s-k+1} & \cdots & a_{s-k+1,s+k} \\ \vdots & \cdots \quad \vdots & \\ a_{s+k,s-k+1} & \cdots & a_{s+k,s+k} \end{pmatrix}_{2k,2k} . \tag{3.3}$$

Theorem 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. A has a ZW factorization, if and only if for every k, $1 \leq k \leq n/2$, the submatrix Λ_k defined by (3.3) be invertible.*

Proof. The proof follows the lines of the proof for Theorem 2 in [17] replacing Δ_i by Λ_i. □

Let Λ_k be nonsingular, for $k = 1, \cdots, n/2$. Let, the rows of H_k generates the null space of Λ_k expect the first and the last rows. Let $e_i \in \mathbb{R}^{2k}$ be the ith unit vector (i.e. the ith element is 1, otherwise 0). Then, we have,

$$e_i^T \Lambda_k {H_k}^T = 0,\ i \neq 1, 2k,\ (e_1^T, e_{2k}^T)\Lambda_k {H_k}^T \neq 0, \tag{3.4}$$

then there exists $1 \leq j_1, j_2 \leq 2k$ such that,

$$\alpha_1 = e_{j_1}^T H_k \Lambda_k^T e_1 \neq 0, \alpha_2 = e_{j_2}^T H_k \Lambda_k^T e_{2k} \neq 0. \tag{3.5}$$

Let $\mathcal{T}_k = (t_1, \cdots, t_{2k}) = {H_k}^T e_{j_1}/\alpha_1$ and $\mathcal{Y}_k = (y_1, \cdots, y_{2k}) = {H_k}^T e_{j_2}/\alpha_2$. Then, we have

$$\Lambda_k t = (1, \underbrace{0...0}_{2k-1})^T,\ \Lambda_k y = (\underbrace{0...0}_{2k-1}, 1)^T. \tag{3.6}$$

Now, let

$$\bar{w}_{\frac{n}{2}-k+1} = (\underbrace{0, \cdots, 0}_{(n-2k)/2}, t_1, \cdots, t_{2k}, \underbrace{0, \cdots, 0}_{(n-2k)/2})^T, \tag{3.7}$$

and

$$\bar{w}_{\frac{n}{2}+k} = (\underbrace{0, \cdots, 0}_{(n-2k)/2}, y_1, \cdots, y_{2k}, \underbrace{0, \cdots, 0}_{(n-2k)/2})^T, \tag{3.8}$$

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then, we have

$$z_{\frac{n}{2}-k+1} = A\bar{w}_{\frac{n}{2}-k+1} = \\ (z_{1,\frac{n}{2}-k+1}, \cdots, z_{\frac{(n-2k)}{2},\frac{n}{2}-k+1}, 1, \underbrace{0, \cdots, 0}_{2k-1}, z_{\frac{(n+2k)}{2}+1,\frac{n}{2}-k+1}, \cdots, z_{n,\frac{n}{2}-k+1})^T \tag{3.9}$$

and

$$z_{\frac{n}{2}+k} = A\bar{w}_{\frac{n}{2}+k} = \\ (z_{1,\frac{n}{2}+k}, \cdots, z_{\frac{(n-2k)}{2},\frac{n}{2}+k}, \underbrace{0, \cdots, 0}_{2k-1}, 1, z_{\frac{(n+2k)}{2}+1,\frac{n}{2}+k}, \cdots, z_{n,\frac{n}{2}+k})^T. \tag{3.10}$$

$$\bar{W} = (\bar{w}_1, \cdots, \bar{w}_n),\ Z = (z_1, \cdots, z_n),$$

then, we have,

$$A\bar{W} = Z \Rightarrow A = ZW,\ W = \bar{W}^{-1}$$

Here, we are ready to present the ZW algorithm. Without loss of generality we assume that A is an even order matrix.

Algorithm 2. *ZW* algorithm

(1) Let $A^{(0)} = A$, $k = 1$, $s = n/2$.

(2) Compute P_k, $A^{(k)} = P_k A^{(k-1)}$ where, P_k is a permutation matrix and Λ_k is nonsingular.

(3) Let the rows of H_k generate the null space of Λ_k expect the first and the last rows.

(4) Determine $1 \leq j_1, j_2 \leq 2k$ such that,

$$\alpha_1 = e_{j_1}^T H_k \Lambda_k^T e_1 \neq 0, \alpha_2 = e_{j_2}^T H_k \Lambda_k^T e_{2k} \neq 0. \tag{3.11}$$

(5) Compute,

$$\mathcal{T}_k = (t_1, \cdots, t_{2k}) = {H_k}^T e_{j_1}/\alpha_1 \text{ and } \mathcal{Y}_k = (y_1, \cdots, y_{2k}) = {H_k}^T e_{j_2}/\alpha_2.$$

(6) Compute,

$$\bar{w}_{\frac{n}{2}-k+1} = (\underbrace{0, \cdots, 0}_{(n-2k)/2}, t_1, \cdots, t_{2k}, \underbrace{0, \cdots, 0}_{(n-2k)/2})^T, \tag{3.12}$$

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and

$$\bar{w}_{\frac{n}{2}+k} = (\underbrace{0,\cdots,0}_{(n-2k)/2}, y_1,\cdots,y_{2k},\underbrace{0,\cdots,0}_{(n-2k)/2})^T, \tag{3.13}$$

(7) **If** $k < s$ **then** k=k+1 and **go to** (2).

(8) Compute

$$PA = ZW,$$

where, $P = P_s \cdots P_1,\ \bar{W} = (\bar{w}_1,\cdots,\bar{w}_n),\ Z = PA\bar{W}$ and $W = \bar{W}^{-1}$.

(9) **Stop**.

We can also calculate the integer ZW factorization of an integer matrix A. The existence conditions are the same as Theorem 2.3 by replacing Δ by Λ.

Theorem 3.2. *Let $A \in \mathbb{Z}^{n\times n}$ and the submatrices Λ_k be unimodular, then A has an integer ZW factorization.*

For computing an integer ZW factorization (if there exits), in the kth step H_k generates the integer null space of Λ_k expect the first and the last rows. Furthermore, in (3.5) we choose two integer vectors j_1 and j_2 such that

$$\alpha_1 = e_{j_1}^T H_k \Lambda_k^T e_1 = gcd(H_k \Lambda_k^T e_1), \alpha_2 = e_{j_2}^T H_k \Lambda_k^T e_{2k} = gcd(H_k \Lambda_k^T e_{2k}). \tag{3.14}$$

Corollary 3.1. *Every totally unimodular symmetric positive definite matrix has an integer ZW factorization.*

4. Examples

In this section, we present some numerical illustrations of our proposed algorithms to compute the WZ and ZW factorizations of real and integer matrices.

Example 4.1. Consider the following matrix,

$$A = \begin{pmatrix} 5 & 4 & 1 & 1 \\ 4 & 5 & 1 & 1 \\ 1 & 1 & 4 & 2 \\ 1 & 1 & 2 & 4 \end{pmatrix}.$$

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Upon an application of Algorithm 1 for computing the WZ factorization, we obtain the following results:

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.7895 & 1 & 0 & 0.0526 \\ 0.1053 & 0 & 1 & 0.4737 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 5 & 4 & 1 & 1 \\ 0 & 1.7895 & 0.1053 & 0 \\ 0 & 0.1053 & 2.9474 & 0 \\ 1 & 1 & 2 & 4 \end{pmatrix}.$$

Example 4.2. Consider the following matrix

$$A = \begin{pmatrix} 1 & 3 & 1.5 & 2 & 2.5 & 2.5 \\ 3 & 3 & 3.5 & 2.5 & 3 & 2.5 \\ 1.5 & 3.5 & 1 & 2.5 & 2 & 2.5 \\ 2 & 2.5 & 2.5 & 4 & 1.5 & 3 \\ 2.5 & 3 & 2 & 1.5 & 2 & 2.5 \\ 2.5 & 2.5 & 2.5 & 3 & 2.5 & 1 \end{pmatrix}.$$

By applying Algorithm 2 for computing the ZW factorization we have

$$Z = \begin{pmatrix} 1 & 0.3219 & 1.5 & 0.5 & 0.6452 & -0.8439 \\ 0 & 1 & 3.5 & 0.6250 & 1.6613 & 0 \\ 0 & 0 & 1 & 0.6250 & 0 & 0 \\ 0 & 0 & 2.5 & 1 & 0 & 0 \\ 0 & 0.7055 & 2 & 0.3750 & 1 & 0 \\ -1.5774 & 0.3425 & 2.5000 & 0.7500 & 0.7419 & 1.0000 \end{pmatrix},$$

and

$$W = \begin{pmatrix} -0.4855 & 0 & 0 & 0 & 0 & 0 \\ 6.2826 & 8.1111 & 0 & 0 & 0 & 8.5331 \\ -0.4444 & -3.4444 & 1 & 0 & -1.8889 & -1.1111 \\ 3.1111 & 11.1111 & 0 & 4 & 6.2222 & 5.7778 \\ -2.2100 & 0 & 0 & 0 & 3.4444 & -3.4644 \\ 0 & 0 & 0 & 0 & 0 & -0.9075 \end{pmatrix}.$$

Example 4.3. Consider the following integer real matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 & -1 & -1 \\ 0 & 2 & 0 & 3 & 1 & 1 \\ -1 & 0 & 5 & -1 & 7 & 2 \\ 1 & 3 & -1 & 8 & 2 & 1 \\ -1 & 1 & 7 & 2 & 15 & 4 \\ -1 & 1 & 2 & 1 & 4 & 2 \end{pmatrix}.$$

Upon an application of Algorithm 1 for computing the integer WZ factorization, we obtain the following results:

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$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & -1 & 2 \\ -1 & -2 & 0 & 0 & 1 & 3 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; Z = \begin{pmatrix} 1 & 0 & -1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 3 & 1 \end{pmatrix}.$$

EXAMPLE 4.4. Consider the following matrix

$$A = \begin{pmatrix} 5 & 4 & -1 & 1 & -3 & -2 \\ 4 & 7 & -1 & 1 & -4 & 0 \\ -1 & -1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & -1 & 0 \\ -3 & -4 & 1 & -1 & 3 & 1 \\ -2 & 0 & 1 & 0 & 1 & 3 \end{pmatrix}.$$

By applying Algorithm 2 for computing the integer ZW factorization we have

$$Z = \begin{pmatrix} 1 & 0 & -1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}, \; W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & -1 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

5. Conclusion

Parallel implicit matrix elimination schemes for the solution of linear systems were introduced by Evans. In this paper we showed how to compute the real (integer) WZ and ZW factorizations by using the null space generators of particular submatrices of a given matrix A.

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