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Local Symmetry of Unit Tangent Sphere Bundle with g-Natural Almost Contact B-metric Structure

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Abstract. We consider the unit tangent sphere bundle of Riemannian manifold (M, g) with g-natural metric $\tilde{G}$ and we equip it to an almost contact B-metric structure. Considering this structure, we show that there is a direct correlation between the Riemannian curvature tensor of (M, g) and local symmetry property of $\tilde{G}$. More precisely, we prove that the flatness of metric g is necessary and sufficient for the g-natural metric $\tilde{G}$ to be locally symmetric.

Keywords: Almost contact structure, Local symmetry, Natural metric.

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1. Introduction

Riemannian symmetric spaces appear in a wide various situations in both mathematics and physics. These spaces were first introduced and classified by E. Cartan in [6]. The central role of these spaces in the theory of holonomy

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was discovered by M. Berger. They are substantial objects of study in representation theory and harmonic analysis as well as in differential geometry. The notion of locally symmetric manifold is one of the most important kind of these classical spaces and has many applications in physics. Especially, this concept plays an enormous role in general relativity. In [12], the authors have presented some sufficient conditions for a Riemannian manifold (M, g) to be locally symmetric. Also, in the context of symmetric spaces, in [8] and [13] the authors obtained valuable results.

In [11], Sasaki provided the notion of the almost contact structure. In recent years, as a counterpart of the almost contact metric structure, the motif of almost contact B-metric structure has been an interesting research field in differential geometry and Manev has elaborate this motif in some paper and has obtained worthful results in this context ([9, 10]).

The concept of lifted metrics on tangent bundle and tangent sphere bundle of a Riemannian manifold (M, g) has been widely considered by many mathematicians in recent years and in [3] the authors introduced the notion of g-natural metrics on tangent bundle, as the most general type of lifted metrics on tangent bundle.

The locally symmetric property of the unit tangent sphere bundle equipped to a g-natural contact metric structure is investigated in [1] by K. M. T. Abbassi. In fact, Riemannian g-natural contact metrics on the unit tangent bundles present a rigidity, with respect to the property of being locally symmetric in the sense that such a metric can not be locally symmetric unless the base manifold is flat ([1]), and in the present paper we show that this rigidity remains true if we consider g-natural almost contact B-metric structures, for which the associated g-natural metrics are not still Riemannian, but only non-degenerate.

The aim of this paper is to prove that the flatness of metric g is necessary and sufficient for metric $\tilde{G}$ to be locally symmetric on the unit tangent sphere bundle with g-natural almost contact B-metric structure.

The work is organized in the following way. In Section 2, we begin with a study on the concept of g-natural metrics on the tangent bundle and unit tangent sphere bundle of a Riemannian manifold (M, g) and we introduce almost contact B-metric structure on T_1M. We proceed in Section 3, to describe and study the local symmetry property on the mentioned structure and then we prove the main theorem of this paper on local symmetry property of unit tangent sphere bundle.

2. g-Natural Metric on the Sphere Bundle

This section contains some necessary information on g-natural metrics on the tangent and unit tangent sphere bundle.

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2.1. **g-Natural metrics on the tangent bundle.** We consider the $(n+1)$-dimensional Riemannian manifold (M,g) and denoting by ∇ its Levi-Civita connection, the tangent space $TM_{(x,u)}$ of the tangent bundle TM at a point (x,u) splits as

$$(TM)_{(x,u)} = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)},$$

where $\mathcal{H}$ and $\mathcal{V}$ are the horizontal and vertical spaces with respect to ∇. The *horizontal lift* of $X \in M_x$ to $(x,u) \in TM$ is a unique vector $X^h \in \mathcal{H}_{(x,u)}$ such that $\pi_* X^h = X$, where $\pi : TM \to M$ is the natural projection. Moreover, for $X \in M_x$, the *vertical lift* of vector X is a vector $X^v \in \mathcal{V}_{(x,u)}$ such that $X^v(\mathrm{d}f) = Xf$, for all functions f on M. Needless to say, 1-forms $\mathrm{d}f$ on M are considered as functions on TM (i.e., $(\mathrm{d}f)(x,u) = uf$). The map $X \to X^h$ is an isomorphism between the vector spaces M_x and $\mathcal{H}_{(x,u)}$. Similarly, the map $X \to X^v$ is an isomorphism between M_x and $\mathcal{V}_{(x,u)}$. As a result of this explanation, one can write each tangent vector $Z \in (TM)_{(x,u)}$ in the form $Z = X^h + Y^v$, where $X, Y \in M_x$, are uniquely determined vectors (see [4], for more details). Also, the geodesic flow vector field on TM is uniquely determined by $u^h_{(x,u)} = u^i(\frac{\partial}{\partial x^i})^h_{(x,u)}$, for any point $x \in M$ and $u \in TM_x$, with respect to the local coordinates $\{\frac{\partial}{\partial x^i}\}$ on M. In [2], the authors bring up a discussion on g-natural metrics on tangent bundle TM of a Riemannian manifold (M,g), including the following characterization.

Proposition 2.1 ([2])**.** *Let (M,g) be a Riemannian manifold and G be the g-natural metric on TM. Then there are six smooth functions $\alpha_i, \beta_i : \mathbb{R}^+ \to \mathbb{R}$, $\quad i = 1,2,3$, such that for every $u, X, Y \in M_x$, we have*

$$\begin{cases} G_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(r^2) g(X,Y) + (\beta_1 + \beta_3)(r^2) g(X,u) g(Y,u), \\ G_{(x,u)}(X^h, Y^v) = G_{(x,u)}(X^v, Y^h) = \alpha_2(r^2) g(X,Y) + \beta_2(r^2) g(X,u) g(Y,u), \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(r^2) g(X,Y) + \beta_1(r^2) g(X,u) g(Y,u), \end{cases}$$

where $r^2 = g(u,u)$.

As a prime example of Riemannian g-natural metrics on the tangent bundle, we express the Sasaki metric obtained from Proposition 2.1 with

$$\alpha_1(t) = 1, \qquad \alpha_2(t) = \alpha_3(t) = \beta_1(t) = \beta_2(t) = \beta_3(t) = 0.$$

The other classical example of g-natural metrics on the tangent bundle is the Cheeger- Gromoll metric g_{CG} for

$$\alpha_1(t) = \beta_1(t) = -\beta_3(t) = \frac{1}{1+t}, \qquad \alpha_2(t) = \beta_2(t) = 0, \qquad \alpha_3(t) = \frac{t}{1+t}.$$

2.2. **g-Natural metric on the unit sphere bundle.** Let (M,g) be a Riemannian manifold. The hyperspace

$$T_1 M = \{(x,u) \in TM \,|\, g_x(u,u) = 1\},$$

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in TM, is called the unit tangent sphere bundle over the Riemannian manifold (M,g). Denoting by $(T_1M)_{(x,u)}$, the tangent space of T_1M at a point $(x,u) \in T_1M$, we have

$$(T_1M)_{(x,u)} = \{X^h + Y^v | X \in M_x, Y \in \{u\}^\perp \subset M_x\}.$$

A g-natural metric on T_1M, is any metric $\tilde{G}$, induced on T_1M by a g-natural metric G on TM. Using [5], we know that $\tilde{G}$ is completely determined by the values of four real constants, namely

$$a = \alpha_1(1), \qquad b = \alpha_2(1), \qquad c = \alpha_3(1), \qquad d = (\beta_1 + \beta_3)(1).$$

Let (M,g) be a $(2n+1)$-dimensional Riemannian manifold. Considering an orthogonal basis $\{X_0 = u, X_1, \ldots, X_n\}$ on $x \in M$, we define $\xi = X_0^h = u^h$. The metric $\tilde{G}$ on T_1M is completely determined by

$$\begin{cases} \tilde{G}_{(x,u)}(X_i^h, X_j^h) = (a+c)g_x(X_i, X_j) + dg_x(X_i, u)g_x(X_j, u), \\ \tilde{G}_{(x,u)}(X_i^h, Y_j^v) = bg_x(X_i, Y_j), \\ \tilde{G}_{(x,u)}(Y_i^v, Y_j^v) = ag_x(Y_i, Y_j), \end{cases}$$

at any point $(x,u) \in T_1M$, for all $X_i, Y_j \in M_x$, with Y_j orthogonal to u ([5]). Obviously, we have $\tilde{G}(X_i^h, X_j^h) = \tilde{G}(X_i^h, Y_j^v) = \tilde{G}(Y_i^v, Y_j^v) = 0$, when $i \neq j$. Moreover, it requires to

$$a + c + d > 0, \qquad \alpha = a(a+c) - b^2 < 0,$$

in order to achieve a B-metric structure with g-natural metric on the unit tangent sphere bundle T_1M over the Riemannian manifold (M,g) (see [5], for further details).

Taking into account $\phi = a(a+c+d) - b^2$, using the Schmidt's orthogonalization process and some minor calculations, it can be shown that whenever $\phi \neq 0$, the following vector field on TM is normal to T_1M and is unitary at any point of T_1M for all $(x,u) \in TM$

$$N^G_{(x,u)} = \frac{1}{\sqrt{|(a+c+d)\phi|}}[-bu^h + (a+c+d)u^v].$$

Moreover, for a vector $X \in M_x$ at $(x,u) \in T_1M$, the *tangential lift* X^{t_G} with respect to G is defined as the tangential projection of the vertical lift of X to (x,u) with respect to N^G, in other words

$$X^{t_G} = X^v - \frac{\phi}{|\phi|}G_{(x,u)}(X^v, N^G_{(x,u)})N^G_{(x,u)} = X^v - \sqrt{\frac{|\phi|}{|a+c+d|}}g_x(X,u)N^G_{(x,u)}.$$

Also, if $X \in M_x$ is orthogonal to u, then $X^{t_G} = X^v$. Assuming that $b = 0$, the tangential lift X^{t_G} and the classical tangential lift X^t defined for the case of the Sasaki metric coincide. In the most general case we have

$$X^{t_G} = X^t + \frac{b}{a+c+d}g(X,u)u^h.$$

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Remark 2.2 ([5]). The tangential lift u^{t_G} to $(x,u) \in T_1M$ of the vector u is given by $u^{t_G} = \frac{b}{a+c+d}u^h$, that is, u^{t_G} is a horizontal vector. Therefore, the tangent space $(T_1M)_{(x,u)}$ of T_1M at (x,u) is spanned by vectors of the form X^h and Y^{t_G} as follows,

$$(T_1M)_{(x,u)} = \{X^h + Y^{t_G} | X \in M_x, Y \in \{u\}^\perp \subset M_x\}, \tag{2.1}$$

hence, the operation of tangential lift from M_x to a point $(x,u) \in T_1M$ will be always applied **only** to those vectors of M_x which are orthogonal to u.

Taking into account Remark 2.2, the Riemannian metric $\tilde{G}$ on T_1M induced from G is completely determined by the following identities.

$$\begin{cases} \tilde{G}(X_1^h, X_2^h) = (a+c)g_x(X_1,X_2) + dg_x(X_1,u)g_x(X_2,u), \\ \tilde{G}(X_1^h, Y_1^{t_G}) = bg_x(X_1,Y_1), \\ \tilde{G}(Y_1^{t_G}, Y_2^{t_G}) = ag_x(Y_1,Y_2), \end{cases}$$

where $X_i, Y_i \in M_x$, for $i = 1,2$ with Y_i orthogonal to u. It should be noted that by the above equations, horizontal and vertical lifts are orthogonal with respect to $\tilde{G}$, if and only if $b = 0$.

2.3. **Almost contact g-natural metric structure on sphere bundle.** In this part, we consider unit tangent sphere bundle of a Riemannian manifold as an odd dimensional manifold and equip it with an almost contact structure with B-metric.

Definition 2.3 ([9])**.** A $(2n+1)$-dimensional manifold M has an almost contact B-metric structure if it admits a tensor field φ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0,$$
$$g(\varphi x, \varphi y) = -g(x,y) + \eta(x)\eta(y).$$

(2.1) yields that the tangent space of T_1M at (x,u) can be written as

$$(T_1M)_{(x,u)} = span(\xi) \oplus \{X^h | X \perp u\} \oplus \{Y^{t_G} | Y \perp u\}.$$

Notice that we have $Y^{t_G} = Y^v$ when $y \perp u$. Now we consider the unit tangent sphere bundle of a Riemannian manifold (M,g) with g- natural metric and equip it to an almost contact B-metric structure, denoted briefly by $(T_1M, \varphi, \xi, \eta, \tilde{G})$, and also, a basis $\{X_i^h, X_i^v, \xi\}$, such that $X_i^h, X_i^v \perp \xi$, with respect to $\tilde{G}$, where $\xi = u^h$. An almost contact structure on T_1M is defined by

$$\eta(X_i^h) = \eta(X_i^v) = 0, \quad \eta(\xi) = 1, \quad \varphi(X_i^h) = X_i^v, \quad \varphi(\tilde{X_i^v}) = -X_i^h, \quad \varphi(\xi) = 0.$$

The adapted g-natural metric on the unit tangent sphere bundle T_1M with almost contact B-metric structure is of following form

$$\begin{cases} \tilde{G}(X_i^h, X_j^h) = (a+c)g(X_i,X_j) + dg(X_i,u)g(X_j,u), \\ \tilde{G}(X_I^h, Y_j^v) = 0, \\ \tilde{G}(Y_i^v, Y_j^v) = ag(Y_i,Y_j). \end{cases} \tag{2.2}$$

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Also, we have following relations

$$\tilde{G}(\varphi X_i^h, \varphi X_j^h) = -\tilde{G}(X_i^h, X_j^h), \qquad \tilde{G}(\varphi X_i^v, \varphi X_j^v) = -\tilde{G}(X_i^v, X_j^v),$$

which give that $\tilde{G}$ is a B-metric.

Remark 2.4. As a result of above relations we have $a + c = -a$. Notice that using $b = 0$ and $a + c = -a$, we conclude that $\tilde{G}$ is of signature $(n, n + 1)$ or $(n + 1, 1)$ (see [7], for more details). Also, it deduces that $\alpha = a(a + c) - b^2 = -a^2 < 0$, therefore, the associated g-natural metric $\tilde{G}$ is a **non-degenerate pseudo**-Riemannian metric. Moreover, taking into account $a + c + d = 1$, we get $\phi = a(a + c + d) - b^2 = a$.

Let $\tilde{\nabla}$ be the Levi-Civita connection of non-degenerate pseudo-Riemannian metric $\tilde{G}$. Using this fact that $(T_1M, \tilde{G})$ is a hypersurface of (TM, G) using Proposition 2 and Proposition 3 in [5] the Levi-Civita connection and the curvature tensor formulas for non-degenerate pseudo-Riemannian metric $\tilde{G}$ are obtained by

Theorem 2.5. *The Levi-Civita connection $\tilde{\nabla}$ associated with $\tilde{G}$ at $(x, u) \in T_1M$ is given by*

$$\begin{cases} (\tilde{\nabla}_{X^h} Y^h)|_{(x,u)} = ((\nabla_X Y)|_x)^h + (A(u, X, Y))^v, \\ (\tilde{\nabla}_{X^h} Y^v)|_{(x,u)} = ((\nabla_X Y)|_x)^v + (B(u, X, Y))^h, \\ (\tilde{\nabla}_{X^v} Y^h)|_{(x,u)} = (B(u, Y, X))^h, \\ (\tilde{\nabla}_{X^v} Y^v)|_{(x,u)} = 0, \end{cases}$$

for all vector fields $X, Y \in M_x$, where A and B are the tensor fields of type $(1, 2)$ on M defined by

$$A(u, X, Y) = -\frac{1}{2} R(Y, u)X, \tag{2.3}$$

$$B(u, X, Y) = \frac{1}{2} R(X, u)Y - \frac{d}{2}[g(R(X, u)Y, u) - g(X, Y)]u,$$

and R denotes the Riemannian curvature tensor on (M, g) defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

Proof. The proof is concluded from Proposition 2 in [5]. □

Theorem 2.6. *Let (M, g) be a Riemannian manifold and $\tilde{G}$ be the pseudo-Riemannian metric on $(T_1M, \varphi, \xi, \eta, \tilde{G})$ defined by (2.2). The Riemannian curvature tensor $\tilde{R}$ of $(T_1M, \varphi, \xi, \eta, \tilde{G})$ is completely determined by*

$$\begin{aligned} \tilde{R}(X^h, Y^h)Z^h =& \{R(X, Y)Z - \frac{1}{4}[R(R(Y, Z)u, u)X - R(R(X, Z)u, u)Y \\ & - 2R(R(X, Y)u, u)Z] - \frac{d}{4}\{g(R(Y, Z)u, R(X, u)u) \\ & - g(R(X, Z)u, R(Y, u)u) - 2g(R(X, Y)u, R(Z, u)u) \\ & + 3R(X, Y, Z, u)\}u\}^h + \{\frac{1}{2}(\nabla_Z R)(X, Y)u\}^v, \end{aligned}$$

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$$\begin{aligned}\tilde{R}(\xi,Y^h)Z^h =&\{R(u,Y)Z-\frac{1}{4}[R(R(Y,Z)u,u)u-R(R(u,Z)u,u)Y\\&-2R(R(u,Y)u,u)Z]+\frac{ad}{4\alpha}R(Y,u)Z\\&+\frac{d}{4\alpha}\{a^2[-g(R(u,Z)u,R(Y,u)u)-2g(R(u,Y)u,R(Z,u)u)]\\&-ad[g(R(Y,u)Z,u)]+3a^2R(u,Y,Z,u)+(a+c)dg(Y,Z)X\}u\}^h\\&+\{\frac{1}{2}(\nabla_ZR)(u,Y)u\}^v,\end{aligned}$$

$$\begin{aligned}\tilde{R}(X^h,Y^v)Z^h =&\{\frac{1}{2}(\nabla_XR)(Y,u)Z-\frac{d}{2}\{g((\nabla_XR)(Y,u)Z,u)\}u\}^h\\&+\{-\frac{1}{4}R(X,R(Y,u)Z)u+\frac{1}{2}R(X,Y)Z\\&+\frac{d}{4}[g(R(Y,u)Z,u)-g(Y,Z)]R_uX\\&+\frac{d}{4}[g(R(Y,u)Z,u)-g(Y,Z)]X+\frac{(a+c)d}{2\alpha}g(X,Y)Z\}^v,\end{aligned}$$

$$\begin{aligned}\tilde{R}(\xi,Y^v)Z^h =&\{\frac{1}{2}(\nabla_uR)(Y,u)Z-\frac{d}{2}\{g((\nabla_uR)(Y,u)Z,u)\}u\}^h\\&+\{-\frac{1}{4}R(u,R(Y,u)Z)u+\frac{1}{2}R(u,Y)Z+\frac{ad}{4\alpha}R(Y,u)Z\}^v,\end{aligned}$$

$$\begin{aligned}\tilde{R}(X^h,Y^v)\xi =&\{\frac{1}{2}(\nabla_XR)(Y,u)u-\frac{d}{4}\{g((\nabla_XR)(Y,u)u,u)\}u\}^h\\&+\{-\frac{1}{4}R(X,R(Y,u)u)u+\frac{1}{2}R(X,Y)u-\frac{ad}{4\alpha}R(X,Y)u\\&+\frac{(a+c)d}{2\alpha}g(X,Y)u\}^v,\end{aligned}$$

$$\begin{aligned}\tilde{R}(\xi,Y^v)\xi =&\{\frac{1}{2}(\nabla_uR)(Y,u)u-\frac{d}{2}\{g((\nabla_uR)(Y,u)u,u)\}u\}^h\\&+\{-\frac{1}{4}R(u,R(Y,u)u)u+\frac{1}{2}R(u,Y)u+\frac{3ad}{4\alpha}Y\}^v,\end{aligned}$$

$$\tilde{R}(X^v,Y^v)Z^v=\frac{d+2}{2}\{g(Y,Z)X-g(X,Z)Y\}^v,$$

for all arbitrary vector fields $X,Y,Z\in M_x$, where $R_uX=R(X,u)u$ denotes the Jacobi operator associated to u.

Proof. The proof is a special case of Proposition 3 in [5]. □

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3. Locally Symmetric pseudo-Riemannian Manifold $(T_1M, \varphi, \xi, \eta, \tilde{G})$

Let (M, g) be a (pseudo) Riemannian manifold and we denote by ∇ the Levi-Civita connection on M. The (pseudo) Riemannian manifold (M, g) is locally symmetric if and only if the Riemannian curvature tensor R is parallel with respect to the Levi-Civita connection. In other words if

$$(\nabla_W R)(X, Y, Z) - R(\nabla_W X, Y)Z - R(X, \nabla_W Y)Z - R(X, Y)\nabla_W Z = 0, \tag{3.1}$$

then (M, g) is called locally symmetric. In the latter equality, X, Y, Z and W stand for arbitrary vector fields on M.

Lemma 3.1. *Let (M, g) be a Riemannian manifold and let $(x^1, \ldots, x^n)$ be the coordinates on the base manifold M and $(x, u) = (x^i, u^i)$ be the corresponding bundle coordinates on* TM. *Also, let $\mathrm{R}(X, u)u = 0$, for all vector fields X on M, where $u = u^i \dfrac{\partial}{\partial x^i}$. Then the base manifold (M, g) is flat.*

Proof. In the coordinate system we have

$$u^j u^k \mathrm{R}_{ijk}{}^l = 0, \tag{3.2}$$

where $\mathrm{R}_{ijk}{}^l$ are the coefficients of $\mathrm{R}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k}$, i.e.,

$$\mathrm{R}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k} = \mathrm{R}_{ijk}{}^l \frac{\partial}{\partial x^l}. \tag{3.3}$$

Notice that $\mathrm{R}_{ijk}{}^l$ does not depend on u^i. Differentiating (3.2) with respect to u^r and then u^s we obtain

$$\mathrm{R}_{irs}{}^l + \mathrm{R}_{isr}{}^l = 0. \tag{3.4}$$

Replacing i, r in the above equation gives us:

$$\mathrm{R}_{ris}{}^l + \mathrm{R}_{rsi}{}^l = 0. \tag{3.5}$$

(3.4)-(3.5) give us

$$2\mathrm{R}_{irs}{}^l + \mathrm{R}_{isr}{}^l - \mathrm{R}_{rsi}{}^l = 0. \tag{3.6}$$

Now from the Bianchi identity we have

$$\mathrm{R}_{isr}{}^l = -\mathrm{R}_{sri}{}^l - \mathrm{R}_{ris}{}^l. \tag{3.7}$$

Setting the above equation into (3.6) we obtain $3\mathrm{R}_{irs}{}^l = 0$, i.e., M is flat. □

Theorem 3.2. *Let (M, g) be a Riemannian manifold and T_1M be its unit tangent sphere bundle with pseudo-Riemannian g-natural metric $\tilde{G}$ given by (2.2). Then $(T_1M, \varphi, \xi, \eta, \tilde{G})$ is locally symmetric if and only if (M, g) is flat. Therefore, $(T_1M, \varphi, \xi, \eta, \tilde{G})$ is flat.*

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Proof. Theorem 2.6 implies that if $R = 0$ then $\tilde{R} = 0$ and so we have (3.1). Let $(T_1M, \varphi, \xi, \eta, \tilde{G})$ be a locally symmetric manifold, that is, (3.1) holds for all vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}$ and $\tilde{W}$ on T_1M. Let $\tilde{X} = X^h$, $\tilde{Y} = Y^v$, $\tilde{Z} = Z^h$ and $\tilde{W} = W^h$. Now taking into account Theorem 2.6, we compute the vertical part of the left side of (3.1) as follows

$$\begin{aligned}
\mathcal{V}(\tilde{\nabla}_{W^h}\tilde{R}(X^h, Y^v)Z^h) &= -\frac{1}{4}R(W, (\nabla_X R)(Y,u)Z)u \\
&- \frac{(a+c)d}{4\alpha}g((\nabla_X R)(Y,u)Z,u)W + \frac{d}{4}g((\nabla_X R)(Y,u)Z,u)R(W,u)u \\
&+ \frac{d}{4}g((\nabla_X R)(Y,u)Z,u)\frac{(a+c)d}{\alpha}W - \frac{1}{4}\nabla_W R(X, R(Y,u)Z)u \\
&+ \frac{1}{2}\nabla_W R(X,Y)Z + \frac{d}{4}[g(\nabla_W R(Y,u)Z,u) + g(R(Y,u)Z, \nabla_W u)]R_u X \\
&+ \frac{d}{4}g(R(Y,u)Z,u)\nabla_W R_u X - \frac{d}{4}[g(\nabla_W Y, Z) + g(Y, \nabla_W Z)]R_u X \\
&- \frac{d}{4}g(Y,Z)\nabla_W R_u X + \frac{d}{4}[g(\nabla_W R(Y,u)Z,u) + g(R(Y,u)Z, \nabla_W u)]X \\
&+ \frac{d}{4}g(R(Y,u)Z,u)\nabla_W X - \frac{d}{4}[g(\nabla_W Y, Z) + g(Y, \nabla_W Z)]X \\
&- \frac{d}{4}g(Y,Z)\nabla_W X + \frac{(a+c)d}{2\alpha}[g(\nabla_W Y, X) + g(Y, \nabla_W X)]Z \\
&+ \frac{(a+c)d}{2\alpha}g(Y,X)\nabla_W Z,
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\mathcal{V}\tilde{R}(\tilde{\nabla}_{W^h}X^h, Y^v)Z^h &= -\frac{1}{4}R(\nabla_W X, R(Y,u)Z)u + \frac{1}{2}R(\nabla_W X, Y)Z \\
&+ \frac{d}{4}[g(R(Y,u)Z,u) - g(Y,Z)]R_u\nabla_W X \\
&+ \frac{d}{4}[g(R(Y,u)Z,u) - g(Y,Z)]\nabla_W X \\
&+ \frac{(a+c)d}{2\alpha}g(\nabla_W X, Y)Z,
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
\mathcal{V}\tilde{R}(X^h, \tilde{\nabla}_{W^h}Y^v)Z^h &= \frac{1}{4}(\nabla_Z R)(X, R(Y,u)W)u \\
&- \frac{d}{4}[g(R(W,u)Y,u) - g(W,Y)](\nabla_Z R)(X,u)u - \frac{1}{4}R(X, R(\nabla_W Y, u)Z)u \\
&+ \frac{1}{2}R(X, \nabla_W Y)Z + \frac{d}{4}[g(R(\nabla_W Y,u)Z,u) - g(\nabla_W Y, Z)]R_u X \\
&+ \frac{d}{4}[g(R(\nabla_W Y,u)Z,u) - g(\nabla_W Y, Z)]X + \frac{(a+c)d}{2\alpha}g(X, \nabla_W Y)Z,
\end{aligned} \tag{3.10}$$

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$$\mathcal{V}\tilde{R}(X^h,Y^v)\tilde{\nabla}_{W^h}Z^h=-\frac{1}{4}R(X,R(Y,u)\nabla_W Z)u+\frac{1}{2}R(X,Y)\nabla_W Z \tag{3.11}$$
$$+\frac{d}{4}[g(R(Y,u)\nabla_W Z,u)-g(Y,\nabla_W Z)]R_uX+\frac{d}{4}[g(R(Y,u)\nabla_W Z,u)$$
$$-g(Y,\nabla_W Z)]X+\frac{(a+c)d}{2\alpha}g(X,Y)\nabla_W Z.$$

Employing (3.8)-(3.11) we establish immediately $\mathcal{V}((\tilde{\nabla}_{W^h}\tilde{R})(X^h,Y^v)Z^h)$. Now we consider

$$\mathcal{V}((\tilde{\nabla}_{W^h}\tilde{R})(X^h,Y^v)Z^h)=0. \tag{3.12}$$

Substituting $Y=u$ into (3.12) we have

$$-\frac{1}{4}R(W,(\nabla_X R)(u,u)Z)u-\frac{(a+c)d}{4\alpha}g((\nabla_X R)(u,u)Z,u)W \tag{3.13}$$
$$+\frac{d}{4}g((\nabla_X R)(u,u)Z,u)R(W,u)u+\frac{d}{4}g((\nabla_X R)(u,u)Z,u)\frac{(a+c)d}{\alpha}W$$
$$-\frac{1}{4}\nabla_W R(X,R(u,u)Z)u+\frac{1}{2}\nabla_W R(X,u)Z+\frac{d}{4}[g(\nabla_W R(u,u)Z,u)+]R_uX$$
$$+\frac{d}{4}[g(\nabla_W R(u,u)Z,u)]X-\frac{1}{2}R(\nabla_W X,u)Z+\frac{1}{4}R(X,R(\nabla_W u,u)Z)u$$
$$-\frac{1}{2}R(X,\nabla_W u)Z-\frac{d}{4}[g(R(\nabla_W u,u)Z,u)-g(\nabla_W u,Z)]R_uX$$
$$-\frac{d}{4}[g(R(\nabla_W u,u)Z,u)-g(\nabla_W u,Z)]X-\frac{1}{2}R(X,u)\nabla_W Z=0.$$

Analogously, substituting $Z=u$ into (3.12) leads us to

$$-\frac{1}{4}R(W,(\nabla_X R)(Y,u)u)u-\frac{(a+c)d}{4\alpha}g((\nabla_X R)(Y,u)u,u)W \tag{3.14}$$
$$+\frac{d}{4}g((\nabla_X R)(Y,u)u,u)R(W,u)u+\frac{d}{4}g((\nabla_X R)(Y,u)u,u)\frac{(a+c)d}{\alpha}W$$
$$-\frac{1}{4}\nabla_W R(X,R(Y,u)u)u+\frac{1}{2}\nabla_W R(X,Y)u+\frac{d}{4}[g(\nabla_W R(Y,u)u,u)$$
$$+g(R(Y,u)u,\nabla_W u)]R_uX-\frac{d}{4}[g(\nabla_W Y,u)+g(Y,\nabla_W u)]R_uX$$
$$+\frac{d}{4}[g(\nabla_W R(Y,u)u,u)+g(R(Y,u)u,\nabla_W u)]X$$
$$+\frac{(a+c)d}{2\alpha}[g(\nabla_W Y,X)+g(Y,\nabla_W X)]u+\frac{1}{4}R(\nabla_W X,R(Y,u)u)u$$
$$-\frac{1}{2}R(\nabla_W X,Y)u+\frac{1}{4}R(X,R(\nabla_W Y,u)u)u-\frac{1}{2}R(X,\nabla_W Y)u$$
$$+\frac{1}{4}R(X,R(Y,u)\nabla_W u)u-\frac{1}{2}R(X,Y)\nabla_W u-\frac{d}{4}[g(R(Y,u)\nabla_W u,u)$$
$$-g(Y,\nabla_W u)]R_uX-\frac{d}{4}[g(R(Y,u)\nabla_W u,u)-g(Y,\nabla_W u)]X=0.$$

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Now we substitute $Y = Z$ into the latter equation and then by summing the result with (3.13) and using Bianchi identity and (2.3) we have

$$\begin{aligned}
&-\frac{1}{2}[R(X,u)A(u,W,Z) + R(W,u)A(u,X,Z)] \\
&+\frac{d}{4}[R(W,A(u,X,Z))u - R(X,A(u,W,Z))u] \\
&-\frac{d}{4}[R(W,B(u,Z,X))u - R(X,B(u,Z,W))u] \\
&= 0.
\end{aligned}$$

Here, taking g-product with u implies that

$$\begin{aligned}
&-\frac{1}{2}[g(R(X,u)A(u,W,Z),u) + g(R(W,u)A(u,X,Z),u)] \qquad (3.15)\\
&+\frac{d}{4}[g(R(W,A(u,X,Z))u,u) - g(R(X,A(u,W,Z))u,u)] \\
&-\frac{d}{4}[g(R(W,B(u,Z,X))u,u) - g(R(X,B(u,Z,W))u,u)] \\
&= \frac{1}{4}[g(R(X,u)R(Z,u)W,u) + g(R(W,u)R(Z,u)X,u)] = 0.
\end{aligned}$$

By replacing $Z = X$ and $W = u$ into (3.15), after some calculations we deduce that $\frac{1}{4}g(R(X,u)u, R(X,u)u) = 0$ and this equation yields $R(X,u)u = 0$. Using Lemma 3.1 it concludes that $R = 0$ and hence, (M,g) is flat and therefore, $(T_1M, \varphi, \xi, \eta, \tilde{G})$ is flat. □

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