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Some Perturbed Inequalities of Ostrowski Type for Functions whose n-th Derivatives Are Bounded

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Abstract. We firstly establish an identity for n time differentiable mappings Then, a new inequality for n times differentiable functions is deduced. Finally, some perturbed Ostrowski type inequalities for functions whose nth derivatives are of bounded variation are obtained.

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1. Introduction

The following result which is now called as Ostrowski inequality [18] is introduced by Ostrowski in 1938.

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

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During the past years, many authors have tackled new refinement, extensions and generalizations of this inequality (1.1). As an instance, novel Ostrowski type inequalities for different kinds of differentiable and twice differentiable functions was worked in [6], [8], [10], [19] and the references included there. Furthermore, in [7], [20] and [21], authors established generalized Ostrowski type inequalities for higher order differentiable functions on L_1, L_p and L_∞. In particular, some mathematicians focus on Ostrowski type inequalities for mappings of bounded variation in addition to the other function species. For example, Dragomir pointed out a new extansion of Ostrowski's inequality for mappings that are of bounded variation in [9]. He also proved an inequality for functions of bounded variation in [11] as foloows:

Theorem 1.2. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt-(b-a)\,f(x)\right|\le\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)\qquad(1.2)$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

Afterwards, it is given Ostrowski type inequalities for mappings whose first derivatives are of bounded variation in [2] and [17]. In [1], weighted versions of Ostrowski type results obtained by using mappings whose first derivatives are of bounded was provided and they also gave midpoint quadrature formula related to these inequalities. What's more, some inequalities for functions whose higher oerder derivatives are of bounded variation are established in [3], [12].

On the other side, it is presented some perturbed inequalities of Ostrowski type by Dragomir in [14] and [15]. Subsequently, some researcher studied on perturbed inequalities for mappings of bounded variation and functions whose second derivatives are bounded and whose first derivative are of bounded variation in [4], [5], [13] and [16].

We also note that Dragomir [14] established an identity in order to obtain some perturbe inequalities of Ostrowski type. This equality is stated as follows:

Theorem 1.3. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$ and $x\in[a,b]$. Then, for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^x(t-a)\left[f'(t)-\lambda_1(x)\right]dt+\frac{1}{b-a}\int_x^b(t-b)\left[f'(t)-\lambda_2(x)\right]dt\\ =\;& f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f'(t)dt\end{aligned}$$

where the integrals in the left hand side are taken in the Lebesgue sense.

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Perturbed integral inequalities for differentiable and twice differentiable mappings was worked by the interested researchers. Also, it is presented a great many integral inequalities for various function types. However, there are problems which higher order derivatives are required. Therefore, we handle perturbed integral inequalities for higher order differentiable functions. For this, we firstly establish an equality for n times differentiable functions. Then, we obtain some perturbed Ostrowski type integral inequalities for functions whose nth derivatives are of bounded variation and mappings whose higher order derivatives are bounded. So, it can be finded upper limit of an integral inequality obtained by using any order differentiable functions. What's more, this study lead to new works involving new integral inequalities derived by using higher order differentiable functions.

2. An Identity for High Degree Differentiable Functions

At first, it is established an equality for $n-$time differantiable functions in order to give our main results.

Lemma 2.1. *Let $f:[a,b]\to\mathbb{C}$ be a $n-$time differantiable function on (a,b) and $x\in[a,b]$. Then, for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have the identity*

$$\begin{aligned}
&\int_a^x \frac{(t-a)^n}{n!}\left[f^{(n)}(t)-\lambda_1(x)\right]dt \qquad (2.1)\\
&+\int_x^b \frac{(t-b)^n}{n!}\left[f^{(n)}(t)-\lambda_2(x)\right]dt\\
=\ &\sum_{k=0}^{n-1}\frac{(-1)^{n+1}(b-x)^{k+1}+(-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)\\
&-\lambda_1(x)\frac{(x-a)^{n+1}}{(n+1)!}-(-1)^n\lambda_2(x)\frac{(b-x)^{n+1}}{(n+1)!}+(-1)^n\int_a^b f(t)dt
\end{aligned}$$

where the integrals in the left hand side of the equality (2.1) are taken in the Lebesgue sense.

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Proof. Using the integration by parts, we get the equality

$$\begin{aligned} &\int_a^x \frac{(t-a)^n}{n!}\left[f^{(n)}(t)-\lambda_1(x)\right]dt+\int_x^b \frac{(t-b)^n}{n!}\left[f^{(n)}(t)-\lambda_2(x)\right]dt \\ =\ & \frac{(x-a)^n}{n!}f^{(n-1)}(x)-\int_a^x \frac{(t-a)^{n-1}}{(n-1)!}f^{(n-1)}(t)dt-\lambda_1(x)\int_a^x \frac{(t-a)^n}{n!}dt \\ &-\frac{(x-b)^n}{n!}f^{(n-1)}(x)-\int_x^b \frac{(t-b)^{n-1}}{(n-1)!}f^{(n-1)}(t)dt-\lambda_2(x)\int_x^b \frac{(t-b)^n}{n!}dt. \end{aligned}$$

If we apply the integration by parts $n-1$ more time and also use elementary analysis, then we can write the equality (2.1). The proof is thus completed. □

3. Inequalities for n Time Differentiable Functions

Now,we recall sets of complex valued functions:

$$\begin{aligned} &U_{[a,b]}(\gamma,\Gamma) \\ &:=\left\{\begin{array}{r} f:[a,b]\to\mathbb{C}|\operatorname{Re}\left[(\Gamma-f(t))\left(f(t)-\gamma\right)\right]\geq 0 \\ \text{for almost every } t\in[a,b] \end{array}\right\} \end{aligned}$$

and

$$\begin{aligned} &\Delta_{[a,b]}(\gamma,\Gamma) \\ &:=\left\{\begin{array}{r} f:[a,b]\to\mathbb{C}|\left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}\left|\Gamma-\gamma\right| \\ \text{for almost every } t\in[a,b] \end{array}\right\}. \end{aligned}$$

We need to the following proposition, in order to prove the next inequality.

Proposition 3.1. *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that $U_{[a,b]}(\gamma,\Gamma)$ and $\Delta_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed convex sets and*

$$U_{[a,b]}(\gamma,\Gamma)=\Delta_{[a,b]}(\gamma,\Gamma).$$

Theorem 3.2. *Let $f:[a,b]\to\mathbb{C}$ be a $n-$time differantiable function on (a,b) and $x\in[a,b]$. Assume that $\gamma_i,\Gamma_i\in\mathbb{C}$ with $\gamma_i\neq\Gamma_i$, $i=1,2$ and $f^{(n)}\in$*

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$U_{[a,x]}(\gamma_1,\Gamma_1)\cap U_{[x,b]}(\gamma_2,\Gamma_2)$, *then we have*

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+1}(b-x)^{k+1}+(-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)\right. \tag{3.1}$$

$$\left.-\frac{\gamma_1+\Gamma_1}{2}\frac{(x-a)^{n+1}}{(n+1)!}-(-1)^n\frac{\gamma_2+\Gamma_2}{2}\frac{(b-x)^{n+1}}{(n+1)!}+(-1)^n\int_a^b f(t)dt\right|$$

$$\leq \frac{|\Gamma_1-\gamma_1|}{2}\frac{(x-a)^{n+1}}{(n+1)!}+\frac{|\Gamma_2-\gamma_2|}{2}\frac{(b-x)^{n+1}}{(n+1)!}$$

$$\leq \frac{1}{2(n+1)!}P(x;n)$$

where $P(x;n)$ *is defined by*

$$P(x;n) = \begin{cases} \left[(x-a)^{n+1}+(b-x)^{n+1}\right]\max\{|\Gamma_1-\gamma_1|,|\Gamma_2-\gamma_2|\} \\ \left[(x-a)^{(n+1)p}+(b-x)^{(n+1)p}\right]^{\frac{1}{p}}\left[|\Gamma_1-\gamma_1|^q+|\Gamma_2-\gamma_2|^q\right]^{\frac{1}{q}} \quad \textit{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^{n+1}\left[|\Gamma_1-\gamma_1|+|\Gamma_2-\gamma_2|\right]. \end{cases}$$

Proof. Taking the modulus identity (2.1) for $\lambda_1(x)=\frac{\gamma_1+\Gamma_1}{2}$ and $\lambda_2(x)=\frac{\gamma_2+\Gamma_2}{2}$, because of $f^{(n)}\in U_{[a,x]}(\gamma_1,\Gamma_1)\cap U_{[x,b]}(\gamma_2,\Gamma_2)$, we have

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+1}(b-x)^{k+1}+(-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)\right.$$

$$\left.-\frac{\gamma_1+\Gamma_1}{2}\frac{(x-a)^{n+1}}{(n+1)!}-(-1)^n\frac{\gamma_2+\Gamma_2}{2}\frac{(b-x)^{n+1}}{(n+1)!}+(-1)^n\int_a^b f(t)dt\right|$$

$$\leq \int_a^x\frac{(t-a)^n}{n!}\left|f^{(n)}(t)-\frac{\gamma_1+\Gamma_1}{2}\right|dt+\int_x^b\frac{(b-t)^n}{n!}\left|f^{(n)}(t)-\frac{\gamma_2+\Gamma_2}{2}\right|dt.$$

We observe that

$$\int_a^x\frac{(t-a)^n}{n!}\left|f^{(n)}(t)-\frac{\gamma_1+\Gamma_1}{2}\right|dt+\int_x^b\frac{(b-t)^n}{n!}\left|f^{(n)}(t)-\frac{\gamma_2+\Gamma_2}{2}\right|dt$$

$$\leq \frac{|\Gamma_1-\gamma_1|}{2}\int_a^x\frac{(t-a)^n}{n!}dt+\frac{|\Gamma_2-\gamma_2|}{2}\int_x^b\frac{(b-t)^n}{n!}dt$$

which completes the first inequality in (3.1).

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The proof of first branch of second inequality given in (3.1) is obvious. If we use the Hölder's inequality

$$mn + pq \leq (m^{\alpha} + p^{\alpha})^{\frac{1}{\alpha}} \left(n^{\beta} + q^{\beta}\right)^{\frac{1}{\beta}} \tag{3.2}$$

where $m, n, p, q \geq 0$ and $\alpha > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then we can easily prove the second branch of second inequality given in (3.1). Finally, using the identity

$$\max\{X, Y\} = \frac{X+Y}{2} + \left|\frac{X-Y}{2}\right| \tag{3.3}$$

and the property of maximum $\max\{a^n, b^n\} = (\max\{a, b\})^n$ for $a, b > 0$ and $n \in \mathbb{N}$, we get the third branch of secand inequality in (3.1). The proof is thus completed. □

Remark 3.3. Under the same assumptions of Theorem 3.2 with $n = 1$, then the following inequalities hold:

$$\begin{aligned} & \left|(b-a) f(x) - \frac{\gamma_1 + \Gamma_1}{4}(x-a)^2 + \frac{\gamma_2 + \Gamma_2}{4}(b-x)^2 - \int_a^b f(t)dt\right| \\ \leq \; & \frac{|\Gamma_1 - \gamma_1|}{4}(x-a)^2 + \frac{|\Gamma_2 - \gamma_2|}{4}(b-x)^2 \\ \leq \; & \frac{1}{4} P(x;1) \end{aligned}$$

which was given by Dragomir in [14].

Remark 3.4. If we choose $n = 2$ in (3.1), the we have the inequalities

$$\begin{aligned} & \left|(b-a)\left(x - \frac{a+b}{2}\right) f'(x) - (b-a) f(x) + \int_a^b f(t)dt \right. \\ & \left. - \frac{(\gamma_1 + \Gamma_1)(x-a)^3 + (\gamma_2 + \Gamma_2)(b-x)^3}{12}\right| \\ \leq \; & \frac{1}{12}\left[|\Gamma_1 - \gamma_1|(x-a)^3 + |\Gamma_2 - \gamma_2|(b-x)^3\right] \\ \leq \; & \frac{1}{12} P(x;2) \end{aligned}$$

which was proved by Budak et al. in [4].

4. Inequalities for Functions Whose nth Derivatives are of Bounded Variation

We begin with the definitions which will be often used in this section.

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Definition 4.1. Let $P : a = x_0 < x_1 < ... < x_n = b$ be any partition of $[a, b]$ and let $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$, then f is said to be of bounded variation if the sum

$$\sum_{i=1}^{m} |\Delta f(x_i)|$$

is bounded for all such partitions.

Definition 4.2. Let f be of bounded variation on $[a, b]$, and $\sum \Delta f(P)$ denotes the sum $\sum\limits_{i=1}^{n} |\Delta f(x_i)|$ corresponding to the partition P of $[a, b]$. The number

$$\bigvee_a^b (f) := \sup\left\{\sum \Delta f(P) : P \in P([a, b])\right\},$$

is called the total variation of f on $[a, b]$. Here $P([a, b])$ denotes the family of partitions of $[a, b]$.

Now, we establish some perturbed inequalities of Ostrowski type for functions whose high order derivatives are of bounded variation. Also, we give some special results of these inequalities in this section.

Theorem 4.3. *Let $f : I \to \mathbb{C}$ be a n time differentiable function on I° and $[a, b] \subset I^\circ$. If the nth derivative $f^{(n)}$ is of bounded variation on $[a, b]$, then we have*

$$\begin{aligned} &\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+1}(b-x)^{k+1} + (-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \right. \\ &+(-1)^n \int_a^b f(t)dt - \frac{f^{(n)}(a) + f^{(n)}(x)}{2} \frac{(x-a)^{n+1}}{(n+1)!} \\ &\left. -(-1)^n \frac{f^{(n)}(x) + f^{(n)}(b)}{2} \frac{(b-x)^{n+1}}{(n+1)!} \right| \\ \le\ & \frac{1}{2} \frac{(x-a)^{n+1}}{(n+1)!} \bigvee_a^x \left(f^{(n)}\right) + \frac{1}{2} \frac{(b-x)^{n+1}}{(n+1)!} \bigvee_x^b \left(f^{(n)}\right) \\ \le\ & \frac{1}{2(n+1)!} R(x; n) \end{aligned} \tag{4.1}$$

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for any $x \in [a,b]$, *where*

$$R(x;n) \tag{4.2}$$
$$= \begin{cases} \left[(x-a)^{n+1} + (b-x)^{n+1}\right]\left[\frac{1}{2}\bigvee_a^b \left(f^{(n)}\right) + \frac{1}{2}\left|\bigvee_a^x \left(f^{(n)}\right) - \bigvee_x^b \left(f^{(n)}\right)\right|\right] \\ \left[(x-a)^{(n+1)p} + (b-x)^{(n+1)p}\right]^{\frac{1}{p}}\left[\left(\bigvee_a^x \left(f^{(n)}\right)\right)^q + \left(\bigvee_x^b \left(f^{(n)}\right)\right)^q\right]^{\frac{1}{q}} \\ \qquad\qquad \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^{n+1}\bigvee_a^b \left(f^{(n)}\right). \end{cases}$$

Proof. We take tabsolte value of the equality (2.1) for $\lambda_1(x) = \frac{f^{(n)}(a)+f^{(n)}(x)}{2}$ and $\lambda_2(x) = \frac{f^{(n)}(x)+f^{(n)}(b)}{2}$, we find that

$$\begin{aligned} &\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+1}(b-x)^{k+1} + (-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \right. \\ &+(-1)^n \int_a^b f(t)dt - \frac{f^{(n)}(a)+f^{(n)}(x)}{2}\frac{(x-a)^{n+1}}{(n+1)!} \\ &\left. -(-1)^n \frac{f^{(n)}(x)+f^{(n)}(b)}{2}\frac{(b-x)^{n+1}}{(n+1)!}\right| \\ \le\ & \int_a^x \frac{(t-a)^n}{n!}\left|f^{(n)}(t) - \frac{f^{(n)}(a)+f^{(n)}(x)}{2}\right| dt \\ &+ \int_x^b \frac{(b-t)^n}{n!}\left|f^{(n)}(t) - \frac{f^{(n)}(x)+f^{(n)}(b)}{2}\right| dt. \end{aligned} \tag{4.3}$$

Since $f^{(n)}: I^\circ \to \mathbb{C}$ is of bounded variation on $[a,x]$, we get

$$\begin{aligned} &\left|f^{(n)}(t) - \frac{f^{(n)}(a)+f^{(n)}(x)}{2}\right| \\ =\ & \left|\frac{f^{(n)}(t) - f^{(n)}(a) + f^{(n)}(t) - f^{(n)}(x)}{2}\right| \\ \le\ & \frac{\left|f^{(n)}(t) - f^{(n)}(a)\right| + \left|f^{(n)}(t) - f^{(n)}(x)\right|}{2} \\ \le\ & \frac{1}{2}\bigvee_a^x \left(f^{(n)}\right). \end{aligned} \tag{4.4}$$

Similarly, Since $f^{(n)}: I^\circ \to \mathbb{C}$ is of bounded variation on $[x,b]$, we have

$$\left|f^{(n)}(t) - \frac{f^{(n)}(x)+f^{(n)}(b)}{2}\right| \le \frac{1}{2}\bigvee_x^b \left(f^{(n)}\right). \tag{4.5}$$

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If we sabstitute the identities (4.4) and (4.5) in (4.3), then we obtain the first inequality given in (4.1). The second inequality in (4.1) can be deduced in a similar way as the proof of Theorem 3.2. □

Remark 4.4. Under the same assumptions of Theorem 4.3 with $n = 1$, then the following inequalities hold:

$$\begin{aligned} &\left|(b-a)\,f(x)+\frac{1}{2}(b-a)\left(\frac{a+b}{2}-x\right)f'(x)-\int_a^b f(t)dt\right.\\ &\left.+\frac{1}{4}\left[f'(b)\,(b-x)^2-f'(a)\,(x-a)^2\right]\right|\\ \leq\;& \frac{1}{4}\left[(x-a)^2\bigvee_a^x (f')+(b-x)^2\bigvee_x^b (f')\right]\\ \leq\;& \frac{1}{4}R\,(x;1) \end{aligned}$$

which was given by Dragomir in [14] and where $R\,(x;1)$ is defined as in (4.2).

Remark 4.5. If we take $n = 2$ in (4.1), then we have

$$\begin{aligned} &\left|(b-a)\left(x-\frac{a+b}{2}\right)f'(x)-(b-a)\,f(x)+\int_a^b f(t)dt\right.\\ &\left.-\frac{f''(a)\,(x-a)^3+f''(b)\,(b-x)^3+f''(x)\left[(x-a)^3+(b-x)^3\right]}{12}\right|\\ \leq\;& \frac{1}{12}(x-a)^3\bigvee_a^x (f'')+\frac{1}{12}(b-x)^3\bigvee_x^b (f'')\\ \leq\;& \frac{1}{12}R\,(x;2) \end{aligned}$$

which was established by Budak et al. in [4] and where $R\,(x;2)$ is defined as in (4.2).

Theorem 4.6. *Let $f : I \to \mathbb{C}$ be a n time differentiable function on I° and $[a,b] \subset I^\circ$. If the nth derivative $f^{(n)}$ is of bounded variation on $[a,b]$, then, for*

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any $x \in [a,b]$, we have

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+1}(b-x)^{k+1} + (-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \right. \tag{4.6}$$
$$\left. +(-1)^n \int_a^b f(t)dt - f^{(n)}(x)\left[\frac{(x-a)^{n+1}}{(n+1)!} + (-1)^n \frac{(b-x)^{n+1}}{(n+1)!}\right]\right|$$
$$\leq \frac{(x-a)^{n+1}}{(n+1)!} \bigvee_a^x \left(f^{(n)}\right) + \frac{(b-x)^{n+1}}{(n+1)!} \bigvee_x^b \left(f^{(n)}\right)$$
$$\leq \frac{1}{(n+1)!} R(x;n)$$

where $R(x;n)$ is defined as in (4.2).

Proof. Taking modulus in (2.1) for $\lambda_1(x) = \lambda_2(x) = f^{(n)}(x)$, we get the inequality

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+1}(b-x)^{k+1} + (-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) + (-1)^n \int_a^b f(t)dt \right.$$
$$\left. -f^{(n)}(x)\left[\frac{(x-a)^{n+1}}{(n+1)!} + (-1)^n \frac{(b-x)^{n+1}}{(n+1)!}\right]\right|$$
$$\leq \int_a^x \frac{(t-a)^n}{n!} \left|f^{(n)}(t) - f^{(n)}(x)\right| dt + \int_x^b \frac{(b-t)^n}{n!} \left|f^{(n)}(t) - f^{(n)}(x)\right| dt.$$

Then, using the methods applying in the proof of Theorem 4.3, the first and second inequalities are easily deduced. □

Remark 4.7. Under the same assumptions of Theorem 4.6 with $n = 1$, then the following inequalities hold:

$$\left|(b-a)f(x) - f'(x)\left[\frac{(x-a)^2}{2} - \frac{(b-x)^2}{2}\right] - \int_a^b f(t)dt\right|$$
$$\leq \frac{(x-a)^2}{2} \bigvee_a^x \left(f^{(n)}\right) + \frac{(b-x)^2}{2} \bigvee_x^b \left(f^{(n)}\right)$$
$$\leq \frac{1}{2} R(x;1)$$

which was given by Dragomir in [15] and where $R(x;1)$ is defined as in (4.2).

Remark 4.8. If we take $n=2$ in (4.6), then we get

$$\begin{aligned}
&\left|(b-a)\left(x-\frac{a+b}{2}\right)f'(x)-(b-a)f(x)-\int_a^b f(t)dt\right.\\
&\left.-\frac{1}{6}f''(x)\left[(x-a)^3+(b-x)^3\right]\right|\\
\leq\ & \frac{(x-a)^3}{6}\bigvee_a^x\left(f^{(n)}\right)+\frac{(b-x)^3}{6}\bigvee_x^b\left(f^{(n)}\right)\\
\leq\ & \frac{1}{6}R(x;2)
\end{aligned}$$

which was proved by Erden et al. in [16] and where $R(x;2)$ is defined as in (4.2).

Theorem 4.9. *Let $f:I\to\mathbb{C}$ be a n time differentiable function on I° and $[a,b]\subset I^\circ$. If the nth derivative $f^{(n)}$ is of bounded variation on $[a,b]$, then we have*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+1}(b-x)^{k+1}+(-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)\right. \qquad (4.7)\\
&\left.+(-1)^n\int_a^b f(t)dt-\frac{(x-a)^{n+1}f^{(n)}(a)+(-1)^n(b-x)^{n+1}f^{(n)}(b)}{(n+1)!}\right|\\
\leq\ & \begin{cases}
\frac{(x-a)^{n+1}}{(n+1)!}\bigvee_a^x\left(f^{(n)}\right)+\frac{(b-x)^{n+1}}{(n+1)!}\bigvee_x^b\left(f^{(n)}\right)\\
\frac{\left[(x-a)^{nq+1}+(b-x)^{nq+1}\right]^{\frac{1}{q}}}{n!(nq+1)^{\frac{1}{q}}}\left[(x-a)\left(\bigvee_a^x\left(f^{(n)}\right)\right)^p dt+(b-x)\left(\bigvee_x^b\left(f^{(n)}\right)\right)^p\right]^{\frac{1}{p}}\\
\frac{(x-a)^{n+1}+(b-x)^{n+1}}{n!}\left[\frac{1}{2}\bigvee_a^b\left(f^{(n)}\right)+\frac{1}{2}\left|\bigvee_a^x\left(f^{(n)}\right)-\bigvee_x^b\left(f^{(n)}\right)\right|\right]
\end{cases}
\end{aligned}$$

for any $x\in[a,b]$.

Proof. If we choose $\lambda_1(x)=f^{(n)}(a)$ and $\lambda_2(x)=f^{(n)}(b)$ and then we take modulus in (2.1), we get

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+1}(b-x)^{k+1}+(-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)+(-1)^n\int_a^b f(t)dt\right.\\
&\left.-\frac{(x-a)^{n+1}f^{(n)}(a)+(-1)^n(b-x)^{n+1}f^{(n)}(b)}{(n+1)!}\right|\\
\leq\ & \int_a^x\frac{(t-a)^n}{n!}\left|f^{(n)}(t)-f^{(n)}(a)\right|dt+\int_x^b\frac{(b-t)^n}{n!}\left|f^{(n)}(b)-f^{(n)}(t)\right|dt
\end{aligned}$$

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Since the nth derivative $f^{(n)}:I^{\circ}\to\mathbb{C}$ is of bounded variation on $[a,x]$, we have

$$\left|f^{(n)}(t)-f^{(n)}(a)\right|\leq\bigvee_{a}^{t}\left(f^{(n)}\right)\qquad\text{for any } t\in[a.x].$$

Hence, we can easily obtain the inequalities

$$\int_{a}^{x}\frac{(t-a)^{n}}{n!}\left|f^{(n)}(t)-f^{(n)}(a)\right|dt\leq\begin{cases}\frac{(x-a)^{n+1}}{(n+1)!}\bigvee_{a}^{x}\left(f^{(n)}\right)\\ \frac{(x-a)^{n+\frac{1}{q}}}{n!(nq+1)^{\frac{1}{q}}}\left(\int_{a}^{x}\left(\bigvee_{a}^{t}\left(f^{(n)}\right)\right)^{p}dt\right)^{\frac{1}{p}}\\ \frac{(x-a)^{n}}{n!}\int_{a}^{x}\left(\bigvee_{a}^{t}\left(f^{(n)}\right)\right)dt\end{cases}\tag{4.8}$$

In a similar way, because we have the inequality

$$\left|f^{(n)}(b)-f^{(n)}(t)\right|\leq\bigvee_{t}^{b}\left(f^{(n)}\right)\qquad\text{for any } t\in[x.b],$$

we get

$$\int_{x}^{b}\frac{(b-t)^{n}}{n!}\left|f^{(n)}(b)-f^{(n)}(t)\right|dt\leq\begin{cases}\frac{(b-x)^{n+1}}{(n+1)!}\bigvee_{x}^{b}\left(f^{(n)}\right)\\ \frac{(b-x)^{n+\frac{1}{q}}}{n!(nq+1)^{\frac{1}{q}}}\left(\int_{x}^{b}\left(\bigvee_{t}^{b}\left(f^{(n)}\right)\right)^{p}dt\right)^{\frac{1}{p}}\\ \frac{(b-x)^{n}}{n!}\int_{x}^{b}\left(\bigvee_{t}^{b}\left(f^{(n)}\right)\right)dt.\end{cases}\tag{4.9}$$

Finally, if we use the expressions (4.8) and (4.9), then we obtain the inequality (4.7) which completes the proof. □

Remark 4.10. If we apply the method using in the theorems 4.3 or 4.6 for first branch of the inequality (4.7), then we have

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+1}(b-x)^{k+1}+(-1)^{n+1-k}(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)+(-1)^{n}\int_{a}^{b}f(t)dt\right.\\&\left.-\frac{(x-a)^{n+1}f^{(n)}(a)+(-1)^{n}(b-x)^{n+1}f^{(n)}(b)}{(n+1)!}\right|\\\leq\;&\frac{(x-a)^{n+1}}{(n+1)!}\bigvee_{a}^{x}\left(f^{(n)}\right)+\frac{(b-x)^{n+1}}{(n+1)!}\bigvee_{x}^{b}\left(f^{(n)}\right)\\\leq\;&\frac{1}{(n+1)!}R(x;n).\end{aligned}$$

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References

1. H. Budak, M. Z. Sarıkaya, New Weighted Ostrowski Type Inequalities for Mappings with First Derivatives of Bounded Variation, *TJMM*, **8**(1), (2016), 21-27.
2. H. Budak, M. Z. Sarikaya, A New Ostrowski Type Inequality for Functions whose First Derivatives are of Bounded Variation, *Moroccan J. Pure Appl. Anal.*, **2**(1), (2016), 1–11.
3. H. Budak, S. Erden, M. Z. Sarikaya, New Weighted Ostrowski Type Inequalities for Mappings whose nth Derivatıves Are of Bounded Variation, *International Journal of Analysis and Applications*, **12**(1), (2016), 71-79.
4. H. Budak, M. Z. Sarikaya, S. S. Dragomir, Some Perturbed Ostrowski Type Inequality for Twice Differentiable Functions, *Advances in Mathematical inequalities and Applications*, Sipringer, 2018, 279-294.
5. H. Budak, M. Z. Sarikaya, Some Perturbed Ostrowski Type Inequality for Functions whose First Derivatives Are of Bounded Variation, *International Journal of Analysis and Applications*, **11**(2), (2016), 146-156.
6. P. Cerone, S. S. Dragomir, J. Roumeliotis, An Inequality of Ostrowski Type for Mappings whose Second Derivatives are Bounded and Applications, *RGMIA Res. Rep. Coll.*, **1**(1), (1998), Article 4.
7. P. Cerone, S. S. Dragomir, J. Roumeliotis, Some Ostrowski Type Inequalities for n-time Differentiable Mappings and Applications, *Demonstratio Math.,* **32**(4), (1999), 697-712.
8. S. S. Dragomir, N. S. Barnett, An Ostrowski Type Inequality for Mappings whose Second Derivatives are Bounded and Applications, *RGMIA Research Report Collection,* **1**(2), (1998) .
9. S. S. Dragomir, The Ostrowski Integral Inequality for Mappings of Bounded Variation, *Bulletin of the Australian Mathematical Society*, **60**(1), (1999), 495-508.
10. S. S. Dragomir, A. Sofo, An Integral Inequality for Twice Differentiable Mappings and Application, *Tamkang J. Math.*, **31**(4), 2000.
11. S. S. Dragomir, On the Ostrowski's Integral Inequality for Mappings with Bounded Variation and Applications, *Mathematical Inequalities & Applications*, **4**(1), (2001), 59–66.
12. S. S. Dragomir, Approximating Real Functions which Possess nth Derivatives of Bounded Variation and Applications, *Computers and Mathematics with Applications,* **56**, (2008) 2268–2278.
13. S. S. Dragomir, Some Perturbed Ostrowski Type Inequalities for Functions of Bounded Variation, *Asian-European Journal of Mathematics*, **8**(4), (2015,), 14 pages. DOI:10.1142/S1793557115500692.
14. S. S. Dragomir, Some Perturbed Ostrowski Type Inequalities for Absolutely Continuous Functions (I), *Acta Universitatis Matthiae Belii, series Mathematics,* **23**, (2015), 71–86.
15. S. S. Dragomir, Some Perturbed Ostrowski Type Inequalities for Absolutely Continuous Functions (II), *Acta Universitatis Apulensis*, **43**, (2015), 209-228.
16. S. Erden, H. Budak, M. Z. Sarıkaya, Some Perturbed Inequalities of Ostrowski Type for Twice Differentiable Functions, *RGMIA Research Report Collection,* **19**, (2016), Article 70, 11 pp.
17. Z. Liu, Some Ostrowski Type Inequalities, *Mathematical and Computer Modelling,* **48**, (2008) 949–960.
18. A. M. Ostrowski, Über Die Absolutabweichung Einer Differentiebaren Funktion von Ihrem Integralmitelwert, *Comment. Math. Helv.,* **10**, (1938), 226-227.

[DOI: 10.52547/ijmsi.17.1.57]

19. M. Z. Sarikaya, E. Set, On New Ostrowski Type Integral Inequalities, *Thai Journal of Mathematics*, **12**(1), (2014) 145-154.
20. A. Sofo, Integral Inequalities for n- times Differentiable Mappings, with Multiple Branches,on the L_p Norm, *Soochow Journal of Mathematics*, **28**(2), (2002), 179-221.
21. M. Wang, X. Zhao, Ostrowski Type Inequalities for Higher-order Derivatives, *J. of Inequalities and App.*, (2009), Article ID 162689, 8 pages.

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