Coincidence Quasi-Best Proximity Points for Quasi-Cyclic-Noncyclic Mappings in Convex Metric Spaces

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ABSTRACT. We introduce the notion of quasi-cyclic-noncyclic pair and its
relevant new notion of coincidence quasi-best proximity points in a con-
vex metric space. In this way we generalize the notion of coincidence-best
proximity point already introduced by M. Gabeleh et al [14]. It turns out
that under some circumstances this new class of mappings contains the
class of cyclic-noncyclic mappings as a subclass. The existence and con-
vergence of coincidence-best and coincidence quasi-best proximity points
in the setting of convex metric spaces are investigated.

1. INTRODUCTION

Let (X, d) be a metric space, and let A, B be subsets of X. A mapping
$T : A \cup B \to A \cup B$ is said to be *cyclic* provided that $T(A) \subseteq B$ and $T(B) \subseteq A$;
similarly, a mapping $S : A \cup B \to A \cup B$ is said to be *noncyclic* if $S(A) \subseteq A$
and $S(B) \subseteq B$. The following theorem is an extension of Banach contraction
principle.

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Theorem 1.1. *([18]) Let A and B be nonempty closed subsets of a complete metric space (X, d). Suppose that T is a cyclic mapping such that*

$$d(Tx, Ty) \leq \alpha d(x, y),$$

for some $\alpha \in (0, 1)$ and for all $x \in A$, $y \in B$. Then T has a unique fixed point in $A \cap B$.

Let A and B be nonempty subsets of a metric space X. A mapping $T : A \cup B \to A \cup B$ is said to be a *cyclic contraction* if T is cyclic and

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha)\mathrm{dist}(A, B)$$

for some $\alpha \in (0, 1)$ and for all $x \in A$, $y \in B$, where

$$\mathrm{dist}(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}.$$

For a cyclic mapping $T : A \cup B \to A \cup B$, a point $x \in A \cup B$ is said to be a best proximity point provided that

$$d(x, Tx) = \mathrm{dist}(A, B).$$

The following existence, uniqueness and convergence result of a best proximity point for cyclic contractions is the main result of [8].

Theorem 1.2. *([8]) Let A and B be nonempty closed convex subsets of a uniformly convex Banach space X and let $T : A \cup B \to A \cup B$ be a cyclic contraction map. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then there exists a unique $x \in A$ such that $x_{2n} \to x$ and*

$$\|x - Tx\| = \mathrm{dist}(A, B).$$

In the theory of best proximity points, one usually considers a cyclic mapping T defined on the union of two (closed) subsets of a given metric space. Here the objective is to minimize the expression $d(x, Tx)$ where x runs through the domain of T; that is $A \cup B$. In other words, we want to find

$$\min\{d(x, Tx) : x \in A \cup B\}.$$

If A and B intersect, the solution is clearly a fixed point of T; otherwise we have

$$d(x, Tx) \geq \mathrm{dist}(A, B), \quad \forall x \in A \cup B,$$

so that the point at which the equality occurs is called a best proximity point of T. This point of view dominates the literature.

Very recently, M. Gabeleh, O. Olela Otafudu, and N. Shahzad [14] considered two mappings T and S simultaneously and established interesting results. For technical reasons, the first map should be cyclic and the second one should be noncyclic. According to [14], for a nonempty pair of subsets (A, B), and a cyclic-noncyclic pair $(T; S)$ on $A \cup B$ (that is, $T : A \cup B \to A \cup B$ is cyclic and

$S : A \cup B \to A \cup B$ is noncyclic); they called a point $p \in A \cup B$ a *coincidence best proximity point* for $(T; S)$ provided that

$$d(Sp, Tp) = \text{dist}(A, B).$$

Note that if $S = I$, the identity map on $A \cup B$, then $p \in A \cup B$ is a best proximity point for T. Also, if $\text{dist}(A, B) = 0$, then p is called a *coincidence point* for $(T; S)$ (see [12] and [15] for more information). With the definition just given, and depending on the situation as to whether S equals the identity map, or if the distance between the underlying sets is zero, one obtains a best proximity point for T, or a coincidence point for the pair $(T; S)$. This was in fact the philosophy behind the phrase *coincidence-best proximity point* coined by Gabeleh et al. They then defined the notion of a cyclic-noncyclic contraction.

Definition 1.3. ([14]) Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and $T, S : A \cup B \to A \cup B$ be two mappings. The pair $(T; S)$ is called a cyclic-noncyclic contraction pair if it satisfies the following conditions:
  (1) $(T; S)$ is a cyclic-noncyclic pair on $A \cup B$.
  (2) For some $r \in (0, 1)$ we have

$$d(Tx, Ty) \le r d(Sx, Sy) + (1 - r) \text{dist}(A, B), \ \forall (x, y) \in A \times B.$$

To state the main result of [14], we need to recall the notion of convexity in the framework of metric spaces. In [26], Takahashi introduced the notion of convexity in metric spaces as follows (see also [24]).

Definition 1.4. Let (X, d) be a metric space and $I := [0, 1]$. A mapping $\mathcal{W} : X \times X \times I \to X$ is said to be a convex structure on X provided that for each $(x, y; \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, \mathcal{W}(x, y; \lambda)) \le \lambda d(u, x) + (1 - \lambda) d(u, y).$$

A metric space (X, d) together with a convex structure $\mathcal{W}$ is called a *convex metric space*, and is denoted by $(X, d, \mathcal{W})$. A Banach space and each of its convex subsets are convex metric spaces.

A subset K of a convex metric space $(X, d, \mathcal{W})$ is said to be a convex set provided that $\mathcal{W}(x, y; \lambda) \in K$ for all $x, y \in K$ and $\lambda \in I$.

Similarly, a convex metric space $(X, d, \mathcal{W})$ is said to be uniformly convex if for any $\varepsilon > 0$, there exists $\alpha = \alpha(\varepsilon)$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \le r, \ d(z, y) \le r$ and $d(x, y) \ge r\varepsilon$,

$$d(z, \mathcal{W}(x, y; \frac{1}{2})) \le r(1 - \alpha) < r.$$

For example every uniformly convex Banach space is a uniformly convex metric space.

Definition 1.5. ([14]) Let (A, B) be a nonempty pair of subsets of a metric space (X, d). A mapping $S : A \cup B \to A \cup B$ is said to be a relatively anti-Lipschitzian mapping if there exists $c > 0$ such that

$$d(x, y) \leq c\, d(Sx, Sy), \ \forall (x, y) \in A \times B.$$

The main result of M. Gabeleh et al reads as follows:

Theorem 1.6. ([14]) *Let (A, B) be a nonempty, closed pair of subsets of a complete uniformly convex metric space $(X, d, \mathcal{W})$ such that A is convex. Let $(T; S)$ be a cyclic-noncyclic contraction pair defined on $A \cup B$ such that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$ and that S is continuous on A and relatively anti-Lipschitzian on $A \cup B$. Then $(T; S)$ has a coincidence best proximity point in A. Further, if $x_0 \in A$ and $Sx_{n+1} := Tx_n$, then (x_{2n}) converges to the coincidence-best proximity point of $(T; S)$.*

Existence of best proximity pairs was first studied in [9] by using a geometric property on a nonempty pair of subsets of a Banach space, called *proximal normal structure*, for noncyclic relatively nonexpansive mappings (Theorem 2.2 of [9]). Some existence results of best proximity pairs can be found in [1, 2, 5, 6, 7, 10, 11, 13, 17, 23, 25].

In the current paper, we study sufficient conditions which ensure the existence and convergence of *coincidence-best and quasi-best proximity point* for a pair of quasi-cyclic-noncyclic contraction mappings in the setting of convex metric spaces.

2. Coincidence quasi-best proximity point

In this section, we introduce the class of quasi-cyclic-noncyclic mappings that contains the class of cyclic-noncyclic mappings as a subclass. Next, we introduce the new notion of quasi-best proximity points for this mappings. Finally, we study the existence and convergence of coincidence quasi-best proximity points for quasi-cyclic-noncyclic contraction mappings in the setting of convex metric spaces.

Definition 2.1. Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and $T, S : X \to X$ be two mappings. The pair $(T; S)$ is called a quasi-cyclic-noncyclic (**QCN**) contraction pair if it satisfies the following conditions:

(1) $(T; S)$ is a quasi-cyclic-noncyclic pair on X; that is,

$$T(A) \subseteq S(B), \ T(B) \subseteq S(A).$$

(2) For some $\alpha \in (0, 1)$ and for each $(x, y) \in A \times B$ we have

$$d(Tx, Ty) \leq \alpha d(Sx, Sy) + (1 - \alpha)\mathrm{dist}(S(A), S(B)).$$

Note that if $S(A) = A$ and $S(B) = B$, then the above definition reduces to Definition 1.3; that is, the pair $(T; S)$ is a cyclic-noncyclic pair.

EXAMPLE 2.2. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, -1]$ and $B = [1, +\infty)$ define $T, S : X \to X$ by

$$Tx := \begin{cases} -x, \ if \ x \in A \cup B \\ 0, \ ow. \end{cases} \quad \text{and} \quad Sx := \begin{cases} 2x + 1, \ if \ x \in A \\ 2x - 1, \ if \ x \in B \\ 0, \ ow. \end{cases}$$

Then $(T; S)$ is a QCN contraction pair with $\alpha = \frac{1}{2}$. Indeed, for all $(x, y) \in A \times B$ we have

$$|Tx - Ty| = (y - x) \leq \frac{1}{2}(2y - 2x - 2) + \frac{1}{2}(2)$$
$$= \alpha|Sx - Sy| + (1 - \alpha)\text{dist}(S(A), S(B)).$$

Also, $T(A) = B \subseteq S(B)$ and $T(B) = A \subseteq S(A)$.

The next example shows that there is a QCN mapping that is not a cyclic-noncyclic mapping.

EXAMPLE 2.3. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, -1]$ and $B = [1, +\infty)$ define $T, S : X \to X$ by

$$Tx := \begin{cases} -x, \ if \ x \in A \cup B \\ 0, \ ow. \end{cases} \quad \text{and} \quad Sx := \begin{cases} x + 1, \ if \ x \in A \\ x - 1, \ if \ x \in B \\ 0, \ ow. \end{cases}$$

Then $(T; S)$ is a quasi-cyclic-noncyclic pair that is not a cyclic-noncyclic pair.

Remark 2.4. Notice that (2) implies that

$$d(Tx, Ty) \leq d(Sx, Sy), \ \forall(x, y) \in A \times B.$$

Moreover, if S is a noncyclic relatively nonexpansive mapping; meaning that

$$d(Sx, Sy) \leq d(x, y), \ \forall(x, y) \in A \times B,$$

then T is a cyclic contraction. In addition, if in the above definition S is assumed to be continuous, then T would be continuous too.

Definition 2.5. Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and $T, S : X \to X$ be a quasi-cyclic-noncyclic pair on X. A point $p \in A \cup B$ is said to be a coincidence quasi-best proximity point for $(T; S)$ provided that

$$d(Sp, Tp) = \text{dist}(S(A), S(B)).$$

Note that if $S = I$, then p reduces to a coincidence-best proximity point for $(T; S)$.

To prove the main result of this section, we need some preparations.

Lemma 2.6. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a quasi-cyclic-noncyclic pair defined on X. Then there exists a sequence $\{x_n\}$ in X such that for all $n \geq 0$ we have $Tx_n = Sx_{n+1}$ where $\{x_{2n}\}$, $\{x_{2n+1}\}$ are subsequences in A and B respectively.*

Proof. Let $x_0 \in A$. Since $Tx_0 \in S(B)$, there exists $x_1 \in B$ such that $Tx_0 = Sx_1$. Again, since $Tx_1 \in S(A)$, there exists $x_2 \in A$ such that $Tx_1 = Sx_2$.

Continuing this process, we obtain a sequence $\{x_n\}$, such that $\{x_{2n}\}$, $\{x_{2n+1}\}$ are in A and B respectively and $Tx_n = Sx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.   □

Lemma 2.7. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a QCN contraction pair defined on X. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. Then we have*

$$d(Sx_{2n}, Sx_{2n+1}) \to \text{dist}(S(A), S(B)).$$

Proof.

$$
\begin{aligned}
d(Sx_{2n+1}, Sx_{2n+2}) &= d(Tx_{2n}, Tx_{2n+1}) \\
&\leq \alpha d(Sx_{2n}, Sx_{2n+1}) + (1 - \alpha)\text{dist}(S(A), S(B)) \\
&= \alpha d(Tx_{2n-1}, Tx_{2n}) + (1 - \alpha)\text{dist}(S(A), S(B)) \\
&\leq \alpha[\alpha d(Sx_{2n-1}, Sx_{2n}) + (1 - \alpha)\text{dist}(S(A), S(B))] \\
&\quad + (1 - \alpha)\text{dist}(S(A), S(B)) \\
&= \alpha^2 d(Sx_{2n-1}, Sx_{2n}) + (1 - \alpha^2)\text{dist}(S(A), S(B)) \\
&= \alpha^2 d(Tx_{2n-2}, Tx_{2n-1}) + (1 - \alpha^2)\text{dist}(S(A), S(B)) \\
&\leq \cdots \\
&\leq \alpha^{2n} d(Tx_0, Tx_1) + (1 - \alpha^2)\text{dist}(S(A), S(B)).
\end{aligned}
$$

Now, if $n \to \infty$ in above relation, we conclude that

$$d(Sx_{2n}, Sx_{2n+1}) \to \text{dist}(S(A), S(B)).$$

□

Theorem 2.8. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a QCN contraction pair defined on X. Assume that S is continuous on A. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. If $\{x_{2n}\}$ has a convergent subsequence in A, then the pair $(T; S)$ has a coincidence quasi-best proximity point in A.*

Proof. Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ such that $x_{2n_k} \to p \in A$. We have

$$
\begin{aligned}
\text{dist}(S(A), S(B)) \leq d(Tx_{2n_k-1}, Tp) &\leq d(Sx_{2n_k-1}, Sp) \\
&\leq d(Sp, Sx_{2n_k}) + d(Sx_{2n_k}, Sx_{2n_k-1}).
\end{aligned}
$$

By Lemma 2.7, if $k \to \infty$, we obtain that

$$d(Tx_{2n_k-1}, Tp) \to \text{dist}(S(A), S(B)).$$

Moreover, we have

$$
\begin{aligned}
\text{dist}(S(A), S(B)) &\leq d(Sp, Tp) \\
&\leq d(Sp, Tx_{2n_k-1}) + d(Tx_{2n_k-1}, Tp) \\
&= d(Sp, Sx_{2n_k}) + d(Tx_{2n_k-1}, Tp) \\
&\to \text{dist}(S(A), S(B)),
\end{aligned}
$$

that is,

$$d(Sp, Tp) = \text{dist}(S(A), S(B)).$$

$\square$

Lemma 2.9. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a QCN contraction pair defined on X. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. Then $\{Sx_{2n}\}$, and $\{Sx_{2n+1}\}$ are bounded sequences in $S(A)$ and $S(B)$ respectively.*

Proof. Since

$$d(Sx_{2n}, Sx_{2n+1}) \to \text{dist}(S(A), S(B)),$$

it suffices to show that $\{Sx_{2n}\}$ is bounded in $S(A)$. Assume to the contrary that there exists $N_0 \in \mathbb{N}$ such that

$$d(Sx_2, Sx_{2N_0+1}) > M, \ d(Sx_2, Sx_{2N_0-1}) \leq M,$$

where,

$$M > \max\left\{\frac{\alpha^2}{1 - \alpha^2}d(Sx_0, Sx_2) + \text{dist}(S(A), S(B)), \ d(Sx_1, Sx_0)\right\}.$$

By the above assumption, we have

$$
\begin{aligned}
\frac{M - \text{dist}(S(A), S(B))}{\alpha^2} &+ \text{dist}(S(A), S(B)) \\
&< \frac{d(Sx_2, Sx_{2N_0+1}) - \text{dist}(S(A), S(B))}{\alpha^2} \\
&+ \text{dist}(S(A), S(B)) \\
&\leq \frac{d(Sx_2, Sx_{2N_0+1}) + (\alpha^2 - 1)d(Sx_2, Sx_{2N_0+1})}{\alpha^2} \\
&= d(Sx_2, Sx_{2N_0+1}) = d(Tx_1, Tx_{2N_0}) \\
&\leq d(Sx_1, Sx_{2N_0}) = d(Tx_0, Tx_{2N_0-1}) \\
&= d(Sx_0, Sx_{2N_0-1}) \\
&\leq d(Sx_0, Sx_2) + d(Sx_2, Sx_{2N_0-1}) \\
&\leq d(Sx_0, Sx_2) + M.
\end{aligned}
$$

This implies that

$$\frac{M - \operatorname{dist}(S(A), S(B))}{\alpha^2} + \operatorname{dist}(S(A), S(B)) < d(Sx_0, Sx_2) + M,$$

hence,

$$M - (1 - \alpha^2)\operatorname{dist}(S(A), S(B)) < \alpha^2[d(Sx_0, Sx_2) + M],$$

and,

$$(1 - \alpha^2)M < \alpha^2 d(Sx_0, Sx_2) + (1 - \alpha^2)\operatorname{dist}(S(A), S(B)).$$

Now, it follows that

$$M < \frac{\alpha^2}{1 - \alpha^2} d(Sx_0, Sx_2) + \operatorname{dist}(S(A), S(B)),$$

which contradicts the choice of M. $\qquad\square$

Before we state the following theorem, we recall that a subset $A \subseteq X$ is said to be boundedly compact if the closure of every bounded subset of A is compact and is contained in A.

Theorem 2.10. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) such that $S(A)$ is boundedly compact and let $(T; S)$ be a QCN contraction pair defined on X. If S is relatively anti-Lipschitzian and continuous on A, then there exists $p \in A$ such that*

$$d(Sp, Tp) = \operatorname{dist}(S(A), S(B)).$$

Proof. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. By Lemma 2.9, $\{Sx_{2n}\}$ is bounded in $S(A)$. On the other hand, $S(A)$ is boundedly compact, so that there exists a subsequence $\{Sx_{2n_k}\}$ of $\{Sx_{2n}\}$ such that

$$Sx_{2n_k} \to Sp,$$

for some $p \in A$. We know that S is relatively anti-Lipschitzian, therefore

$$d(x_{2n_k}, p) \leq c\, d(Sx_{2n_k}, Sp) \to 0, \ \ k \to \infty.$$

This implies that $\{x_{2n_k}\}$ is a convergent subsequence of $\{x_{2n}\}$. Now, the result follows from Theorem 2.8. $\qquad\square$

EXAMPLE 2.11. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, 0]$ and $B = [0, +\infty)$ define $T, S : X \to X$ by

$$Tx := \begin{cases} -x, \ if \ x \in A \cup B \\ 0, \ ow. \end{cases} \quad \text{and} \quad Sx := \begin{cases} 2x, \ if \ x \in A \cup B \\ 0, \ ow. \end{cases}$$

Then $(T; S)$ is a QCN contraction pair with $\alpha = \frac{1}{2}$. Indeed, for all $(x, y) \in A \times B$ we have

$$|Tx - Ty| = (y - x) \leq \frac{1}{2}(2y - 2x) + \frac{1}{2}(0)$$
$$= \alpha|Sx - Sy| + (1 - \alpha)\operatorname{dist}(S(A), S(B)).$$

Also, $T(A) = B \subseteq S(B)$ and $T(B) = A \subseteq S(A)$. Moreover, S is continuous on A and $S(A)$ is boundedly compact in X. Besides, S is relatively anti-Lipschitzian on $A \cup B$ with $c = 1$. In fact, for all $(x, y) \in A \times B$ we have

$$|Sx - Sy| = 2y - 2x \geq |x - y|.$$

Finally, the existence of coincidence quasi-best proximity point of the pair $(T; S)$ follows from Theorem 2.10; that is, there exists $p \in A$ such that

$$|Tp - Sp| = \text{dist}(S(A), S(B)) = 0 \ or \ -p - 2p = 0,$$

which implies that $p = 0$. In this case, p is a fixed point of S.

In the following we supply an example which shows that there exists a coincidence quasi-best proximity point that is not a fixed point of S.

EXAMPLE 2.12. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, 0]$ and $B = [0, +\infty)$ define $T, S : X \to X$ by

$$Tx := \begin{cases} -(x+1), \ if \ x \in A \cup B \\ 0, \ ow. \end{cases} \quad \text{and} \quad Sx := \begin{cases} 2x, \ if \ x \in A \cup B \\ 0, \ ow. \end{cases}$$

Then $(T; S)$ is a QCN contraction pair with $\alpha = \frac{1}{2}$. Indeed, for all $(x, y) \in A \times B$ we have

$$|Tx - Ty| = (y - x) \leq \frac{1}{2}(2y - 2x) + \frac{1}{2}(0)$$
$$= \alpha|Sx - Sy| + (1 - \alpha)\text{dist}(S(A), S(B)).$$

Also, $T(A) = [1, +\infty) \subseteq S(B)$ and $T(B) = (-\infty, -1] \subseteq S(A)$. Moreover, S is continuous on A and $S(A)$ is boundedly compact in X. Besides, S is relatively anti-Lipschitzian on $A \cup B$ with $c = 1$. In fact, for all $(x, y) \in A \times B$ we have

$$|Sx - Sy| = 2y - 2x \geq |x - y|.$$

Finally, the existence of coincidence quasi-best proximity point of the pair $(T; S)$ follows from Theorem 2.10; that is, there exists $p \in A$ such that

$$|Tp - Sp| = \text{dist}(S(A), S(B)) = 0 \ or \ -(p+1) - 2p = 0,$$

which implies that $p = -\frac{1}{3}$.

Lemma 2.13. *Let (A, B) be a nonempty pair of subsets of a uniformly convex metric space $(X, d, \mathcal{W})$ such that $S(A)$ is convex. Let $(T; S)$ be a QCN contraction pair defined on X. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. Then*

$$d(Sx_{2n+2}, Sx_{2n}) \to 0, \ d(Sx_{2n+3}, Sx_{2n+1}) \to 0.$$

Proof. We prove that $d(Sx_{2n+2}, Sx_{2n}) \to 0$. To the contrary, assume that there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$, there exists $n_k \geq k$ such that

$$d(Sx_{2n_k+2}, Sx_{2n_k}) \geq \varepsilon_0.$$

Choose $0 < \gamma < 1$ such that $\frac{\varepsilon_0}{\gamma} > \text{dist}(S(A), S(B))$ and choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \min\left\{\frac{\varepsilon_0}{\gamma} - \text{dist}(S(A), S(B)), \frac{\text{dist}(S(A), S(B))\alpha(\gamma)}{1 - \alpha(\gamma)}\right\}.$$

By Lemma 2.7, since $d(Sx_{2n_k}, Sx_{2n_k+1}) \to \text{dist}(S(A), S(B))$, there exists $N \in \mathbb{N}$ such that

$$d(Sx_{2n_k}, Sx_{2n_k+1}) \leq \text{dist}(S(A), S(B)) + \varepsilon,$$
$$d(Sx_{2n_k+2}, Sx_{2n_k+1}) \leq \text{dist}(S(A), S(B)) + \varepsilon$$

and

$$d(Sx_{2n_k}, Sx_{2n_k+2}) \geq \varepsilon_0 > \gamma(\text{dist}(S(A), S(B)) + \varepsilon).$$

It now follows from the uniform convexity of X and the convexity of $S(A)$ that

$$\begin{aligned}
\text{dist}(S(A), S(B)) &\leq d(Sx_{2n_k+1}, \mathcal{W}(Sx_{2n_k}, Sx_{2n_k+2}, \tfrac{1}{2})) \\
&\leq (\text{dist}(S(A), S(B)) + \varepsilon)(1 - \alpha(\gamma)) \\
&< \text{dist}(S(A), S(B)) + \frac{\text{dist}(S(A), S(B))\alpha(\gamma)}{1 - \alpha(\gamma)}(1 - \alpha(\gamma)) \\
&= \text{dist}(S(A), S(B)),
\end{aligned}$$

which is a contradiction. Similarly, we see that $d(Sx_{2n+3}, Sx_{2n+1}) \to 0$. $\square$

The following Theorem guarantees the existence and convergence of coincidence quasi-best proximity points for QCN contraction mappings in the setting of uniformly convex metric spaces.

Theorem 2.14. *Let (A, B) be a nonempty, closed pair of subsets of a complete uniformly convex metric space $(X, d; \mathcal{W})$ such that $S(A)$ is convex. Let $(T; S)$ be a QCN contraction pair defined on X such that S is continuous on A and relatively anti-Lipschitzian on $A \cup B$. Then there exists $p \in A$ such that*

$$d(Sp, Tp) = \text{dist}(S(A), S(B)).$$

Further, if $x_0 \in A$ and $Tx_n = Sx_{n+1}$, then $\{x_{2n}\}$ converges to the coincidence quasi-best proximity point of $(T; S)$.

Proof. For $x_0 \in A$ define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. We prove that $\{Sx_{2n}\}$ and $\{Sx_{2n+1}\}$ are Cauchy sequences. First, we verify that for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$d(Sx_{2l}, Sx_{2n+1}) < \text{dist}(S(A), S(B)) + \varepsilon, \ \forall l > n \geq N_0. \qquad (*)$$

Assume to the contrary that there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$ there exists $l_k > n_k \geq k$ satisfying

$$d(Sx_{2l_k}, Sx_{2n_k+1}) \geq \text{dist}(S(A), S(B)) + \varepsilon_0$$

and

$$d(Sx_{2l_k-2}, Sx_{2n_k+1}) < \text{dist}(S(A), S(B)) + \varepsilon_0.$$

We have

$$\text{dist}(S(A), S(B)) + \varepsilon_0 \leq d(Sx_{2l_k}, Sx_{2n_k+1})$$
$$\leq d(Sx_{2l_k}, Sx_{2l_k-2}) + d(Sx_{2l_k-2}, Sx_{2n_k+1})$$
$$\leq d(Sx_{2l_k}, Sx_{2l_k-2}) + \text{dist}(S(A), S(B)) + \varepsilon_0.$$

Letting $k \to \infty$, we obtain

$$d(Sx_{2l_k}, Sx_{2n_k+1}) \to \text{dist}(S(A), S(B)) + \varepsilon_0.$$

Moreover, we have

$$\text{dist}(S(A), S(B)) + \varepsilon_0 \leq d(Sx_{2l_k}, Sx_{2n_k+1}) = d(Tx_{2l_k-1}, Tx_{2n_k})$$
$$\leq \alpha d(Sx_{2l_k-1}, Sx_{2n_k}) + (1-\alpha)\text{dist}(S(A), S(B))$$
$$= \alpha d(Tx_{2l_k-2}, Tx_{2n_k-1}) + (1-\alpha)\text{dist}(S(A), S(B))$$
$$\leq \alpha d(Sx_{2l_k-2}, Sx_{2n_k-1}) + (1-\alpha)\text{dist}(S(A), S(B)).$$

Therefore, by letting $k \to \infty$ we obtain

$$\text{dist}(S(A), S(B)) + \varepsilon_0 \leq \alpha(\text{dist}(S(A), S(B)) + \varepsilon_0) + (1-\alpha)\text{dist}(S(A), S(B))$$
$$\leq \text{dist}(S(A), S(B)) + \varepsilon_0.$$

This implies that $\alpha = 1$, which is a contradiction. That is, $(*)$ holds. Now, assume $\{Sx_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$ there exists $l_k > n_k \geq k$ such that

$$d(Sx_{2l_k}, Sx_{n_k}) \geq \varepsilon_0.$$

Choose $0 < \gamma < 1$ such that $\frac{\varepsilon_0}{\gamma} > \text{dist}(S(A), S(B))$ and choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \min\left\{\frac{\varepsilon_0}{\gamma} - \text{dist}(S(A), S(B)), \frac{\text{dist}(S(A), S(B))\alpha(\gamma)}{1 - \alpha(\gamma)}\right\}.$$

Let $N \in \mathbb{N}$ be such that

$$d(Sx_{2n_k}, Sx_{2n_k+1}) \leq \text{dist}(S(A), S(B)) + \varepsilon, \ \forall n_k \geq N$$

and

$$d(Sx_{2l_k}, Sx_{2n_k+1}) \leq \text{dist}(S(A), S(B)) + \varepsilon, \ \forall l_k > n_k \geq N.$$

Uniform convexity of X implies that

$$\text{dist}(S(A), S(B)) \leq d(Sx_{2n_k+1}, \mathcal{W}(Sx_{2n_k}, Sx_{2l_k}, \frac{1}{2}))$$
$$\leq (\text{dist}(S(A), S(B)) + \varepsilon)(1 - \alpha(\gamma)) < \text{dist}(S(A), S(B)),$$

which is a contradiction. Therefore, $\{Sx_{2n}\}$ is a Cauchy sequence in $S(A)$. By the fact that S is relatively anti-Lipschitzian on $A \cup B$, we have

$$d(x_{2l}, x_{2n}) \leq c d(Sx_{2l}, Sx_{2n}) \to 0, \ l, n \to \infty,$$

that is, $\{x_{2n}\}$ is a Cauchy sequence. Since A is complete, there exists $p \in A$ such that $x_{2n} \to p$. Now, the result follows from a similar argument as in Theorem 2.8.   □

3. QUASI-CYCLIC-NONCYCLIC RELATIVELY CONTRACTION MAPPINGS

In this section, we introduce the class of quasi-cyclic-noncyclic relatively contraction mappings that contains the class of cyclic-noncyclic contraction mappings as a subclass. Next, we study the existence and convergence of coincidence best proximity points in the setting of convex metric spaces for quasi-cyclic-noncyclic relatively contraction mappings.

Definition 3.1. Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and $T, S : X \to X$ be two mappings. The pair $(T; S)$ is called a quasi-cyclic-noncyclic relatively contraction pair if it satisfies the following conditions:

(1) $(T; S)$ is a quasi-cyclic-noncyclic pair on X; that is,

$$T(A) \subseteq S(B), \ T(B) \subseteq S(A).$$

(2) For some $\alpha \in (0, 1)$ and for each $(x, y) \in A \times B$ we have

$$d(Tx, Ty) \le \alpha d(Sx, Sy) + (1 - \alpha)\operatorname{dist}(A, B).$$

Note that in the above definition we do not have the inequality

$$\operatorname{dist}(A, B) \le d(Sx, Sy),$$

that is,

$$d(Tx, Ty) \le d(Sx, Sy)$$

is not always true.

We emphasize that if $S = I$ or if $S(A) = A$ and $S(B) = B$, then the above definition reduces to Definition 1.3.

EXAMPLE 3.2. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, -3]$ and $B = [3, +\infty)$ define $T, S : X \to X$ by

$$Tx := \begin{cases} -(x+1), \ if \ x \in A \cup B \\ 0, \ ow. \end{cases} \quad \text{and} \quad Sx := \begin{cases} 3x + 5, \ if \ x \in A \\ 3x - 7, \ if \ x \in B \\ 0, \ ow. \end{cases}$$

Then $(T; S)$ is a QCN relatively contraction pair with $\alpha = \frac{1}{3}$. Indeed, for all $(x, y) \in A \times B$ we have

$$|Tx - Ty| = (y - x) \le \frac{1}{3}(3y - 3x - 12) + \frac{2}{3}(6)$$
$$= \alpha|Sx - Sy| + (1 - \alpha)\operatorname{dist}(A, B).$$

Also, $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$.

Lemma 3.3. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a QCN relatively contraction pair defined on X and $dist(A, B) \leq dist(S(A), S(B))$. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. Then we have*

$$d(Sx_{2n}, Sx_{2n+1}) \to \text{dist}(A, B).$$

Proof. We note that

$$\begin{aligned}
\text{dist}(A, B) \leq \text{dist}(S(A), S(B)) &\leq d(Sx_{2n+1}, Sx_{2n+2}) = d(Tx_{2n}, Tx_{2n+1}) \\
&\leq \alpha d(Sx_{2n}, Sx_{2n+1}) + (1 - \alpha)\text{dist}(A, B) \\
&= \alpha d(Tx_{2n-1}, Tx_{2n}) + (1 - \alpha)\text{dist}(A, B) \\
&\leq \alpha[\alpha d(Sx_{2n-1}, Sx_{2n}) + (1 - \alpha)\text{dist}(A, B)] \\
&\quad + (1 - \alpha)\text{dist}(A, B) \\
&= \alpha^2 d(Sx_{2n-1}, Sx_{2n}) + (1 - \alpha^2)\text{dist}(A, B) \\
&= \alpha^2 d(Tx_{2n-2}, Tx_{2n-1}) + (1 - \alpha^2)\text{dist}(A, B) \\
&\leq \cdots \\
&\leq \alpha^{2n} d(Tx_0, Tx_1) + (1 - \alpha^2)\text{dist}(A, B).
\end{aligned}$$

Now, if $n \to \infty$, we conclude that

$$d(Sx_{2n}, Sx_{2n+1}) \to \text{dist}(A, B).$$

$\square$

Remark 3.4. If the pair $(T; S)$ is a QCN relatively contraction pair such that

$$S(A) \subseteq A \text{ and } S(B) \subseteq B,$$

then we have

$$dist(A, B) \leq \text{dist}(S(A), S(B)).$$

Thus, by this assumption, the Lemma holds true.

Theorem 3.5. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a QCN relatively contraction pair defined on X and $\text{dist}(A, B) \leq \text{dist}(S(A), S(B))$. Assume S is continuous on A. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. If $\{x_{2n}\}$ has a convergent subsequence in A, then the pair $(T; S)$ has a coincidence best proximity point in A.*

Proof. Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ such that $x_{2n_k} \to p \in A$. we have

$$\begin{aligned}
\text{dist}(A, B) \leq \text{dist}(S(A), S(B)) &\leq d(Tx_{2n_k-1}, Tp) \leq d(Sx_{2n_k-1}, Sp) \\
&\leq d(Sp, Sx_{2n_k}) + d(Sx_{2n_k}, Sx_{2n_k-1}).
\end{aligned}$$

By Lemma 3.3, if $k \to \infty$, we obtain that

$$d(Tx_{2n_k-1}, Tp) \to \text{dist}(A, B).$$

Moreover,

$$\mathrm{dist}(A,B) \leq \mathrm{dist}(S(A),S(B)) \leq d(Sp,Tp)$$
$$\leq d(Sp,Tx_{2n_k-1}) + d(Tx_{2n_k-1},Tp)$$
$$= d(Sp,Sx_{2n_k}) + d(Tx_{2n_k-1},Tp)$$
$$\to \mathrm{dist}(A,B),$$

that is,

$$d(Sp,Tp) = \mathrm{dist}(A,B).$$

$$\square$$

Lemma 3.6. *Let* (A,B) *be a nonempty pair of subsets of a metric space* (X,d). *Suppose* $(T;S)$ *is a QCN relatively contraction pair defined on* X *and* $\mathrm{dist}(A,B) \leq \mathrm{dist}(S(A),S(B))$. *For* $x_0 \in A$, *define* $Tx_n = Sx_{n+1}$ *for each* $n \geq 0$. *Then* $\{Sx_{2n}\}$, *and* $\{Sx_{2n+1}\}$ *are bounded sequences in* $S(A)$ *and* $S(B)$ *respectively.*

Proof. Since

$$d(Sx_{2n}, Sx_{2n+1}) \to \mathrm{dist}(A,B),$$

it suffices to verify that $\{Sx_{2n}\}$ is bounded in $S(A)$. Assume to the contrary that there exists $N_0 \in \mathbb{N}$ such that

$$d(Sx_2, Sx_{2N_0+1}) > M, \ d(Sx_2, Sx_{2N_0-1}) \leq M,$$

where,

$$M > \max \left\{ \frac{\alpha^2}{1-\alpha^2} d(Sx_0, Sx_2) + \mathrm{dist}(A,B), \ d(Sx_1, Sx_0) \right\}.$$

By the above assumption, we have

$$\frac{M - \mathrm{dist}(A,B)}{\alpha^2} + \mathrm{dist}(A,B) < \frac{d(Sx_2, Sx_{2N_0+1}) - \mathrm{dist}(A,B)}{\alpha^2} + \mathrm{dist}(A,B)$$
$$\leq \frac{d(Sx_2, Sx_{2N_0+1}) + (\alpha^2-1)d(Sx_2, Sx_{2N_0+1})}{\alpha^2}$$
$$= d(Sx_2, Sx_{2N_0+1}) = d(Tx_1, Tx_{2N_0})$$
$$\leq d(Sx_1, Sx_{2N_0}) = d(Tx_0, Tx_{2N_0-1})$$
$$= d(Sx_0, Sx_{2N_0-1})$$
$$\leq d(Sx_0, Sx_2) + d(Sx_2, Sx_{2N_0-1})$$
$$\leq d(Sx_0, Sx_2) + M.$$

This implies that

$$\frac{M - \mathrm{dist}(A,B)}{\alpha^2} + \mathrm{dist}(A,B) < d(Sx_0, Sx_2) + M,$$

or,

$$M - (1-\alpha^2)\mathrm{dist}(A,B) < \alpha^2[d(Sx_0, Sx_2) + M].$$

and finally,

$$(1 - \alpha^2)M < \alpha^2 d(Sx_0, Sx_2) + (1 - \alpha^2)\mathrm{dist}(A, B).$$

Now, we conclude that

$$M < \frac{\alpha^2}{1 - \alpha^2} d(Sx_0, Sx_2) + \mathrm{dist}(A, B),$$

which is a contradiction by the choice of M. $\square$

Theorem 3.7. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) such that $S(A)$ is boundedly compact. Suppose $(T; S)$ is a QCN relatively contraction pair defined on X and $\mathrm{dist}(A, B) \leq \mathrm{dist}(S(A), S(B))$. If S is relatively anti-Lipschitzian and continuous on A, then there exists $p \in A$ such that*

$$d(Sp, Tp) = \mathrm{dist}(A, B).$$

Proof. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. According to Lemma 3.6, $\{Sx_{2n}\}$ is bounded in $S(A)$, on the other hand $S(A)$ is boundedly compact, so that there exists a subsequence $\{Sx_{2n_k}\}$ of $\{Sx_{2n}\}$ such that

$$Sx_{2n_k} \to Sp,$$

for some $p \in A$. We know that S is relatively anti-Lipschitzian, therefore

$$d(x_{2n_k}, p) \leq cd(Sx_{2n_k}, Sp) \to 0, \ k \to \infty.$$

This implies that $\{x_{2n_k}\}$ is a convergent subsequence of $\{x_{2n}\}$, hence the result follows from Theorem 3.5. $\square$

In the following we give examples to show that there exists a coincidence best proximity point that is not a fixed point for S.

EXAMPLE 3.8. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, -3]$ and $B = [3, +\infty)$ define $T, S : X \to X$ by

$$Tx := \begin{cases} 3 - x, \ if \ x \in A \cup B \\ 0, \ ow. \end{cases} \quad \text{and} \quad Sx := \begin{cases} 2x + 6, \ if \ x \in A \\ 2x, \ if \ x \in B \\ 0, \ ow. \end{cases}$$

Then $(T; S)$ is a QCN relatively contraction pair with $\alpha = \frac{1}{2}$. Indeed, for all $(x, y) \in A \times B$ we have

$$|Tx - Ty| = (y - x) \leq \frac{1}{2}(2y - 2x - 6) + \frac{1}{2}(6)$$
$$= \alpha|Sx - Sy| + (1 - \alpha)\mathrm{dist}(A, B).$$

Also, $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Finally, the existence of coincidence best proximity point of the pair $(T; S)$ follows from Theorem 3.7; that is, there exists $p \in A$ such that

$$|Tp - Sp| = \mathrm{dist}(A, B) = 0 \ or \ 3 - p - 2p - 6 = 6,$$

which implies that $p = -3$.

EXAMPLE 3.9. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, -4]$ and $B = [4, +\infty)$ define $T, S : X \to X$ by

$$Tx := \begin{cases} 4 - x, \ if \ x \in A \cup B \\ 0, \ ow. \end{cases} \quad \text{and} \quad Sx := \begin{cases} 4x + 16, \ if \ x \in A \\ 4x - 8, \ if \ x \in B \\ 0, \ ow. \end{cases}$$

Then $(T; S)$ is a QCN relatively contraction pair with $\alpha = \frac{1}{4}$. Indeed, for all $(x, y) \in A \times B$ we have

$$|Tx - Ty| = (y - x) \leq \frac{1}{4}(4y - 4x - 24) + \frac{3}{4}(8)$$
$$= \alpha|Sx - Sy| + (1 - \alpha)\text{dist}(A, B).$$

Also, $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Finally, the existence of coincidence best proximity point of the pair $(T; S)$ follows from Theorem 3.7; that is, there exists $p \in A$ such that

$$|Tp - Sp| = \text{dist}(A, B) = 8 \ or \ 4 - p - 4p - 16 = 8,$$

which implies that $p = -4$.

Lemma 3.10. *Let (A, B) be a nonempty pair of subsets of a uniformly convex metric space $(X, d, \mathcal{W})$ such that $S(A)$ is convex. Suppose $(T; S)$ is a QCN relatively contraction pair defined on X and $\text{dist}(A, B) \leq \text{dist}(S(A), S(B))$. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. Then*

$$d(Sx_{2n+2}, Sx_{2n}) \to 0, \ d(Sx_{2n+3}, Sx_{2n+1}) \to 0.$$

Proof. We prove that $d(Sx_{2n+2}, Sx_{2n}) \to 0$. Assume to the contrary that there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$, there exists $n_k \geq k$ such that

$$d(Sx_{2n_k+2}, Sx_{2n_k}) \geq \varepsilon_0.$$

Choose $0 < \gamma < 1$ such that $\frac{\varepsilon_0}{\gamma} > \text{dist}(A, B)$ and choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \min \left\{ \frac{\varepsilon_0}{\gamma} - \text{dist}(A, B), \frac{\text{dist}(A, B)\alpha(\gamma)}{1 - \alpha(\gamma)} \right\}.$$

By Lemma 3.3, we know that $d(Sx_{2n_k}, Sx_{2n_k+1}) \to \text{dist}(A, B)$, so there exists $N \in \mathbb{N}$ such that

$$d(Sx_{2n_k}, Sx_{2n_k+1}) \leq \text{dist}(A, B) + \varepsilon,$$

$$d(Sx_{2n_k+2}, Sx_{2n_k+1}) \leq \text{dist}(A, B) + \varepsilon$$

and

$$d(Sx_{2n_k}, Sx_{2n_k+2}) \geq \varepsilon_0 > \gamma(\text{dist}(A, B) + \varepsilon).$$

It now follows from the uniformly convexity of X and the convexity of $S(A)$ that

$$\text{dist}(A,B) \leq \text{dist}(S(A),S(B)) \leq d(Sx_{2n_k+1}, \mathcal{W}(Sx_{2n_k}, Sx_{2n_k+2}, \frac{1}{2}))$$
$$\leq (\text{dist}(A,B) + \varepsilon)(1 - \alpha(\gamma))$$
$$< \text{dist}(A,B) + \frac{\text{dist}(A,B)\alpha(\gamma)}{1 - \alpha(\gamma)}(1 - \alpha(\gamma))$$
$$= \text{dist}(A,B),$$

which is a contradiction. Similarly, we see that $d(Sx_{2n+3}, Sx_{2n+1}) \to 0$. $\square$

The following Theorem guarantees the existence and convergence of coincidence best proximity points for QCN relatively contraction mappings in the setting of uniformly convex metric spaces.

Theorem 3.11. *Let (A,B) be a nonempty, closed pair of subsets of a complete uniformly convex metric space $(X,d;\mathcal{W})$ such that $S(A)$ is convex. Suppose $(T;S)$ is a QCN relatively contraction pair defined on X such that S is continuous on A and relatively anti-Lipschitzian on $A \cup B$. Assume that $\text{dist}(A,B) \leq \text{dist}(S(A),S(B))$. Then there exists $p \in A$ such that*

$$d(Sp, Tp) = \text{dist}(A,B).$$

Further, if $x_0 \in A$ and $Tx_n = Sx_{n+1}$, then $\{x_{2n}\}$ converges to the coincidence best proximity point of $(T;S)$.

Proof. For $x_0 \in A$ define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. We prove that $\{Sx_{2n}\}$ and $\{Sx_{2n+1}\}$ are Cauchy sequences. First, we verify that for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$d(Sx_{2l}, Sx_{2n+1}) < \text{dist}(A,B) + \varepsilon, \ \forall l > n \geq N_0. \qquad (*)$$

Assume the contrary. Then there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$ there exists $l_k > n_k \geq k$ satisfying

$$d(Sx_{2l_k}, Sx_{2n_k+1}) \geq \text{dist}(A,B) + \varepsilon_0, \ d(Sx_{2l_k-2}, Sx_{2n_k+1}) < \text{dist}(A,B) + \varepsilon_0.$$

Note that

$$\text{dist}(A,B) + \varepsilon_0 \leq d(Sx_{2l_k}, Sx_{2n_k+1})$$
$$\leq d(Sx_{2l_k}, Sx_{2l_k-2}) + d(Sx_{2l_k-2}, Sx_{2n_k+1})$$
$$\leq d(Sx_{2l_k}, Sx_{2l_k-2}) + \text{dist}(A,B) + \varepsilon_0.$$

Letting $k \to \infty$, we obtain

$$d(Sx_{2l_k}, Sx_{2n_k+1}) \to \text{dist}(A,B) + \varepsilon_0.$$

Moreover, we have

$$
\begin{aligned}
\mathrm{dist}(A,B) + \varepsilon_0 \leq d(Sx_{2l_k}, Sx_{2n_k+1}) &= d(Tx_{2l_k-1}, Tx_{2n_k}) \\
&\leq \alpha d(Sx_{2l_k-1}, Sx_{2n_k}) + (1-\alpha)\mathrm{dist}(A,B) \\
&= \alpha d(Tx_{2l_k-2}, Tx_{2n_k-1}) + (1-\alpha)\mathrm{dist}(A,B) \\
&\leq \alpha d(Sx_{2l_k-2}, Sx_{2n_k-1}) + (1-\alpha)\mathrm{dist}(A,B).
\end{aligned}
$$

Therefore, by letting $k \to \infty$ we obtain

$$
\mathrm{dist}(A,B) + \varepsilon_0 \leq \alpha(\mathrm{dist}(A,B)+\varepsilon_0) + (1-\alpha)\mathrm{dist}(A,B) \leq \mathrm{dist}(A,B) + \varepsilon_0.
$$

This implies that $\alpha = 1$, which is a contradiction. That is, $(*)$ holds. Now, assume that $\{Sx_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$ there exists $l_k > n_k \geq k$ such that

$$
d(Sx_{2l_k}, Sx_{n_k}) \geq \varepsilon_0.
$$

Choose $0 < \gamma < 1$ such that $\frac{\varepsilon_0}{\gamma} > \mathrm{dist}(A,B)$ and choose $\varepsilon > 0$ such that

$$
0 < \varepsilon < \min\left\{ \frac{\varepsilon_0}{\gamma} - \mathrm{dist}(A,B), \frac{\mathrm{dist}(A,B)\alpha(\gamma)}{1-\alpha(\gamma)} \right\}.
$$

Let $N \in \mathbb{N}$ be such that

$$
d(Sx_{2n_k}, Sx_{2n_k+1}) \leq \mathrm{dist}(A,B) + \varepsilon, \ \forall n_k \geq N
$$

and

$$
d(Sx_{2l_k}, Sx_{2n_k+1}) \leq \mathrm{dist}(A,B) + \varepsilon, \ \forall l_k > n_k \geq N.
$$

Uniformly convexity of X implies that

$$
\begin{aligned}
\mathrm{dist}(A,B) \leq \mathrm{dist}(S(A), S(B)) &\leq d(Sx_{2n_k+1}, \mathcal{W}(Sx_{2n_k}, Sx_{2l_k}, \tfrac{1}{2})) \\
&\leq (\mathrm{dist}(A,B)+\varepsilon)(1-\alpha(\gamma)) < \mathrm{dist}(A,B),
\end{aligned}
$$

which is a contradiction. Therefore, $\{Sx_{2n}\}$ is a Cauchy sequence in $S(A)$. By the fact that S is relatively anti-Lipschitzian on $A \cup B$, we have

$$
d(x_{2l}, x_{2n}) \leq cd(Sx_{2l}, Sx_{2n}) \to 0, \ l, n \to \infty,
$$

that is, $\{x_{2n}\}$ is Cauchy. Since A is complete, there exists $p \in A$ such that $x_{2n} \to p$. Now, the result follows from a similar argument as in the proof of Theorem 3.5. $\square$

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