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An Optimal Algorithm for the δ-ziti Method to Solve Some Mathematical Problems

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Abstract. The numerical approximation methods of the differential problems solution are numerous and various. Their classifications are based on several criteria: Consistency, precision, stability, convergence, dispersion, diffusion, speed and many others. For this reason a great interest must be given to the construction and the study of the associated algorithm: indeed the algorithm must be simple, robust, less expensive and fast. In this paper, after having recalled the δ-ziti method, we reformulat it to obtain an algorithm that does not require as many calculations as many nodes knowing that they are counted by thousands. We have, therefore, managed to optimize the number of iterations by passing for example from 10^3 at 10 iterations.

Keywords: Algorithm, Meshing, $\delta-$ziti, Optimal, Operations number.

2000 Mathematics subject classification: 65D15, 65N06, 65Q10.

1. Introduction

The δ-ziti is a new method of approximation, which allows:

(1) to interpolate, approach and integrate a function of one or more variables,

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(2) to solve a differential equation of any order,
(3) to solve an equation or a system with partial derivatives.

Generally, in concrete problems, one confronts with singularities that generate horrible instabilities in the form of oscillations where the classical numerical methods are very sensitive [5](ie improper integrals (generalized), functions which are not sufficiently regular or distributions as Dirac type , singular differential problem, presence of natural phenomena such as shocks, boundary layers, Blow-up, turbulence and others ...). The δ-ziti method has shown its effectiveness in solving such problems. When presenting the δ-ziti method in [1, 2, 3, 4], we did several tests on several mathematical models. We have established algorithms without considering the cost of calculation and their complications. In this work, we will show how to minimize the cost of calculation by a very small number of iterations. First, we will recall, in the following paragraph, the main lines of the δ-ziti method.

2. Overview of the δ-ziti method.

As it was presented in [3], the method is based, essentially on the famous function

$$\phi(x) = \begin{cases} exp\left(\frac{1}{|x|^2 - R^2}\right) & ; if\ |x|^2 := \Sigma_{i=1}^n x_i^2 < R, \\ 0 & otherwise. \end{cases} \tag{2.1}$$

where R is a positive constant.

This function is very often encountered in Numerical Analysis, especially in distributions and functional analysis. It is characterized by its power to approach the Dirac measurement:

indeed, if we put,

$$\varphi_\epsilon(x) = \frac{C}{\epsilon^n}.\phi\left(\frac{x}{\epsilon}\right)\ \forall \epsilon > 0, \tag{2.2}$$

where, $C := \frac{1}{\int_{\mathbb{R}^n} \phi(x)dx}$, then this sequence φ_ϵ converges to Dirac in the sense of distributions.

The Dirac distribution is a very significant symbol of a singularity, which made us think of using $\phi(x)$ to deal with singularities. At first, we are interested in the one dimensional case (n = 1). For the multidimensional case, we will see how we can reduce ourselves to the dimensional one. Consider, for the moment, an interval bounded [a, b]. We consider a uniform meshing, the interval [a, b] is subdivided into $(m + 1)$ points (where m is the number of subintervals: $(m + 1)$ nodes) with a constant step h: $x_1 = a$, $x_{m+1} = b$, $x_i = a + (i - 1)h$ for $i = 1, \ldots, m + 1$.

where, $h = \frac{b-a}{m}$.

At each x_i, we associate the function φ_i defined by:

$$\varphi_i(x) = \frac{C}{h}.\phi\left(\frac{x - x_i}{h}\right)\ for\ x \in [x_{i-1}, x_{i+1}]\ (i = 2, \ldots, m), \tag{2.3}$$

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$$\varphi_1(x) = \frac{C}{h}.\phi\left(\frac{x-x_1}{h}\right) \ for\ x \in [x_1, x_2], \tag{2.4}$$

and,

$$\varphi_{m+1}(x) = \frac{C}{h}.\phi\left(\frac{x-x_{m+1}}{h}\right) \ for\ x \in [x_m, x_{m+1}]. \tag{2.5}$$

This family of functions is represented in the following figures:

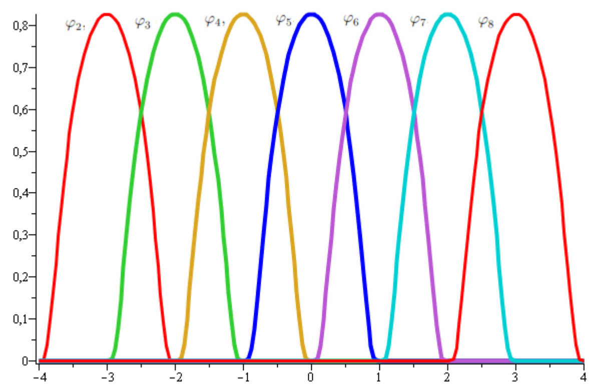

FIGURE 1. Superposition of some elements of the family $(\varphi_i)_{2\leq i\leq m}$ for a=-4,b=4,h=1 and C=2.252283621.

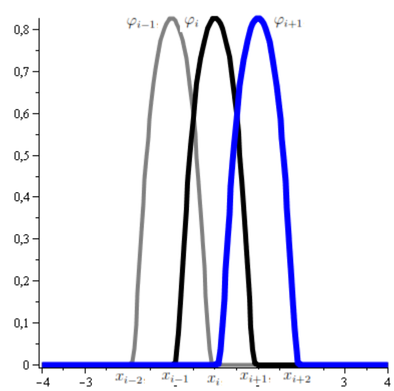

FIGURE 2. Superposition of the consecutive elements φ_{i-1}, φ_i and φ_{i+1} for a=-4,b=4,h=1 and C=2.252283621.

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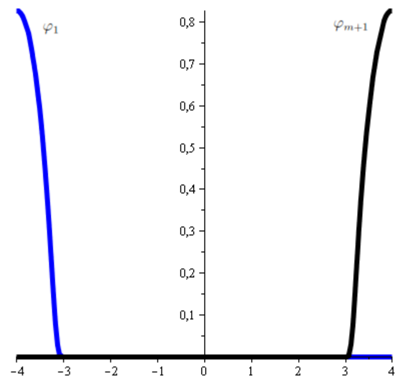

FIGURE 3. Graphical representation of extreme elements φ_1 and φ_{m+1} for a=-4,b=4,h=1 and C=2.252283621.

By choosing a suitable Prehilbert space with the scalar product $<,>$, the norm associated is $\|.\|$, we use the Gram-Schmidt process, to construct the orthogonal family $\left(\tilde{\psi}_i\right)_i$.

We have shown that the family $\left(\tilde{\psi}_i\right)_i$ satisfies the following recurrence relation:

$$\left\{\begin{array}{l} \tilde{\psi}_1(x) = \varphi_1(x)\ for\ all\ x \in\ [x_1, x_{m+1}] \\ \tilde{\psi}_i(x) = \varphi_i(x) + \lambda_{i-1}\tilde{\psi}_{i-1}(x)\ for\ all\ x \in\ [x_1, x_{m+1}],\ 2 \leq i \leq m+1 \\ supp\tilde{\psi}_i = [x_1, x_{i+1}]\ pour\ tout\ indice\ i, \\ with:\ \lambda_{i-1} = -\frac{<\varphi_i, \tilde{\psi}_{i-1}>}{\left\|\tilde{\psi}_{i-1}\right\|^2}, \end{array}\right. \tag{2.6}$$

and that

$$-1 < \lambda_k < 0\ for\ 1 \leq k \leq m,$$

From this relation of recurrence, one can develop $\tilde{\psi}_i$ according to φ_k $(1 \leq k \leq i)$:

$$\tilde{\psi}_i(x) = \varphi_i(x) + \lambda_{i-1}\varphi_{i-1}(x) + \lambda_{i-1}\lambda_{i-2}\varphi_{i-2}(x) + \cdots + \lambda_{i-1}\lambda_{i-2}\cdots\lambda_1\varphi_1(x).$$

We construct an algorithm with (2.6) to compute $\tilde{\psi}_i$, λ_i and $<\tilde{\psi}_i, \tilde{\psi}_i>$.

The functions $\left(\tilde{\psi}_i\right)_i$ are represented in the following figures: Finally, we orthonormalize $\left(\tilde{\psi}_i\right)$ to obtain the family (ψ_i) :

$$\psi_i(x) = \frac{\tilde{\psi}_i(x)}{\left\|\tilde{\psi}_i\right\|}\ pour\ 1 \leq i \leq m+1 \tag{2.7}$$

Remark 2.1. The algorithm we are going to build aims to calculate λ_i, ψ_i and roots r_i and to simulate the results.

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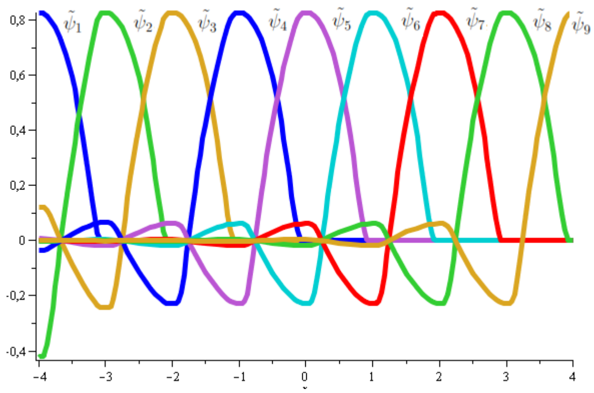

FIGURE 4. Superposition of some elements of the orthogonal family $\left(\tilde{\psi}_i\right)_{1\leq i\leq m+1}$ for a=-4,b=4,h=1 and c=0.1330861208.

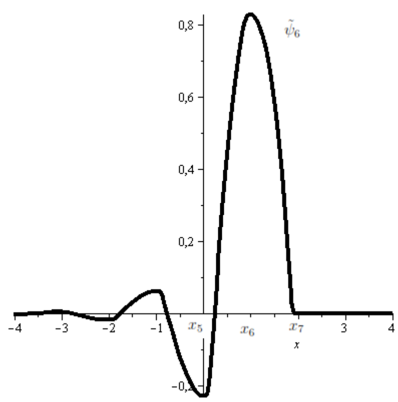

FIGURE 5. Graphical representation of $\tilde{\psi}_6$ for a=-4,b=4,h=1 and c=0.1330861208.

2.1. **algorithm 1.**
$a = -4$, $b = 4$; (the border of the interval [a, b])
$m = 800$; (number of intervals after subdivision of [a, b])
$h = \frac{b-a}{m}$; (the step of discretization)
Discretization of [a, b]

$$\begin{cases} for\ i = 1 : m+1 \\ x_i = a + (i-1)h \\ \quad end\ i \end{cases}$$

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Construction of (φ_i)

$$\phi(x) = \begin{cases} exp\left(\frac{1}{|x|^2-R^2}\right) & ; if\ |x|^2 := \Sigma_{i=1}^{n} x_i^2 < R, \\ 0 & otherwise. \end{cases}$$

$$\varphi_1(x) = \frac{C}{h}.\phi\left(\frac{x-x_1}{h}\right)\ for\ x \in [x_1, x_2],$$

$$\begin{cases} for\ i =\ 2:m \\ \varphi_i(x) = \frac{C}{h}.\phi\left(\frac{x-x_i}{h}\right)\ for\ x \in [x_{i-1}, x_{i+1}] \\ end\ i \end{cases}$$

$$\varphi_{m+1}(x) = \frac{C}{h}.\phi\left(\frac{x-x_{m+1}}{h}\right)\ for\ x \in [x_m, x_{m+1}].$$

Construction of $\left(\tilde{\psi}_i\right)$

$$\begin{cases} for\ i = 1 \\ \tilde{\psi}_1(x) = \varphi_1(x), \\ \|\tilde{\psi}_1\|^2 = \int_a^b \tilde{\psi}_1(x)\tilde{\psi}_1(x)dx, \\ \lambda_1 = \frac{\int_a^b \varphi_1(x)\varphi_2(x)dx}{\|\tilde{\psi}_1\|^2} \end{cases}$$

$$\begin{cases} for\ i = 2:m \\ \tilde{\psi}_i(x) = \varphi_i(x) + \lambda_{i-1}\tilde{\psi}_{i-1}(x), \\ \|\tilde{\psi}_i\|^2 = \int_a^b \tilde{\psi}_i(x)\tilde{\psi}_i(x)dx, \\ \lambda_i = \frac{\int_a^b \varphi_i(x)\varphi_{i+1}(x)dx}{\|\tilde{\psi}_i\|^2}, \\ end\ i \end{cases}$$

$$\begin{cases} for\ i =\ m+1 \\ \tilde{\psi}_{m+1}(x) = \varphi_{m+1}(x) + \lambda_m\tilde{\psi}_m(x), \\ \|\tilde{\psi}_{m+1}\|^2 = \int_a^b \tilde{\psi}_{m+1}(x)\tilde{\psi}_{m+1}(x)dx, \end{cases}$$

Calculating the roots of $\left(\tilde{\psi}_i\right)$

$$r = Roots(\tilde{\psi}_i(x), x = a..b)$$

Remark 2.2. To compute any integral, we use any approximation's method as Simpson-method for example.

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i	λ_i	r_i
1	-0.508960181	-3.996667413
2	-0.292344636	-3.987512758
3	-0.274934054	-3.977583248
4	-0.273624280	-3.967588576
5	-0.273526252	-3.957588975
6	-0.273518918	-3.947589005
7	-0.273518369	-3.937589007
8	-0.273518328	-3.927589007
9	-0.273518325	-3.917589007
10	-0.273518325	-3.907589007
11	-0.273518325	-3.897589007
12	-0.273518325	-3.887589007
13	-0.273518325	-3.877589007
14	-0.292344636	-3.987512758
$\cdots$	$\cdots$	$\cdots$
795	-0.273518325	3.942410993
796	-0.273518325	3.952410993
797	-0.273518325	3.962410993
798	-0.273518325	3.972410993
799	-0.273518325	3.982410993
800	-0.273518325	3.992410993
801	$---$	4

TABLE 1. Results obtained by the first algorithm with 801 nodes.

2.2. Roots of ψ_i.

The δ-ziti method mainly uses the roots (r_i) of the functions $(\psi_i)_i$ instead of the mesh point (x_i). It is not a question of replacing (x_i) by the roots (r_i) but

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it is about a radical transformation of the approximation.

We studied minorly, the roots (r_i) and we showed that

(1) the roots of ψ_i is, also, root of ψ_k for: $k \geq i-1$, in particular are also roots of ψ_{m+1}
(2) ψ_i admits (i-1) distinct real root(s) r_k $(k = 1; \cdots ; i-1)$ in the interval $]x_1, x_i[$ precisely $r_k \in]x_k, x_{k+1}[$ for $k = 1, \ldots, i-1$,
(3) $r_j \in]x_j, x_j + \frac{h}{2}[$,
(4)
$$\lambda_{i-1} = -\frac{\varphi_{i+1}(r_i)}{\varphi_i(r_i)}. \tag{2.8}$$
What allowed us to write
$$r_k = x_k + h.X_k, \tag{2.9}$$
where: X_k is solution of the following polynom
$$\Lambda_k X^4 - 2\Lambda_k X^3 - \Lambda_k X^2 + 2(\Lambda_k - 1)X + 1 = 0, \tag{2.10}$$
where:$\Lambda_k = Ln(-\lambda_k)$ and $\lambda_{i-1} = -\frac{<\varphi_i, \varphi_{i-1}>}{\left\|\tilde{\psi}_{i-1}\right\|^2}$.

Remark 2.3. A second algorithm differs from the previous one can be built using the results (2.8), (2.9) and (2.10) but still with always the same number of iterations as the number of nodes.

The functions (ψ_i) and the position of their roots are represented in the following figures:

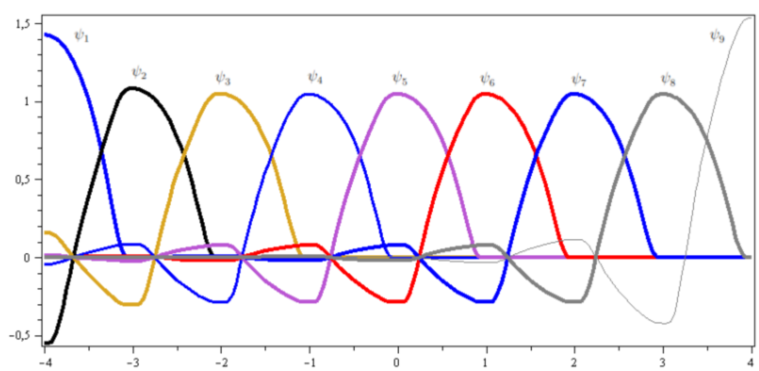

FIGURE 6. Superposition of some elements of the family $(\psi_k)_{1\leq k\leq m+1}$ and position of their roots for a=-4,b=4,h=1 and c=0.1330861208.

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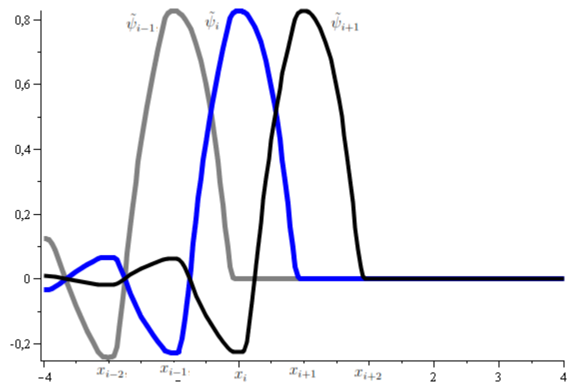

FIGURE 7. Superposition of consecutive elements ψ_{i-1}, ψ_i, ψ_{i+1} and position of its roots for a=-4,b=4,h=1 and c=0.1330861208.

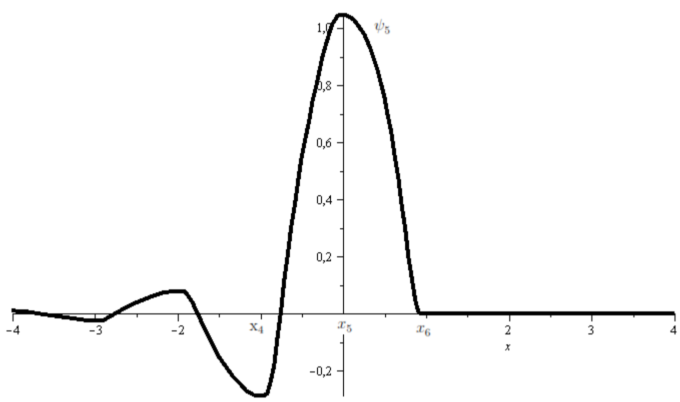

FIGURE 8. Representation of element ψ_5 and position of its roots for a=-4, b=4, h=1 and C=2.252283621.

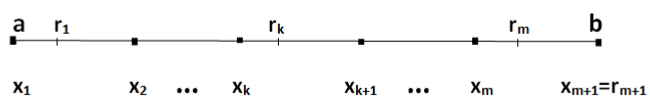

FIGURE 9. The position of the roots r_i of ψ_{m+1}.

2.3. **Description of δ-ziti method.**
The δ-ziti method starts, like the finite element method [6, 7, 8], with a variational formulation to have an adequate functional framework. To avoid the mesh $(x_i\ )$ and its complications in the variational formulation, we used the

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roots (r_i) of the extreme function ψ_{m+1} which made it possible to give a better approximation of a function and an integral one by a simple formula.

Remark 2.4. The use of only ψ_{m+1} and not the others ψ_i is because every root of ψ_i is also root of ψ_{m+1}.

2.3.1. *Interpolation and approximation of a function with a single variable.*
For a given function f, we approached it as follows:

$$f(x) \simeq \sum_{i=1}^{i=m+1} \alpha_i \psi_i(x). \tag{2.11}$$

where,

$$\alpha_i =< f,\ \psi_i >,$$

We showed that,

$$\alpha_i = \frac{f(r_i)}{\psi_i(r_i)}.$$

2.3.2. *The numerical integration.*
We used the following approximation:

$$\int_a^b f(x)\psi_i(x)dx \simeq \frac{f(r_i)}{\psi_i(r_i)}\ for\ i = 1...m+1. \tag{2.12}$$

$$\int_a^b f(x)dx \simeq \sum_{i=1}^{i=m+1} \frac{f(r_i)}{\psi_i^2(r_i)}. \tag{2.13}$$

2.3.3. *Function with several variables.*
Let us take, Ω in the form: $\Omega = [a,b] \times [c,d]$, and use an uniform mesh of the interval $[a,b]$ and $[c,d]$ with the step $h_1 = \frac{b-a}{m_1}$ and $h_2 = \frac{d-c}{m_2}$ where m_1, m_2 are integers such $x_1 = a$, $x_{m_1+1} = b$ and $y_1 = c$, $y_{m_2+1} = d$, $x_k = a+(k-1)h_1$ for: $k = 1, \ldots, m_1+1$, $y_k = c+(l-1)h_2$ for: $l = 1, \ldots, m_2+1$.
From (2.6) and (2.7), we construct two orthonormal families:

$$(\psi_k^1)_{1\leq k\leq m+1}\ and\ (\psi_k^2)_{1\leq k\leq m+1}$$

Our strategy is to take:

$$\psi_{i,j}(x,y) = \psi_i^1(x)\psi_j^2(y)\ for\ (x,y) \in \Omega. \tag{2.14}$$

- **i: Approximation of a function of two variables.**
  Let f be a function of two variables, $f(x,y)$ can be approximated by:

$$f(x,y) \simeq \sum_{i=1}^{m_1+1} \sum_{j=1}^{m_2+1} \alpha_{ij}\psi_i^1(x)\psi_j^2(y). \tag{2.15}$$

  where,

$$\alpha_{ij} = \frac{f(r_i^1, r_j^2)}{\psi_i^1(r_i^1)\psi_j^2(r_j^2)}.$$

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ii: The numerical integration.

We have approached a multiple integral as follows:

$$\int_a^b \int_c^d f(x,y)\psi_i^1(x)\psi_j^2(y)dxdy \simeq \frac{f(r_i^1, r_j^2)}{\psi_i^1(r_i^1)\psi_j^2(r_j^2)},$$

$$\int_a^b \int_c^d f(x,y)dxdy \simeq \sum_{i=1}^{m_1+1} \sum_{j=1}^{m_2+1} \frac{f(r_i^1, r_j^2)}{\left(\psi_i^1(r_i^1)\psi_j^2(r_j^2)\right)^2}.$$

Remark 2.5. The extension of the δ-ziti method with several variables is done in the same way:

i: Approximation of a function.

$$f(y_1, \cdots, y_n) \simeq \sum_{1\leq i_1\leq m_1+1} \cdots \sum_{1\leq i_n\leq m_n+1} \alpha_{i_1\cdots i_n}\psi_{i_1,\cdots,i_n}(y_1,\cdots,y_n),$$

where,

$$\alpha_{i_1\cdots i_n} = \frac{f(r_{i_1}^1, \cdots, r_{i_n}^n)}{\psi_{i_1,\cdots,i_n}(r_{i_1}^1, \cdots, i_{i_n}^n)}.$$

ii: The numerical integration.

We have approached a multiple integral as follows:

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1,\cdots,x_n)\psi_{i_1}^1(x_1)\cdots\psi_{i_n}^n(x_n)dx_1\cdots dx_n \simeq \frac{f(r_{i_1}^1,\cdots,r_{i_n}^n)}{\psi_i^1(r_i^1)\cdots\psi_i^n(r_{i_n}^n)},$$

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1,\cdots,x_n)dx_1\cdots dx_n \simeq \sum_{1\leq i_1\leq m_1+1} \cdots \sum_{1\leq i_n\leq m_n+1} \frac{f(r_{i_1}^1,\cdots,r_{i_n}^n)}{\left(\psi_i^1(r_i^1)\cdots\psi_i^n(r_{i_n}^n)\right)^2},$$

where, $r_{i_j}^j$ is the $(i_j)^{th}$ root of ψ_{m+1}^j on $[a_j, b_j]$ (for $i_j = 1, \ldots, m_j + 1$, $j = 1, \ldots, n$).

2.3.4. *Differential equation and partial differential equation.*
We consider, therefore, an ordinary differential equation of order 1,

$$(\mathcal{P}) \begin{cases} u'(x) = f(x, u(x)) \ ; x \in [a,b], & (1) \\ u(a) = \xi_0, & (2) \end{cases} \tag{2.16}$$

where, ξ_0 is a given constant, $u : \mathbb{R} \to \mathbb{R}$ is the unknown function and $f : \mathbb{R}^2 \to \mathbb{R}$ is a given function.
The variational formulation of $(\mathcal{P})$ consists to multiply the differential equation by ψ_k and integrating:

$$\int_a^b u'(x).\psi_i(x)dx = \int_a^b f(x, u(x)).\psi_i(x)dx \ for \ i = 1, \ldots, m + 1. \tag{2.17}$$

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The use of the previous integration formula (2.15) made it possible to ignore the points of meshing x_i and to take into account only the roots r_i.
From the integration formula (2.15) we have:

$$\int_a^b u'(x).\psi_i(x)dx \simeq \frac{u'(r_i)}{\psi_i(r_i)},$$

and,

$$\int_a^b f(x,u(x)).\psi_i(x)dx \simeq \frac{f(r_i,u(r_i))}{\psi_i(r_i)}.$$

Therefore,

$$u'(r_i) = f(r_i,u(r_i)) \ for \ i = 1,\ldots,m+1, \tag{2.18}$$

Remark 2.6. For the differential equations of order one, the δ-ziti method resembles, at first sight, the collocation method (moreover the finite element method also coincides with the collocation method in dimension one), which is not the case for the order greater than two.

If we approach the derivative $u'(r_k)$ in (2.18) by (for example) the following finite difference approximations:

$$u'(r_i) \simeq \frac{u(r_{i+1}) - u(r_i)}{r_{i+1} - r_i} \ for \ i = 1,\ldots,m,$$

and let's use the fact that,

$$u(r_i) = \alpha_i\psi_i(r_i) \ for \ i = 1,\ldots,m,$$

then the (α_i) will be solutions of the following discrete problem:

$$(\mathcal{S}) \left\{ \begin{array}{l} \alpha_1 = \frac{\xi_0}{\psi_1(r_1)} \\ \alpha_{i+1} = \frac{\alpha_i\psi_i(r_i)+(r_{i+1}-r_i)f(r_i,\alpha_i\psi_i(r_i))}{\psi_{i+1}(r_{i+1})} \ for \ i = 1,\ldots,m. \end{array} \right. \tag{2.19}$$

In all of the above, the calculations of r_i and λ_i are essential to solve (S) but with what cost?
In all our work where we used the δ-ziti method we obtained very satisfactory results and even surprising in comparison with the exact results or with the well-known numerical results. We performed the calculation by following the formulas found mathematically without taking into account their complications and the cost (for example the number of iterations). In the next paragraph we will reduce the cost considerably.

3. Reformulation of λ_i and r_i.

3.1. **Calculation of λ_i.**

We recall that,

$$\alpha =< \varphi_1, \varphi_2 >, \ \beta =< \varphi_1, \varphi_1 > .$$

The following theorem shows that (λ_i) follows a simple iterative process:

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Theorem 3.1. *λ_i verifies the following recurrence relation:*

$$\begin{array}{rcl} \lambda_1 & = & -\frac{\alpha}{\beta}\ , \\ \lambda_{i+1} & = & f\left(\lambda_i\right)\ pour\ 1 \leq i \leq m-1, \\ & & where\ f(x) = \frac{\lambda_1}{2-\lambda_1 x}. \end{array}$$

Proof. We have,

$$\lambda_1 = -\frac{< \varphi_1, \varphi_2 >}{\left\|\tilde{\psi}_1\right\|^2} = -\frac{< \varphi_1, \varphi_2 >}{\left\|\varphi_1\right\|^2} = -\frac{\alpha}{\beta},$$

and,

$$\lambda_i = -\frac{< \varphi_i, \varphi_{i+1} >}{\left\|\tilde{\psi}_i\right\|^2},$$

then,

$$< \tilde{\psi}_i, \tilde{\psi}_i >= -\frac{< \varphi_i, \varphi_{i+1} >}{\lambda_i}, \tag{3.1}$$

but,

$$\lambda_{i+1} = -\frac{< \varphi_{i+1}, \varphi_{i+2} >}{\left\|\tilde{\psi}_{i+1}\right\|^2},$$

from (2.6) we have,

$$< \varphi_{i+1}, \tilde{\psi}_i >=< \varphi_{i+1}, \varphi_i >,$$

and

$$\tilde{\psi}_{i+1}(x) = \varphi_{i+1}(x) + \lambda_i \tilde{\psi}_i(x).$$

Therefore,

$$\begin{array}{rcl} < \tilde{\psi}_{i+1}, \tilde{\psi}_{i+1} > & = & < \varphi_{i+1}, \varphi_{i+1} > +2\lambda_i < \varphi_{i+1}, \tilde{\psi}_i > +\lambda_i^2 < \tilde{\psi}_i, \tilde{\psi}_i > \\ & = & < \varphi_{i+1}, \varphi_{i+1} > +2\lambda_i < \varphi_{i+1}, \varphi_i > +\lambda_i^2 < \tilde{\psi}_i, \tilde{\psi}_i >, \end{array}$$

and

$$\lambda_{i+1} = -\frac{< \varphi_{i+1}, \varphi_{i+2} >}{< \varphi_{i+1}, \varphi_{i+1} > +2\lambda_i < \varphi_{i+1}, \varphi_i > +\lambda_i^2 < \tilde{\psi}_i, \tilde{\psi}_i >}. \tag{3.2}$$

By injecting (3.1) into (3.2), we obtain,

$$\lambda_{i+1} = -\frac{< \varphi_{i+1}, \varphi_{i+2} >}{< \varphi_{i+1}, \varphi_{i+1} > +\lambda_i < \varphi_{i+1}, \varphi_i >}, \tag{3.3}$$

according to [3], for every index i, we have (in fact φ_i is a translation of φ_2)

$$< \varphi_{i+1}, \varphi_i >=< \varphi_1, \varphi_2 >= \alpha\ et\ \ < \varphi_{i+1}, \varphi_{i+1} >= 2 < \varphi_1, \varphi_1 >= 2\beta,$$

(3.3) then becomes

$$\lambda_{i+1} = -\frac{\alpha}{2\beta + \lambda_i \alpha} = \frac{-\frac{\alpha}{\beta}}{2 + \lambda_i \frac{\alpha}{\beta}} = \frac{\lambda_1}{2 - \lambda_1 \lambda_i}.$$

□

The following result will make it possible to estimate $|\lambda_{i+1} - \lambda_i|$.

Corollary 3.2. *For $\lambda_1 = -\frac{\alpha}{\beta}$ we have,*

(1)
$$0 < \left(\frac{\lambda_1}{2+\lambda_1}\right)^2 < 1. \tag{3.4}$$

(2)
$$For\ all\ i\ such\ that\ i \geq \left[\frac{ln\left(\frac{\epsilon(2-\lambda_1^2)}{\lambda_1^3-\lambda_1}\right)}{ln\left(\frac{\lambda_1}{2+\lambda_1}\right)^2}\right] + 1, \tag{3.5}$$

we have
$$|\lambda_{i+1} - \lambda_i| < \epsilon,\ for\ any\ arbitrary\ smal\ real\ \epsilon > 0,$$

where [x] is the floor function of x.

Proof. (1) We know that
$$\lambda_1 = -\frac{\alpha}{\beta}\ and\ -1 < \lambda_1 < 0,$$
so,
$$0 < \frac{-\lambda_1}{2+\lambda_1} < 1.$$
Therefore,
$$0 < \left(\frac{\lambda_1}{2+\lambda_1}\right)^2 < 1.$$

(2) Let f be the function defined by:
$$f(X) = \frac{\lambda_1}{2-\lambda_1.X}\ avec\ -1 < X < 0,$$
it's easy to verify that
$$f^{''}(X) < 0\ pour\ tout\ X \in]-1, 0[.$$
Therefore, $f^{'}$ is strictly decreasing on $]-1, 0[$,
a simple calculation shows that
$$0 < \frac{\lambda_1^2}{4} = f^{'}(0) < f^{'}(X) < f^{'}(-1) = \frac{\lambda_1^2}{(2+\lambda_1)^2},$$
and
$$f^{'}(X) < \left(\frac{\lambda_1}{2+\lambda_1}\right)^2\ for\ all\ X \in]-1, 0[.$$
By applying the theorem of finite increments on $]\lambda_i, \lambda_{i+1}[$, we obtain,
$$\begin{array}{rcl} |\lambda_{i+1} - \lambda_i| & = & |f(\lambda_i) - f(\lambda_{i-1})|, \\ & = & |f^{'}(\xi)|\,|\lambda_i - \lambda_{i-1}|\ \xi\ entre\ \lambda_{i+1}\ et\ \lambda_i, \end{array}$$
Therefore,
$$\begin{array}{rcl} |\lambda_{i+1} - \lambda_i| & < & \left(\frac{\lambda_1}{2+\lambda_1}\right)^2 |f(\lambda_{i-1}) - f(\lambda_{i-2})|, \\ & < & ... \end{array}$$

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so,

$$|\lambda_{i+1}-\lambda_i| < \left(\frac{\lambda_1}{2+\lambda_1}\right)^{2(i-1)} \frac{\lambda_1(\lambda_1^2-1)}{2-\lambda_1^2}.$$

For the condition $|\lambda_{i+1}-\lambda_i| \leq \epsilon$ to be satisfactory, it suffices to take,

$$\left(\frac{\lambda_1}{2+\lambda_1}\right)^{2(i-1)} \frac{\lambda_1(\lambda_1^2-1)}{2-\lambda_1^2} \leq \epsilon,$$

that is to say

$$\left(\frac{\lambda_1}{2+\lambda_1}\right)^{2(i-1)} \leq \frac{\epsilon}{\frac{\lambda_1(\lambda_1^2-1)}{2-\lambda_1^2}}. \tag{3.6}$$

Using the function Ln in (3.6) and the fact that

$$(i-1)ln\left(\left(\frac{\lambda_1}{2+\lambda_1}\right)^2\right) \leq ln\left(\frac{\epsilon(2-\lambda_1^2)}{\lambda_1(\lambda_1^2-1)}\right),$$

so,

$$0 < \left(\frac{\lambda_1}{2+\lambda_1}\right)^2 < 1,$$

(3.6) becomes

$$i \geq \frac{ln\left(\frac{\epsilon(2-\lambda_1^2)}{\lambda_1^3-\lambda_1}\right)}{ln\left(\frac{\lambda_1}{2+\lambda_1}\right)^2} + 1.$$

So $|\lambda_{i+1}-\lambda_i| < \epsilon$, starting from

$$N_0 = \left[\frac{ln\left(\frac{\epsilon(2-\lambda_1^2)}{\lambda_1^3-\lambda_1}\right)}{ln\left(\frac{\lambda_1}{2+\lambda_1}\right)^2}\right] + 1.$$

where [x] is the floor function of x. □

In practice, we take $\epsilon = 10^{-N}$, we then show the following result:

Corollary 3.3. *Once*

$$i \geq \left[\frac{-Nln(10) + ln\left(\frac{(2-\lambda_1^2)}{\lambda_1^3-\lambda_1}\right)}{ln\left(\frac{\lambda_1}{2+\lambda_1}\right)^2}\right] + 1, \tag{3.7}$$

we have

$$|\lambda_{i+1}-\lambda_i| < 10^{-N}$$

where [x] is the floor function of x.

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3.1.1. *Idea on some estimates of N_0.*
For

$$\begin{array}{lcl} m = 1000, & & \\ \alpha & = & <\varphi_1, \varphi_2>= 21,47547348, \\ \beta & = & <\varphi_1, \varphi_1>= 42,1948008, \\ \lambda_1 & = & -\frac{\alpha}{\beta} = -050896018165347, \\ N_0 & = & \left[\frac{-N ln(10) + ln\left(\frac{(2-\lambda_1^2)}{\lambda_1^3-\lambda_1}\right)}{ln\left(\frac{\lambda_1}{2+\lambda_1}\right)^2} \right] + 1. \end{array}$$

$Precision\ 10^{-N}$	$iteration\ number\ N_0$
$1 \leq N \leq 10$	N
$11 \leq N \leq 24$	$N+1$
$25 \leq N \leq 38$	$N+2$
$39 \leq N \leq 52$	$N+3$
$53 \leq N \leq 66$	$N+4$
$67 \leq N \leq 80$	$N+5$
$81 \leq N \leq 94$	$N+6$
$95 \leq N \leq 108$	$N+7$
$109 \leq N \leq 122$	$N+8$
$123 \leq N \leq 136$	$N+9$

TABLE 2. Examples the calculation of N_0: to reduce the iteration number.

Corollary 3.4. *(λ_i) is stationary from N_0.*

Proof. if $i \geq N_0$
we have

$$|\lambda_{i+1} - \lambda_i| < 10^{-N},$$

so

$$|\lambda_{N_0+p} - \lambda_{N_0}| < |\lambda_{N_0+p} - \lambda_{N_0+p-1}| + ... + |\lambda_{N_0+1} - \lambda_{N_0}| .$$

Therefore,

$$|\lambda_{N_0+p} - \lambda_{N_0}| < p10^{-N},$$

where, $p = m - N_0$. □

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EXAMPLE 3.5. For $m = 10^3$, $N = N_0 = 10$,

$$|\lambda_{N_0+p} - \lambda_{N_0}| < 0,999.10^{-7} < 10^{-7}.$$

3.2. **Calculation of r_i.**

The theorem makes it possible to calculate the values of the roots r_i.

Theorem 3.6. *For $i = 2, \ldots, m+1$ et $k = 1, \ldots, i-1$ if we put $r_k = x_k + h.X_k$, then, X_k is the solution of ,*

$$\Lambda_k X^4 - 2\Lambda_k X^3 - \Lambda_k X^2 + 2(\Lambda_k - 1)X + 1 = 0, \tag{3.8}$$

where: $X_k \in \left]0, \frac{1}{2}\right[$ and $\Lambda_k = Ln(-\lambda_k)$.

Remark 3.7.

- The fact that the function:
$$F(X) = \Lambda_k X^4 - 2\Lambda_k X^3 - \Lambda_k X^2 + 2(\Lambda_k - 1)X + 1,$$
is strictly decreasing and concave on $[0, \frac{1}{2}]$, then any numerical approximation method of the root of F is valid.
- The fact that $\Lambda_k = Ln(-\lambda_k)$ and since, from N_0 the sequence λ_i is considered as a stationary sequence $(|\lambda_{i+1} - \lambda_i| < 10^{-N_0})$, then all the functions $F(X)$ are the same for $k \geq N_0$.

All the calculations that were done with the previous algorithm work with a very large number of nodes m (m = 10000) which gave very large tables (table1) as a database to do the following.

Now, we will reduce the number of calculations: from m to N_0 depending on the precision required.

4. δ-ZITI ALGORITHM.

We give ourselves a precision $\epsilon = 10^{-N}$ (ϵ or N),
We take the step of subdivision h of the interval [a,b], (a,b,m),
We compute

$$\begin{aligned}
\alpha &= <\varphi_1, \varphi_2>, \\
\beta &= <\varphi_1, \varphi_1>, \\
\lambda_1 &= -\frac{\alpha}{\beta}\ , \\
N_0 &= \left[\frac{-N ln(10) + ln\left(\frac{(2-\lambda_1^2)}{\lambda_1^3 - \lambda_1}\right)}{ln\left(\frac{\lambda_1}{2+\lambda_1}\right)^2} \right] + 1.
\end{aligned}$$

$$\begin{cases} For\ i = 1, N_0 - 1 & \\ & \lambda_{i+1} = \frac{\lambda_1}{2 - \lambda_1 \lambda_i}. \\ \quad End\ i & \end{cases} \tag{4.1}$$

For $i \geq N_0 + 1$

$$\lambda_i \simeq \lambda_{N_0}$$

$$\left\{\begin{array}{l} For\ k=1:N_0 \\ \qquad\qquad Calculation\ of\ the\ root\ X_k\ of\ the\ function \\ \qquad\qquad F(X)=AX^4-2AX^3-AX^2+2(A-1)X+1,\ on\ \left]0,\frac{1}{2}\right[\\ \qquad\qquad where\ A=ln(-\lambda_k), \\ \qquad\qquad r_k=x_k+h.X_k. \\ \quad end\ k \end{array}\right.$$

For $k \geq N_0+1$
$X_k = X_{N_0}$,
$r_k = x_k + hX_k$, $r_k = r_{k-1}+h$.

4.1. **Application.**
We consider the following Cauchy problem:

$$\left\{\begin{array}{ll} x^2.u'(x)=4.u(x)\ x\in[-4,4], & (1) \\ u(-4)=1, & (2) \end{array}\right. \tag{4.2}$$

According to (2.19) the following algorithm has been obtained:

$$\left\{\begin{array}{l} \alpha_1=\frac{1}{\psi_1(r_1)} \\ \alpha_{k+1}=\frac{\alpha_k.\psi_k(r_k)(4(r_{k+1}-r_k)+r_k^2)}{r_k^2.\psi_{k+1}(r_{k+1})}\ for\ k=1,\dots,m. \end{array}\right. \tag{4.3}$$

This algorithm of the δ-ziti method that we have to build aims to calculate λ_i and roots r_i.
$m=1000$, $a=-4$, $b=4$.
precision $\epsilon=10^{-N}$ with $N=8$. (from table2 we have $N_0=N=8$)
We calculate

$$\begin{array}{rcl} \alpha &=& <\varphi_1,\varphi_2>=21,47547348, \\ \beta &=& <\varphi_1,\varphi_1>=42,19480080, \\ \lambda_1 &=& -\frac{\alpha}{\beta}=-0.508960181653470, \\ N_0 &=& \left[\frac{-Nln(10)+ln\left(\frac{(2-\lambda_1^2)}{\lambda_1^3-\lambda_1}\right)}{ln\left(\frac{\lambda_1}{2+\lambda_1}\right)^2}\right]+1=8. \end{array}$$

Calculation of λ_i. Calculation of r_i.
For example, the secant method is used to solve:

$$\left\{\begin{array}{l} For\ k=1:N_0=8 \\ \qquad\qquad Calculation\ of\ the\ root\ X_k\ of\ the\ function \\ \qquad\qquad F(X)=AX^4-2AX^3-AX^2+2(A-1)X+1,\ on\ \left]0,\frac{1}{2}\right[\\ \qquad\qquad where\ A=ln(-\lambda_k), \\ \qquad\qquad r_k=x_k+h.X_k. \end{array}\right.$$

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λ_1	-0.508960181653470
λ_2	-0.292344636312607
λ_3	-0.274934054384423
λ_4	-0.273624280111297
λ_5	-0.273526252120255
λ_6	-0.273518918194867
λ_7	-0.273518369525952
λ_8	-0.273518328478775
$if\ N > N_0,\ \lambda_N = \lambda_{N_0}$	-0.273518325407943

TABLE 3. Results using the optimal algorithm.

X_1	0.333258836280244
X_2	0.248728737840737
X_3	0.241680753511322
X_4	0.241148006382392
X_5	0.241108118629783
X_6	0.241105134358228
X_7	0.241104911097089
X_8	0.241104894394410
$if\ N > N_0,\ X_N = X_{N_0}$	0.241104894394410

TABLE 4. Results using the optimal algorithm.

The two graphs in Fig.10 present the comparison between the exact and approximate solution of (4.2).

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$r_1 = x_1 + hX_1$	-3.997333929309758
$r_2 = x_2 + hX_2$	-3.990010170097274
$r_3 = x_3 + hX_3$	-3.982066553971909
$r_4 = x_4 + hX_4$	-3.974070815948941
$r_5 = x_5 + hX_5$	-3.966071135050962
$r_6 = x_6 + hX_6$	-3.958071158925134
$r_7 = x_7 + hX_7$	-3.950071160711223
$r_8 = x_8 + hX_8$	-3.942071160844845
$if\ N > N_0,\ r_N = r_{N_0} + h$	$\cdots$
$r_{m+1} = b$	4

TABLE 5. Results using the optimal algorithm.

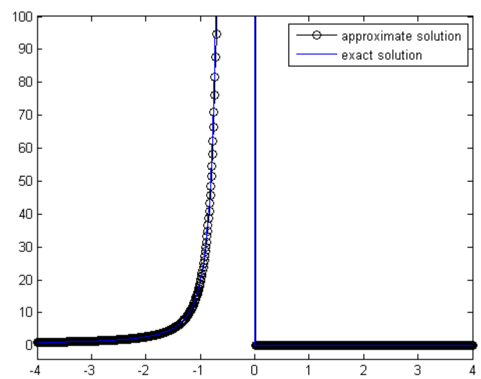

FIGURE 10. Comparison the approximate solution obtained by δ-ziti's scheme with the exact solution in presence a singularity.

5. Conclusion

The method has shown its effectiveness in comparison with other results (exact or obtained with classical methods). In this work, we have established two algorithms that have resulted in the same results. Table 1 obtained by the first algorithm required as many iterations as number of nodes (likely to be very large, for example 10^6).

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Table 2 obtained by the second algorithm deduces from a careful study (reformulation of λ_i, ψ_i and r_i) to reduce the number of iterations with an acceptable stopping test for example for a test of 10^{-5} on number of iterations is reduced to 8 iterations.

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