

## A Functional Characterization of the Hurewicz Property

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**ABSTRACT.** For a Tychonoff space  $X$ , we denote by  $C_p(X)$  the space of all real-valued continuous functions on  $X$  with the topology of point-wise convergence. We study a functional characterization of the covering property of Hurewicz.

**Keywords:**  $U_{fin}(\mathcal{O}, \Omega)$ , Hurewicz property, Selection principles,  $C_p$  theory,  $U_{fin}(\mathcal{O}, \Gamma)$ .

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### 1. INTRODUCTION

Many topological properties are defined or characterized in terms of the following classical selection principles. Let  $\mathcal{A}$  and  $\mathcal{B}$  be sets consisting of families of subsets of an infinite set  $X$ . Then:

$S_1(\mathcal{A}, \mathcal{B})$  is the selection hypothesis: for each sequence  $(A_n : n \in \mathbb{N})$  of elements of  $\mathcal{A}$  there is a sequence  $\{b_n\}_{n \in \mathbb{N}}$  such that for each  $n$ ,  $b_n \in A_n$ , and  $\{b_n : n \in \mathbb{N}\}$  is an element of  $\mathcal{B}$ .

$S_{fin}(\mathcal{A}, \mathcal{B})$  is the selection hypothesis: for each sequence  $(A_n : n \in \mathbb{N})$  of elements of  $\mathcal{A}$  there is a sequence  $\{B_n\}_{n \in \mathbb{N}}$  of finite sets such that for each  $n$ ,  $B_n \subseteq A_n$ , and  $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$ .

$U_{fin}(\mathcal{A}, \mathcal{B})$  is the selection hypothesis: whenever  $\mathcal{U}_1, \mathcal{U}_2, \dots \in \mathcal{A}$  and none contains a finite subcover, there are finite sets  $\mathcal{F}_n \subseteq \mathcal{U}_n$ ,  $n \in \mathbb{N}$ , such that  $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathcal{B}$ .

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Many equivalences hold among these properties, and the surviving ones appear in the following Scheepers Diagram (Fig. 1) (where an arrow denotes implication) [11].

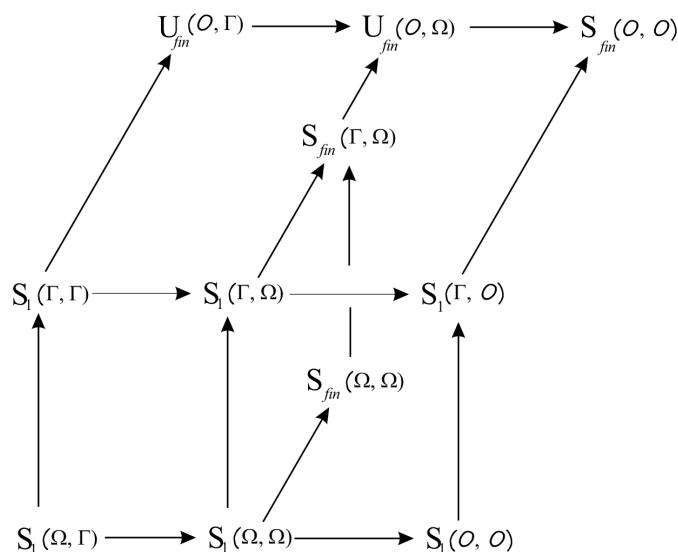


Fig. 1. The Scheepers Diagram for Lindelöf space.

The papers [11, 15, 32, 35, 36, 37, 38] have initiated the simultaneous consideration of these properties in the case where  $\mathcal{A}$  and  $\mathcal{B}$  are important families of open covers of a topological space  $X$ .

In papers [1-8, 12-32] (and many others) were investigated the applications of selection principles in the study of the properties of function spaces. In particular, the properties of the space  $C_p(X)$  were investigated.

In [9] (see also [10]), Hurewicz introduced a covering property of a space  $X$ , nowadays called the *Hurewicz property* in this way: for each sequence  $(U_n : n \in \mathbb{N})$  of open covers of  $X$  there is a sequence  $(V_n : n \in \mathbb{N})$  such that for each  $n \in \mathbb{N}$ ,  $V_n$  is a finite subset of  $U_n$  and each  $x \in X$  belongs to  $\cup V_n$  for all but finitely many  $n$  (i.e.,  $X$  satisfies  $U_{fin}(\mathcal{O}, \Gamma)$ ).

In this paper we continue to study different selectors for sequences of dense sets of  $C_p(X)$  and we study a functional characterization of the covering property of Hurewicz.

## 2. MAIN DEFINITIONS AND NOTATION

Throughout this paper, all spaces are assumed to be Tychonoff. The set of positive integers is denoted by  $\mathbb{N}$ . Let  $\mathbb{R}$  be the real line and  $\mathbb{Q}$  be the rational numbers. For a space  $X$ , we denote by  $C_p(X)$  the space of all real-valued continuous functions on  $X$  with the topology of pointwise convergence.

Basic open sets of  $C_p(X)$  are of the form

$[x_1, \dots, x_k, U_1, \dots, U_k] = \{f \in C(X) : f(x_i) \in U_i, i = 1, \dots, k\}$ , where each  $x_i \in X$  and each  $U_i$  is a non-empty open subset of  $\mathbb{R}$ . Sometimes we will write the basic neighborhood of the point  $f$  as  $\langle f, A, \epsilon \rangle$  where  $\langle f, A, \epsilon \rangle := \{g \in C(X) : |f(x) - g(x)| < \epsilon \forall x \in A\}$ ,  $A$  is a finite subset of  $X$  and  $\epsilon > 0$ .

The symbol  $\mathbf{0}$  denote the constantly zero function in  $C_p(X)$ . Because  $C_p(X)$  is homogeneous we can work with  $\mathbf{0}$  to study local properties of  $C_p(X)$ .

If  $X$  is a space and  $A \subseteq X$ , then the sequential closure of  $A$ , denoted by  $[A]_{seq}$ , is the set of all limits of sequences from  $A$ . A set  $D \subseteq X$  is said to be sequentially dense if  $X = [D]_{seq}$ . A space  $X$  is called sequentially separable if it has a countable sequentially dense set.

We recall that a subset of  $X$  that is the complete preimage of zero for a certain function from  $C(X)$  is called a zero-set. A subset  $O \subseteq X$  is called a cozero-set of  $X$  if  $X \setminus O$  is a zero-set.

In this paper, by a cover we mean a nontrivial one, that is,  $\mathcal{U}$  is a cover of  $X$  if  $X = \bigcup \mathcal{U}$  and  $X \notin \mathcal{U}$ .

An open cover  $\mathcal{U}$  of a space  $X$  is:

- an  $\omega$ -cover if every finite subset of  $X$  is contained in a member of  $\mathcal{U}$ .
- a  $\gamma$ -cover if it is infinite and each  $x \in X$  belongs to all but finitely many elements of  $\mathcal{U}$ . Note that every  $\gamma$ -cover contains a countably  $\gamma$ -cover.
- a  $\gamma_F$ -shrinkable cover  $\mathcal{U}$  if it is a  $\gamma$ -cover  $\mathcal{U}$  of  $X$  by co-zero sets and there exists a  $\gamma$ -cover  $\{F(U) : U \in \mathcal{U}\}$  of  $X$  by zero-sets with  $F(U) \subset U$  for some  $U \in \mathcal{U}$ .

For a topological space  $X$  we denote:

- $\mathcal{O}$  — the family of all open covers of  $X$ ;
- $\Gamma$  — the family of all countable open  $\gamma$ -covers of  $X$ ;
- $\Omega$  — the family of all open  $\omega$ -covers of  $X$ ;
- $\Gamma_F$  — the family of all  $\gamma_F$ -shrinkable covers of  $X$ .

For a topological space  $C_p(X)$  we denote:

- $\mathcal{D}$  — the family of all dense subsets of  $C_p(X)$ ;
- $\mathcal{S}$  — the family of all sequentially dense subsets of  $C_p(X)$ .

In the case of  $U_{fin}$  note that for any class of covers  $\mathcal{B}$  of Lindelöf space  $X$ ,  $U_{fin}(\mathcal{O}, \mathcal{B})$  is equivalent to  $U_{fin}(\Gamma, \mathcal{B})$  because given an open cover  $\{U_n : n \in \mathbb{N}\}$  we may replace it by  $\{\bigcup_{i < n} U_i : n \in \mathbb{N}\}$ , which is a  $\gamma$ -cover (unless it contains a finite subcover) of  $X$ .

Recall that the  $i$ -weight  $iw(X)$  of a space  $X$  is the smallest infinite cardinal number  $\tau$  such that  $X$  can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than  $\tau$ .

**Theorem 2.1.** (Noble [18]) *A space  $C_p(X)$  is separable iff  $iw(X) = \aleph_0$ .*

Let  $X$  be a topological space, and  $x \in X$ . A subset  $A$  of  $X$  *converges* to  $x$ ,  $x = \lim A$ , if  $A$  is infinite,  $x \notin A$ , and for each neighborhood  $U$  of  $x$ ,  $A \setminus U$  is finite. Consider the following collection:

- $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}$ ;
- $\Gamma_x = \{A \subseteq X : x = \lim A\}$ .

Note that if  $A \in \Gamma_x$ , then there exists  $\{a_n\} \subset A$  converging to  $x$ . So, simply  $\Gamma_x$  may be the set of non-trivial convergent sequences to  $x$ .

We write  $\Pi(\mathcal{A}_x, \mathcal{B}_x)$  without specifying  $x$ , we mean  $(\forall x)\Pi(\mathcal{A}_x, \mathcal{B}_x)$ .

So we have three types of topological properties described through the selection principles:

- local properties of the form  $S_*(\Phi_x, \Psi_x)$ ;
- global properties of the form  $S_*(\Phi, \Psi)$ ;
- semi-local properties of the form  $S_*(\Phi, \Psi_x)$ .

### 3. $U_{fin}(\mathcal{O}, \Omega)$

For a function space  $C_p(X)$ , we represent the following selection principle

$F_{fin}(\mathcal{S}, \mathcal{D})$ : whenever  $\mathcal{S}_1, \mathcal{S}_2, \dots \in \mathcal{S}$  there are finite sets  $\mathcal{F}_n \subseteq \mathcal{S}_n$ ,  $n \in \mathbb{N}$ , such that for each  $f \in C_p(X)$  and a base neighborhood  $\langle f, K, \epsilon \rangle$  of  $f$  where  $\epsilon > 0$  and  $K = \{x_1, \dots, x_k\}$  is a finite subset of  $X$ , there is  $n' \in \mathbb{N}$  such that for each  $j \in \{1, \dots, k\}$  there is  $g \in \mathcal{F}_{n'}$  such that  $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$ .

It is clear that the condition of the selection principle  $F_{fin}(\mathcal{S}, \mathcal{D})$  can be written more briefly: whenever  $\mathcal{S}_1, \mathcal{S}_2, \dots \in \mathcal{S}$  there are finite sets  $\mathcal{F}_n \subseteq \mathcal{S}_n$ ,  $n \in \mathbb{N}$ , such that for each  $f \in C_p(X)$ ,  $\epsilon > 0$  and  $K \in [X]^{<\omega}$ , there is  $n' \in \mathbb{N}$  such that  $\min_{h \in \mathcal{F}_{n'}} \{|f(x) - h(x)|\} < \epsilon$  for each  $x \in K$ .

Similarly,  $F_{fin}(\Gamma_0, \Omega_0)$ : whenever  $\mathcal{S}_1, \mathcal{S}_2, \dots \in \Gamma_0$  there are finite sets  $\mathcal{F}_n \subseteq \mathcal{S}_n$ ,  $n \in \mathbb{N}$ , such that for  $\epsilon > 0$  and  $K \in [X]^{<\omega}$ , there is  $n' \in \mathbb{N}$  such that  $\min_{h \in \mathcal{F}_{n'}} \{|h(x)|\} < \epsilon$  for each  $x \in K$ .

**Theorem 3.1.** *For a space  $X$ , the following statements are equivalent:*

- (1)  $C_p(X)$  satisfies  $F_{fin}(\Gamma_0, \Omega_0)$ ;
- (2)  $X$  satisfies  $U_{fin}(\Gamma_F, \Omega)$ .

*Proof.* (1)  $\Rightarrow$  (2). Let  $\{\mathcal{U}_i\}_{i \in \mathbb{N}} \subset \Gamma_F$  and let  $\mathcal{U}_i = \{U_i^m\}$  for each  $i \in \mathbb{N}$ . We consider  $\mathcal{K}_i = \{f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = 0 \text{ and } f_i^m \upharpoonright (X \setminus U_i^m) = 1 \text{ for } m \in \mathbb{N}\}$ .

Since  $\mathcal{F}_i = \{F(U_i^m) : m \in \mathbb{N}\}$  is a  $\gamma$ -cover of  $X$ , we have that  $\mathcal{K}_i$  converges to  $\mathbf{0}$  for each  $i \in \mathbb{N}$ . Since  $C_p(X)$  satisfies  $F_{fin}(\Gamma_0, \Omega_0)$ , there are finite sets  $F_i = \{f_i^{m_1}, \dots, f_i^{m_{s(i)}}\} \subseteq \mathcal{K}_i$  such that for a base neighborhood  $O(f) = \langle f, K, \epsilon \rangle$  of  $f = \mathbf{0}$  where  $\epsilon > 0$  and  $K = \{x_1, \dots, x_k\}$  is a finite subset of  $X$ , there is  $n' \in \mathbb{N}$  such that for each  $j \in \{1, \dots, k\}$  there is  $g \in \mathcal{F}_{n'}$  such that  $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$ . Note that  $\{\bigcup\{U_i^{m_1}, \dots, U_i^{m_{s(i)}}\} : i \in \mathbb{N}\} \in \Omega$ .

(2)  $\Rightarrow$  (1). Let  $X$  satisfies  $U_{fin}(\Gamma_F, \Omega)$  and  $A_i \in \Gamma_0$  for each  $i \in \mathbb{N}$ . Consider  $\mathcal{U}_i = \{U_{i,f} = f^{-1}(-\frac{1}{i}, \frac{1}{i}) : f \in A_i\}$  for each  $i \in \mathbb{N}$ . Without loss of generality we can assume that a set  $U_{i,f} \neq X$  for any  $i \in \mathbb{N}$  and  $f \in A_i$ . Otherwise there is sequence  $\{f_{i_k}\}_{k \in \mathbb{N}}$  such that  $\{f_{i_k}\}_{k \in \mathbb{N}}$  uniform converges to  $\mathbf{0}$  and  $\{f_{i_k} : k \in \mathbb{N}\} \in \Omega_0$ .

Note that  $\mathcal{F}_i = \{F_{i,m}\}_{m \in \mathbb{N}} = \{f_{i,m}^{-1}[-\frac{1}{i+1}, \frac{1}{i+1}] : m \in \mathbb{N}\}$  is a  $\gamma$ -cover of  $X$  and  $F_{i,m} \subset U_{i,m}$  for each  $i, m \in \mathbb{N}$ . It follows that  $\mathcal{U}_i \in \Gamma_F$  for each  $i \in \mathbb{N}$ .

Since  $X$  satisfies  $U_{fin}(\Gamma_F, \Omega)$ , there is a sequence  $\{U_{i,m(1)}, U_{i,m(2)}, \dots, U_{i,m(i)} : i \in \mathbb{N}\}$  such that for each  $i$  and  $k \in \{m(1), \dots, m(i)\}$ ,  $U_{i,m(k)} \in \mathcal{U}_i$ , and

$$\{\bigcup\{U_{i,m(1)}, \dots, U_{i,m(i)}\} : i \in \mathbb{N}\} \in \Omega.$$

Let  $\langle \mathbf{0}, K, \epsilon \rangle$  be a base neighborhood of  $\mathbf{0}$  where  $\epsilon > 0$  and  $K = \{x_1, \dots, x_s\}$  is a finite subset of  $X$ , then there is  $i_1 \in \mathbb{N}$  such that  $\frac{1}{i_1} < \epsilon$  and  $\bigcup_{k=m(1)}^{m(i_1)} U_{i_1,k} \supset K$ . It follows that for each  $j \in \{1, \dots, s\}$  there is  $g \in \{f_{i_1,m(1)}, \dots, f_{i_1,m(i_1)}\}$  such that  $g(x_j) \in (-\epsilon, \epsilon)$ .  $\square$

**Lemma 3.2.** (Lemma 6.5 in [20]) Let  $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$  be a  $\gamma_F$ -shrinkable co-zero cover of a space  $X$ . Then the set  $S = \{f \in C(X) : f \upharpoonright (X \setminus U_n) \equiv 1 \text{ for some } n \in \mathbb{N}\}$  is sequentially dense in  $C_p(X)$ .

**Theorem 3.3.** For a space  $X$  with  $iw(X) = \aleph_0$ , the following statements are equivalent:

- (1)  $C_p(X)$  satisfies  $F_{fin}(\mathcal{S}, \mathcal{D})$ ;
- (2)  $X$  satisfies  $U_{fin}(\Gamma_F, \Omega)$ ;
- (3)  $C_p(X)$  satisfies  $F_{fin}(\Gamma_0, \Omega_0)$ ;
- (4)  $C_p(X)$  satisfies  $F_{fin}(\mathcal{S}, \Omega_0)$ .

*Proof.* (1)  $\Rightarrow$  (2). Let  $\mathcal{U}_i = \{U_i^j : j \in \mathbb{N}\} \in \Gamma_F$  for each  $i \in \mathbb{N}$ . Then, by Lemma 3.2, each  $S_i = \{f \in C(X) : f \upharpoonright (X \setminus U_i^j) \equiv 1 \text{ for some } m \in \mathbb{N}\}$  is sequentially dense in  $C_p(X)$ . Since  $C_p(X)$  satisfies  $F_{fin}(\mathcal{S}, \mathcal{D})$ , there are finite sets  $F_i = \{f_i^{m_1}, \dots, f_i^{m_{s(i)}}\} \subseteq \mathcal{S}_i$  such that for each  $f \in C_p(X)$  and a base neighborhood  $\langle f, K, \epsilon \rangle$  of  $f$  where  $\epsilon > 0$  and  $K = \{x_1, \dots, x_k\}$  is a finite subset of  $X$ , there is  $n' \in \mathbb{N}$  such that for each  $j \in \{1, \dots, k\}$  there is  $g \in F_{n'}$  such that  $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$ . Note that  $\{\bigcup\{U_i^{m_1}, \dots, U_i^{m_{s(i)}}\} : i \in \mathbb{N}\} \in \Omega$ .

(2)  $\Rightarrow$  (3). By Theorem 3.1.

(3)  $\Rightarrow$  (4) is immediate.

(4)  $\Rightarrow$  (1). Suppose that  $C_p(X)$  satisfies  $F_{fin}(\mathcal{S}, \Omega_0)$ .

Let  $D = \{d_n : n \in \mathbb{N}\}$  be a dense subspace of  $C_p(X)$  and  $S_i \in \mathcal{S}$  for each  $i \in \mathbb{N}$ . Given a sequence of sequentially dense subspace of  $C_p(X)$ , enumerate it as  $\{S_{n,m} : n, m \in \mathbb{N}\}$ . For each  $n \in \mathbb{N}$ , pick

$\mathcal{F}_{n,m} = \{d_{n,m,1}, \dots, d_{n,m,k(n,m)}\} \subset S_{n,m}$  so that for a base neighborhood  $\langle d_n, K, \epsilon \rangle$  of  $d_n$  where  $\epsilon > 0$  and  $K = \{x_1, \dots, x_k\}$  is a finite subset of  $X$ , there is  $m' \in \mathbb{N}$  such that for each  $j \in \{1, \dots, k\}$  there is  $g \in \mathcal{F}_{n,m'}$  such that  $g(x_j) \in (d_n(x_j) - \epsilon, d_n(x_j) + \epsilon)$ . It follows that  $C_p(X)$  satisfies  $F_{fin}(\mathcal{S}, \mathcal{D})$ .  $\square$

**Theorem 3.4.** *For a space  $X$  the following statements are equivalent:*

- (1)  $X$  is Lindelöf and  $X$  satisfies  $U_{fin}(\Gamma_F, \Omega)$ ;
- (2)  $X$  satisfies  $U_{fin}(\mathcal{O}, \Omega)$ .

*Proof.* It is proved similarly to the proof of Theorem 4.1.  $\square$

**Theorem 3.5.** *For a separable metrizable space  $X$ , the following statements are equivalent:*

- (1)  $C_p(X)$  satisfies  $F_{fin}(\mathcal{S}, \mathcal{D})$ ;
- (2)  $X$  satisfies  $U_{fin}(\mathcal{O}, \Omega)$ ;
- (3)  $C_p(X)$  satisfies  $F_{fin}(\Gamma_0, \Omega_0)$ ;
- (4)  $C_p(X)$  satisfies  $F_{fin}(\mathcal{S}, \Omega_0)$ .

#### 4. $U_{fin}(\mathcal{O}, \Gamma)$ - HUREWICZ PROPERTY

**Theorem 4.1.** *For a space  $X$  the following statements are equivalent:*

- (1)  $X$  satisfies  $U_{fin}(\Gamma_F, \Gamma)$  and is Lindelöf;
- (2)  $X$  has the Hurewicz property.

*Proof.* (1)  $\Rightarrow$  (2). Let  $(\mathcal{U}_n : n \in \mathbb{N})$  be a sequence of open covers of  $X$ . For every  $n$ ,  $U \in \mathcal{U}_n$  and  $x \in X$  we find co-zero sets  $W_{0,n,U,x}$  and  $W_{2,n,U,x}$ , and, a zero-set  $W_{1,n,U,x}$  such that  $x \in W_{0,n,U,x} \subset W_{1,n,U,x} \subset W_{2,n,U,x} \subset U$ . Since  $X$  is Lindelöf, there is a sequence  $(x_k^n : k \in \mathbb{N})$  such that  $X$  is covered by  $\{W_{0,n,U,x_k^n} : k \in \mathbb{N}\}$ . Look at the cover  $\mathcal{W}_n$  of  $X$  consisting of sets  $W_k^n = \bigcup_{i \leq k} W_{2,n,U,x_i^n}$ ,  $k \in \mathbb{N}$ . Note that  $\mathcal{W}_n \in \Gamma_F$  because  $\bigcup_{i \leq k} W_{1,n,U,x_i^n}$  is a zero-set contained in  $W_k^n$ , and  $\{\bigcup_{i \leq k} W_{1,n,U,x_i^n} : k \in \mathbb{N}\}$  is a  $\gamma$ -cover of  $X$  because even  $\{\bigcup_{i \leq k} W_{0,n,U,x_i^n} : k \in \mathbb{N}\}$  is a  $\gamma$ -cover of  $X$ .

Now use the property  $U_{fin}(\Gamma_F, \Gamma)$  to the sequence  $(\mathcal{W}_n : n \in \mathbb{N})$  together with the fact that  $\mathcal{W}_n$  is a finer cover than  $\mathcal{U}_n$  for all  $n$ .  $\square$

For a function space  $C_p(X)$ , we represent the following selection principle  $F_{fin}(\mathcal{S}, \mathcal{S})$ : whenever  $\mathcal{S}_1, \mathcal{S}_2, \dots \in \mathcal{S}$  there are finite sets  $\mathcal{F}_n \subseteq \mathcal{S}_n$ ,  $n \in \mathbb{N}$ , such that for each  $f \in C_p(X)$  there is  $\{\mathcal{F}_{n_k} : k \in \mathbb{N}\}$  such that for a base neighborhood  $\langle f, K, \epsilon \rangle$  of  $f$  where  $\epsilon > 0$  and  $K = \{x_1, \dots, x_m\}$  is a finite subset of  $X$ , there is  $k' \in \mathbb{N}$  such that for each  $k > k'$  and  $\forall j \in \{1, \dots, m\}$  there is  $g \in \mathcal{F}_{n_k}$  such that  $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$ .

It is clear that the condition of the selection principle  $F_{fin}(\mathcal{S}, \mathcal{S})$  can be written more briefly: whenever  $\mathcal{S}_1, \mathcal{S}_2, \dots \in \mathcal{S}$  there are finite sets  $\mathcal{F}_n \subseteq \mathcal{S}_n$ ,  $n \in \mathbb{N}$ , such that for each  $f \in C_p(X)$ ,  $\epsilon > 0$  and  $K \in [X]^{<\omega}$ , there is  $n' \in \mathbb{N}$  such that for every  $n > n'$   $\min_{h \in \mathcal{F}_n} \{|f(x) - h(x)|\} < \epsilon$  for each  $x \in K$ .

Similarly,  $F_{fin}(\Gamma_0, \Gamma_0)$ : whenever  $\mathcal{S}_1, \mathcal{S}_2, \dots \in \Gamma_0$ , there are finite sets  $\mathcal{F}_n \subseteq \mathcal{S}_n$ ,  $n \in \mathbb{N}$ , such that for  $\epsilon > 0$  and  $K \in [X]^{<\omega}$ , there is  $n' \in \mathbb{N}$  such that for every  $n > n'$   $\min_{h \in \mathcal{F}_n} \{|h(x)|\} < \epsilon$  for each  $x \in K$ .

**Theorem 4.2.** *For a space  $X$ , the following statements are equivalent:*

- (1)  $C_p(X)$  satisfies  $F_{fin}(\Gamma_0, \Gamma_0)$ ;
- (2)  $X$  satisfies  $U_{fin}(\Gamma_F, \Gamma)$ .

*Proof.* (1)  $\Rightarrow$  (2). Let  $\{\mathcal{U}_i\}_{i \in \mathbb{N}} \subset \Gamma_F$ ,  $\mathcal{U}_i = \{U_i^m\}_{m \in \mathbb{N}}$  for each  $i \in \mathbb{N}$ . We consider a subset  $\mathcal{S}_i$  of  $C_p(X)$  where

$$\mathcal{S}_i = \{f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = 0 \text{ and } f_i^m \upharpoonright (X \setminus U_i^m) = 1 \text{ for } m \in \mathbb{N}\}.$$

Since  $\mathcal{F}_i = \{F(U_i^m) : m \in \mathbb{N}\}$  is a  $\gamma$ -cover of  $X$ , we have that  $\mathcal{S}_i$  converges to  $\mathbf{0}$ , i.e.  $\mathcal{S}_i \in \Gamma_0$  for each  $i \in \mathbb{N}$ .

Since  $C(X)$  satisfies  $F_{fin}(\Gamma_0, \Gamma_0)$ , there is a sequence  $\{\mathcal{F}_i\}_{i \in \mathbb{N}} = \{f_i^{m_1}, \dots, f_i^{m_{k(i)}} : i \in \mathbb{N}\}$  such that for each  $i$ ,  $\mathcal{F}_i \subseteq \mathcal{S}_i$ , and for a base neighborhood  $\langle \mathbf{0}, K, \epsilon \rangle$  of  $\mathbf{0}$  where  $\epsilon > 0$  and  $K = \{x_1, \dots, x_k\}$  is a finite subset of  $X$ , there is  $n' \in \mathbb{N}$  such that for each  $n > n'$  and  $j \in \{1, \dots, k\}$  there is  $g \in \mathcal{F}_n$  such that  $g(x_j) \in (-\epsilon, \epsilon)$ .

Consider the sequence  $\{W_i\}_{i \in \mathbb{N}} = \{U_i^{m_1}, \dots, U_i^{m_{k(i)}} : i \in \mathbb{N}\}$ .

- (a).  $W_i \subset \mathcal{U}_i$  for each  $i \in \mathbb{N}$ .
- (b).  $\{\bigcup W_i : i \in \mathbb{N}\}$  is a  $\gamma$ -cover of  $X$ .

Let  $K = \{x_1, \dots, x_s\}$  be a finite subset of  $X$  and  $\langle \mathbf{0}, K, \frac{1}{2} \rangle$  be a base neighborhood of  $\mathbf{0}$ , then there exists  $i_0 \in \mathbb{N}$  such that for each  $i > i_0$  and  $j \in \{1, \dots, s\}$  there is  $g \in \mathcal{F}_i$  such that  $g(x_j) \in (-\frac{1}{2}, \frac{1}{2})$ .

It follows that  $K \subset \bigcup_{j=1}^{k(i)} U_i^{m_j}$  for  $i > i_0$ . We thus get that  $X$  satisfies

$U_{fin}(\Gamma_F, \Gamma)$ .

(2)  $\Rightarrow$  (1). Fix  $\{\mathcal{S}_i : i \in \mathbb{N}\} \subset \Gamma_0$  where  $\mathcal{S}_i = \{f_k^i : k \in \mathbb{N}\}$  for each  $i \in \mathbb{N}$ .

For each  $i, k \in \mathbb{N}$ , we put  $U_{i,k} = \{x \in X : |f_k^i(x)| < \frac{1}{i}\}$ ,  $Z_{i,k} = \{x \in X : |f_k^i(x)| \leq \frac{1}{i+1}\}$ .

Each  $U_{i,k}$  (resp.,  $Z_{i,k}$ ) is a cozero-set (resp., zero-set) in  $X$  with  $Z_{i,k} \subset U_{i,k}$ . Let  $\mathcal{U}_i = \{U_{i,k} : k \in \mathbb{N}\}$  and let  $\mathcal{Z}_i = \{Z_{i,k} : k \in \mathbb{N}\}$ . So without loss of generality, we may assume  $U_{i,k} \neq X$  for each  $i, k \in \mathbb{N}$ . We can easily check that the condition  $f_k^i \rightarrow \mathbf{0}$  ( $k \rightarrow \infty$ ) implies that  $\mathcal{Z}_i$  is a  $\gamma$ -cover of  $X$ .

Since  $X$  satisfies  $U_{fin}(\Gamma_F, \Gamma)$  there is a sequence  $\{\mathcal{F}_i\}_{i \in \mathbb{N}} = (U_{i,k_1}, \dots, U_{i,k_i} : i \in \mathbb{N})$  such that for each  $i$ ,  $\mathcal{F}_i \subset \mathcal{U}_i$ , and  $\{\bigcup \mathcal{F}_i : i \in \mathbb{N}\}$  is an element of  $\Gamma$ .

Let  $K = \{x_1, \dots, x_s\}$  be a finite subset of  $X$ ,  $\epsilon > 0$ , and  $\langle \mathbf{0}, K, \epsilon \rangle$  be a base neighborhood of  $\mathbf{0}$ , then there exists  $i' \in \mathbb{N}$  such that for every  $i > i'$   $K \subset \bigcup \mathcal{F}_i$ . It follow that for every  $i > i'$  and  $j \in \{1, \dots, s\}$  there is  $g \in \mathcal{S}_i$  such that  $g(x_j) \in (-\epsilon, \epsilon)$ . So  $C_p(X)$  satisfies  $F_{fin}(\Gamma_0, \Gamma_0)$ .  $\square$

**Theorem 4.3.** For a Lindelöf space  $X$ , the following statements are equivalent:

- (1)  $C_p(X)$  satisfies  $F_{fin}(\Gamma_0, \Gamma_0)$ ;
- (2)  $X$  has the Hurewicz property.

A space  $X$  has *Velichko property* ( $X \models V$ ), if there exists a condensation (one-to-one continuous mapping)  $f : X \mapsto Y$  from the space  $X$  on a separable metric space  $Y$ , such that  $f(U)$  is an  $F_\sigma$ -set of  $Y$  for any cozero-set  $U$  of  $X$ .

**Theorem 4.4.** (Velichko [40]). Let  $X$  be a space. A space  $C_p(X)$  is sequentially separable iff  $X \models V$ .

**Theorem 4.5.** For a space  $X$  with  $X \models V$ , the following statements are equivalent:

- (1)  $C_p(X)$  satisfies  $F_{fin}(\mathcal{S}, \mathcal{S})$ ;
- (2)  $X$  satisfies  $U_{fin}(\Gamma_F, \Gamma)$ ;
- (3)  $C_p(X)$  satisfies  $F_{fin}(\Gamma_0, \Gamma_0)$ ;
- (4)  $C_p(X)$  satisfies  $F_{fin}(\mathcal{S}, \Gamma_0)$ .

*Proof.* (1)  $\Rightarrow$  (2). Let  $\mathcal{U}_i = \{U_i^j : j \in \mathbb{N}\} \in \Gamma_F$  for each  $i \in \mathbb{N}$ . Then, by Lemma 3.2, each  $S_i = \{f \in C(X) : f \upharpoonright (X \setminus U_i^j) \equiv 1 \text{ for some } m \in \mathbb{N}\}$  is sequentially dense in  $C_p(X)$ .

Since  $C(X)$  satisfies  $U_{fin}(\mathcal{S}, \mathcal{S})$ , there is a sequence  $\{\mathcal{F}_i\} = \{f_i^{m_1}, \dots, f_i^{m_s} : i \in \mathbb{N}\}$  such that for  $f = \mathbf{0}$  there is  $\{F_{i_k} : k \in \mathbb{N}\}$  such that for a base neighborhood  $\langle f, K, \epsilon \rangle$  of  $f$  where  $\epsilon > 0$  and  $K = \{x_1, \dots, x_m\}$  is a finite subset of  $X$ , there is  $k' \in \mathbb{N}$  such that for each  $k > k'$  and  $j \in \{1, \dots, m\}$  there is  $g \in F_{i_k}$  such that  $g(x_j) \in (-\epsilon, \epsilon)$ .

Let  $\epsilon = \frac{1}{2}$  and  $\mathbb{N}' = \mathbb{N} \setminus \{k'\}$ . Consider a sequence  $\{Q_k\}_{k \in \mathbb{N}'} = \{U_{i_k}^{m_1}, \dots, U_{i_k}^{m_s} : k \in \mathbb{N}'\}$  for corresponding to  $\{F_{i_k}\} = \{f_{i_k}^{m_1}, \dots, f_{i_k}^{m_s} : k \in \mathbb{N}'\}$ .

- (a).  $Q_k \subset \mathcal{U}_{i_k}$  for  $k \in \mathbb{N}'$ .
- (b).  $\{\bigcup Q_k : k \in \mathbb{N}'\}$  is a  $\gamma$ -cover of  $X$ . We thus get  $X$  satisfies  $U_{fin}(\Gamma_F, \Gamma)$ .
- (2)  $\Rightarrow$  (3). By Theorem 4.2.
- (3)  $\Rightarrow$  (4) is immediate.

(4)  $\Rightarrow$  (1). For each  $n \in \mathbb{N}$ , let  $S_n$  be a sequentially dense subset of  $C_p(X)$  and let  $\{h_n : n \in \mathbb{N}\}$  be sequentially dense in  $C_p(X)$ . Take a sequence  $\{f_n^m : m \in \mathbb{N}\} \subset S_n$  such that  $f_n^m \mapsto h_n$  ( $m \mapsto \infty$ ). Then  $f_n^m - h_n \mapsto \mathbf{0}$  ( $m \mapsto \infty$ ). Hence, there exist  $\mathcal{F}_n = \{f_n^{m_1}, \dots, f_n^{m_{k(n)}}\} \subset S_n$  such that  $\{\bigcup \{f_n^{m_1} - h_n, \dots, f_n^{m_{k(n)}} - h_n\} : n \in \mathbb{N}\} \in \Gamma_0$ , i.e. for a base neighborhood  $\langle f, K, \epsilon \rangle$  of  $f = \mathbf{0}$  where  $\epsilon > 0$  and  $K = \{x_1, \dots, x_m\}$  is a finite subset of  $X$ , there is  $n' \in \mathbb{N}$  such that for each  $n > n'$  and  $\forall j \in \{1, \dots, m\}$  there is  $g \in \{f_n^{m_1} - h_n, \dots, f_n^{m_{k(n)}} - h_n\}$  such that  $g(x_j) \in (-\epsilon, \epsilon)$ .

Let  $h \in C_p(X)$  and take a sequence  $\{h_{n_j} : j \in \mathbb{N}\} \subset \{h_n : n \in \mathbb{N}\}$  converging to  $h$ . Let  $K = \{x_1, \dots, x_m\}$  be a finite subset of  $X$  and  $\epsilon > 0$ . Consider a base neighborhood  $\langle h, K, \epsilon \rangle$  of  $h$ . Then there is  $j' \in \mathbb{N}$  such that  $h_{n_j} \in \langle h, K, \frac{\epsilon}{2} \rangle$  and  $\forall s \in \{1, \dots, m\}$  there is  $g \in \{f_{n_j}^{m_1} - h_{n_j}, \dots, f_{n_j}^{m_{k(n_j)}} - h_{n_j}\}$  such that  $g(x_s) \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$  for  $j > j'$ . It follows that for each  $s \in \{1, \dots, m\}$  there is  $l(j) \in \mathbb{N}$  such that  $((f_{n_j}^{m_{l(j)}} - h_{n_j}) + (h_{n_j} - h))(x_s) \in (-\epsilon, \epsilon)$  for  $j > j'$ . Hence  $C_p(X)$  satisfies  $F_{fin}(\mathcal{S}, \mathcal{S})$ .  $\square$

**Theorem 4.6.** For a separable metrizable space  $X$ , the following statements are equivalent:



- (1)  $C_p(X)$  satisfies  $F_{fin}(\mathcal{S}, \mathcal{S})$ ;
- (2)  $X$  satisfies  $U_{fin}(\mathcal{O}, \Gamma)$  [Hurewicz property];
- (3)  $C_p(X)$  satisfies  $F_{fin}(\Gamma_0, \Gamma_0)$ ;
- (4)  $C_p(X)$  satisfies  $F_{fin}(\mathcal{S}, \Gamma_0)$ .

Recall that a space  $X$  is said to be Rothberger [27] (or, [17]) if for every sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of open covers of  $X$ , there is a sequence  $(V_n : n \in \mathbb{N})$  such that for each  $n$ ,  $V_n \in \mathcal{U}_n$ , and  $\{V_n : n \in \mathbb{N}\}$  is an open cover of  $X$ .

A space  $X$  is said to be Menger [9] (or, [30]) if for every sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of open covers of  $X$ , there are finite subfamilies  $\mathcal{V}_n \subset \mathcal{U}_n$  such that  $\bigcup \{\mathcal{V}_n : n \in \mathbb{N}\}$  is a cover of  $X$ .

Every  $\sigma$ -compact space is Menger, and a Menger space is Lindelöf.

In [21], we gave the functional characterizations of Rothberger and Menger properties.

Recall that if  $C_p(X)$  and  $C_p(Y)$  are homeomorphic (linearly homeomorphic, uniformly homeomorphic), we say that the spaces  $X$  and  $Y$  are  $t$ -equivalent ( $l$ -equivalent,  $u$ -equivalent). The properties preserved by  $t$ -equivalence ( $l$ -equivalence,  $u$ -equivalence) we call  $t$ -invariant ( $l$ -invariant,  $u$ -invariant) [2].

**Question 1.** Is the Hurewicz (Rothberger, Menger) property  $t$ -invariant?  $l$ -invariant?  $u$ -invariant?

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