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Approximation by (p,q)-Lupaş Stancu Operators

Asif Khan, Vinita Sharma*

Department of Mathematics, Aligarh Muslim University, Aligarh–202002, India.

E-mail: asifjnu07@gmail.com E-mail: vinita.sha23@gmail.com

ABSTRACT. In this paper, (p,q)-Lupas Bernstein Stancu operators are constructed. Statistical as well as other approximation properties of (p,q)-Lupaş Stancu operators are studied. Rate of statistical convergence by means of modulus of continuity and Lipschitz type maximal functions has been investigated.

Keywords: (p,q)-Integers, Lupaş (p,q)-Bernstein Stancu operators, Statistical approximation, Korovkin's type approximation.

2000 Mathematics subject classification: 65D17, 41A10, 41A25, 41A36.

1. Introduction and Preliminaries

In 1912, S.N. Bernstein [6] introduced his famous operators $B_n: C[0,1] \to C[0,1]$ defined for any $n \in \mathbb{N}$ and for any function $f \in C[0,1]$

$$B_n(f;x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \ x \in [0,1].$$
 (1.1)

and named it Bernstein polynomials to provide a constructive proof of the Weierstrass theorem [20].

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^{*}Corresponding Authors

Further, based on q-integers, Lupaş [21] introduced the first q-Bernstein operators [6] and investigated its approximating and shape-preserving properties. Another q-analogue of the Bernstein polynomials is due to Phillips [38]. Since then several generalizations of well-known positive linear operators based on q-integers have been introduced and their approximation properties studied.

Recently, the applications of (p,q)-calculus (post quantum calculus) emerged as a new area in the field of approximation theory [20]. The development of post quantum calculus has led to the discovery of various generalizations of Bernstein polynomials involving (p,q)-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design [7] and solutions of differential equations.

Mursaleen et al [27] introduced the concept of post quantum calculus in approximation theory and constructed the (p,q)-analogue of Bernstein operators defined as follows for $0 < q < p \le 1$:

$$B_{n,p,q}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^{k} \prod_{s=0}^{n-k-1} (p^{s} - q^{s}x) f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right), x \in [0,1].$$

$$(1.2)$$

Note when p = 1, (p,q)-Bernstein Operators given by (1.2) turns out to be Phillips q-Bernstein Operators [38].

Also, we have

$$(1-x)_{p,q}^{n} = \prod_{s=0}^{n-1} (p^{s} - q^{s}x) = (1-x)(p-qx)(p^{2} - q^{2}x)...(p^{n-1} - q^{n-1}x)$$
$$= \sum_{k=0}^{n} (-1)^{k} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^{k}.$$

Further, they applied the concept of (p,q)-calculus in approximation theory and studied approximation properties based on (p,q)-integers for Bernstein-Stancu operators, (p,q)-analogue of Bernstein-Kantorovich, (p,q)-analogue of Bernstein-Shurer operators, (p,q)-analogue of Bleimann-Butzer-Hahn operators and (p,q)-analogue of Lorentz polynomials on a compact disk in [28, 31, 32, 33, 35].

On the other hand, Khalid and Lobiyal defined (p,q)-analogue of Lupaş Bernstein operators [17] as follows :

For any p>0 and q>0, the linear operators $L_{p,q}^n:C[0,1]\to C[0,1]$ as

$$L_{p,q}^{n}(f;x) = \sum_{k=0}^{n} \frac{f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right) {n \choose k}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} x^{k} (1-x)^{n-k}}{\prod_{j=1}^{n} \{p^{j-1}(1-x) + q^{j-1}x\}},$$
(1.3)

are (p,q)-analogue of Lupas Bernstein operators.

Again when p = 1, Lupaş (p,q)-Bernstein operators turns out to be Lupaş q-Bernstein operators as given in [22, 37].

When p = q = 1, Lupaş (p, q)-Bernstein operators turns out to be classical Bernstein operators [6].

They studied two different techniques as de-Casteljau's algorithm and Korovkin's type approximation properties [17]: de-Casteljau's algorithm and related results of degree elevation reduction for Bèzier curves and surfaces holds for all p>0 and q>0. However to study Korovkin's type approximation properties for Lupaş (p,q)-analogue of the Bernstein operators, $0< q< p\leq 1$ is needed.

Based on Korovkin's type approximation, they proved that the sequence of (p,q)-analogue of Lupaş Bernstein operators $L^n_{p_n,q_n}(f,x)$ converges uniformly to $f(x) \in C[0,1]$ if and only if $0 < q_n < p_n \le 1$ such that $\lim_{n \to \infty} q_n = 1$, $\lim_{n \to \infty} p_n = 1$ and $\lim_{n \to \infty} q_n^n = 1$. On the other hand, for any p > 0 fixed and $p \ne 1$, the sequence $L^n_{p,q}(f,x)$ converges uniformly to $f(x) \in C[0,1]$ if and only if f(x) = ax + b for some $a, b \in \mathbb{R}$.

Furthermore, in comparison to q-Bèzier curves and surfaces based on Lupaş q-Bernstein rational functions, their generalization gives more flexibility in controlling the shapes of curves and surfaces.

Some advantages of using the extra parameter p have been discussed in the field of approximations on compact disk [35] and in computer aided geometric design [17].

For more details related to approximation theory [20], one can refer [1, 2, 3, 5, 8, 9, 12, 13, 14, 15, 18, 19, 22, 23, 24, 34, 36, 39, 40, 42, 43, 44, 45, 46, 47, 48].

Let us recall certain notations of (p,q)-calculus. For any p>0 and q>0, the (p,q) integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \dots + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n - q^n}{p - q}, & \text{when } p \neq q \neq 1 \\ n \ p^{n-1}, & \text{when } p = q \neq 1 \\ [n]_q, & \text{when } p = 1 \\ n, & \text{when } p = q = 1 \end{cases}$$

where $[n]_q$ denotes the q-integers and $n = 0, 1, 2, \cdots$.

The formula for (p, q)-binomial expansion is as follows:

$$(ax + by)_{p,q}^{n} := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} b^{k} x^{n-k} y^{k},$$

$$(x+y)_{p,q}^{n} = (x+y)(px+qy)(p^{2}x+q^{2}y) \cdots (p^{n-1}x+q^{n-1}y),$$

$$(1-x)_{p,q}^{n} = (1-x)(p-qx)(p^{2}-q^{2}x) \cdots (p^{n-1}-q^{n-1}x),$$

where (p,q)-binomial coefficients are defined by

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

Details on (p,q)-calculus can be found in [10, 11, 27].

Also, we have (p,q)-analogue of Euler's identity as:

$$(1-x)_{p,q}^{n} = \prod_{s=0}^{n-1} (p^{s} - q^{s}x) = (1-x)(p-qx)(p^{2} - q^{2}x)...(p^{n-1} - q^{n-1}x)$$
$$= \sum_{k=0}^{n} (-1)^{k} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^{k}.$$

Again by some simple calculations and using the property of (p,q)-integers, we get (p,q)-analogue of Pascal's relation as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} + p^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q}$$
 (1.4)

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = p^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q}. \tag{1.5}$$

We recall some results from [17] for Lupas (p,q)-Bernstein operators, which reproduces linear and constant functions.

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Some auxillary results:

$$(1) L_{p,q}^n(1,\frac{u}{u+1}) = 1$$

(2)
$$L_{p,q}^n(t, \frac{u}{u+1}) = \frac{u}{u+1}$$

(3)
$$L_{p,q}^n(t^2, \frac{u}{u+1}) = \frac{u}{u+1} \frac{p^{n-1}}{[n]_{p,q}} + \frac{qu}{u+1} (\frac{qu}{p+qu}) \frac{[n-1]_{p,q}}{[n]_{p,q}}$$

or equivalently for $x = \frac{u}{u+1}$

$$L_{p,q}^{n}(1,x) = 1, (1.6)$$

$$L_{p,q}^n(t,x) = x, (1.7)$$

$$L_{p,q}^{n}(t^{2},x) = \frac{p^{n-1}x}{[n]_{p,q}} + \frac{q^{2}x^{2}}{p(1-x)+qx} \frac{[n-1]_{p,q}}{[n]_{p,q}}.$$
 (1.8)

2. Construction of (p,q)-Lupaş Stancu Operators

In this section, we introduce (p,q)-Lupaş Stancu operators as follows:

For any p>0 and q>0, the linear operators $L^n_{p,q}:C[0,1]\to C[0,1]$

$$L_{n,p,q}^{\alpha,\beta}(f;x) = \sum_{k=0}^{n} f\left(\frac{p^{n-k} [k]_{p,q} + \alpha}{[n]_{p,q} + \beta}\right) b_{p,q}^{k,n}(t)$$
 (2.1)

and $b_{p,q}^{k,n}(t)$ is given by

$$b_{p,q}^{k,n}(t) = \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^n \{p^{j-1}(1-t) + q^{j-1}t\}},$$
 (2.2)

where $0 < \alpha < \beta$.

We give some equalities for operators (2.1) in the following lemma.

Lemma 4.1. The following equalities are true:

(i)
$$L_{n,n,q}^{\alpha,\beta}(1;x) = 1$$
,

(ii)
$$L_{n,p,q}^{\alpha,\beta}(t;x) = \frac{[n]_{p,q}x + \alpha}{[n]_{n,q} + \beta}$$
,

$$\begin{split} &\text{(i)}\ \ L_{n,p,q}^{\alpha,\beta}(1;x)=1,\\ &\text{(ii)}\ \ L_{n,p,q}^{\alpha,\beta}(t;x)=\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta},\\ &\text{(iii)}\ \ L_{n,p,q}^{\alpha,\beta}(t^2;x)=\frac{1}{([n]_{p,q}+\beta)^2}\frac{q^2[n]_{p,q}[n-1]_{p,q}}{p(1-x)+qx}x^2\ +\ \frac{[n]_{p,q}(2\alpha+p^{n-1})}{([n]_{p,q}+\beta)^2}x+\frac{\alpha^2}{([n]_{p,q}+\beta)^2}. \end{split}$$

Proof. Proof of part (i) is obvious.

$$\begin{split} L_{n,p,q}^{\alpha,\beta}(t;x) &= \sum_{k=0}^{n} \left(\frac{p^{n-k}\ [k]_{p,q} + \alpha}{[n]_{p,q} + \beta}\right) b_{p,q}^{k,n}(t) \\ &= \frac{[n]_{p,q}}{[n]_{p,q} + [\beta]} L_{p,q}^{n}(t;x) \ + \ \frac{[\alpha]}{[n]_{p,q} + [\beta]} L_{p,q}^{n}(1;x). \end{split}$$

So from inequalities (1.6) and (1.7), we get the result

Proof (iii)

$$\begin{split} L_{n,p,q}^{\alpha,\beta}(t^2;x) &= \sum_{k=0}^n \, \left(\frac{p^{n-k} \, [k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) b_{p,q}^{k,n}(t) \\ &= \frac{1}{([n]_{p,q} \, + \, \beta)^2} \bigg[p^{2n-2k} [k]_{p,q}^2 b_{p,q}^{k,n}(t) \\ &+ 2\alpha p^{n-k} [k]_{p,q} b_{p,q}^{k,n}(t) \, + \, \alpha^2 b_{p,q}^{k,n}(t) \bigg] \\ &= \frac{1}{([n]_{p,q} \, + \, \beta)^2} \bigg[\, A \, + \, B \, + \, C \, \bigg]. \\ A \, &= \, p^{2n} \sum_{k=0}^n \frac{[k]^2}{p^{2k}} \frac{\bigg[\, n \, \bigg]_{p,q} \, p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \, t^k \, (1-t)^{n-k}}{\prod\limits_{j=1}^n \{p^{j-1}(1-t) + q^{j-1}t\}} \\ A \, &= \, [n] p^{2n} \sum_{k=1}^n \frac{[k]}{p^{2k}} \frac{\bigg[\, n-1 \, \bigg]_{p,q} \, p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \, u^k}{\prod\limits_{j=1}^n \{p^{j-1} + q^{j-1}u\}}. \end{split}$$

On shifting the limits and on replacing k by k+1, we get

$$\begin{split} A \ = \ [n] p^{2n} \sum_{k=1}^n \frac{[k+1]}{p^{2k+2}} \frac{\left[\begin{array}{c} n-1 \\ k \end{array} \right]_{p,q}^{p\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k+1)}{2}} \, u^k}{\prod\limits_{j=1}^{n-1} \{p^j + q^j u\}}, \\ \\ = \ [n] p^n \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{[k+1]}{p^{k+2}} \frac{\left[\begin{array}{c} n-1 \\ k \end{array} \right]_{p,q}^{p\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k+1)}{2}} \, (\frac{qu}{p})^k}{\prod\limits_{j=0}^{n-2} \{p^j + q^j (\frac{qu}{p})\}}. \end{split}$$

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Using $[k+1]_{p,q} = p^k + q[k]_{p,q}$, we get our desired result:

$$\begin{split} A \ = \ [n] p^n \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{[p^k + q[k]]}{p^{k+2}} \frac{\left[\begin{array}{c} n-1 \\ k \end{array} \right]_{p,q}^{p \frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k+1)}{2}} \left(\frac{qu}{p} \right)^k}{\prod\limits_{j=0}^{n-2} \{p^j + q^j \left(\frac{qu}{p} \right) \}}, \\ = \ [n]_{p,q} p^{n-1} \frac{u}{u+1} \ + \ \frac{q^2 u^2 [n]_{p,q} [n-1]_{p,q}}{(u+1)(p+qu)}, \end{split}$$

equivalently

$$A = [n]_{p,q} p^{n-1} x + \frac{q^2 [n]_{p,q} [n-1]_{p,q}}{(p(1-x)+qx)} x^2.$$

Similarly

$$B = 2\alpha p^{n} \sum_{k=0}^{n} \frac{[k]}{p^{k}} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^{k} (1-t)^{n-k}}{\prod_{j=1}^{n} \{p^{j-1}(1-t) + q^{j-1}t\}},$$

$$= 2\alpha [n] p^{n} \sum_{k=1}^{n} \frac{1}{p^{k}} \frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} u^{k}}{\prod_{j=1}^{n} \{p^{j-1} + q^{j-1}u\}}.$$

After shifting the limits and on replacing k by k+1, we get

$$B = 2\alpha[n]p^{n} \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{1}{p} \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q}^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k-1)}{2}} (\frac{qu}{p})^{k}}{\prod_{j=0}^{n-2} \{p^{j} + q^{j}(\frac{qu}{p})\}},$$

which implies

$$B = 2\alpha [n]_{p,q} x.$$

Similarly

$$C = \alpha^2 \sum_{k=0}^{n} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^{n} \{p^{j-1}(1-t) + q^{j-1}t\}},$$

= α^2 .

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Theorem 2.1. Let $0 < q_n < p_n \le 1$ such that $\lim_{n \to \infty} p_n = 1$, $\lim_{n \to \infty} q_n = 1$, $\lim_{n \to \infty} q_n = 1$ and for $f \in C[0,1]$, we have $\lim_{n \to \infty} |L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)| = 0$.

Proof. Let us recall the following Korovkin's theorem see [20]. Let (T_n) be a sequence of positive linear operators from C[a,b] into C[a,b]. Then $\lim_n \|T_n(f,x) - f(x)\|_{C[a,b]} = 0$, for all $f \in C[a,b]$ if and only if $\lim_n \|T_n(f_i,x) - f_i(x)\|_{C[a,b]} = 0$, for i = 0, 1, 2, where $f_0(t) = 1$, $f_1(t) = t$ and $f_2(t) = t^2$.

3. The Rate of Convergence

In this section, we compute the rates of convergence of the operators $L_{n,p,q}^{\alpha,\beta}(f;x)$ to the functions f by means of modulus of continuity, elements of Lipschitz class and peetre's K-functional.

Let $f \in C[0,1]$. The modulus of continuity of f denoted by $\omega(f,\delta)$ is defined as:

$$\omega(f, \delta) = \sup_{y, x \in [0, 1], |y - x| < \delta} |f(y) - f(x)|.$$

where $w(f; \delta)$ satisfies the following conditions: for all $f \in C[0, 1]$,

$$\lim_{\delta \to 0} w(f; \delta) = 0. \tag{3.1}$$

and

$$|f(y) - f(x)| \le w(f;\delta) \left(\frac{|y - x|}{\delta} + 1\right). \tag{3.2}$$

Theorem 3.1. Let $0 < q < p \le 1$, and $f \in C[0,1]$, and $\delta > 0$, we have

$$||L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)||_{C[0,1]} \le 2\omega(f;\delta_n)$$

where

$$\delta_{n} = \left[\left(\frac{q^{2}[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^{2}(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \left(\frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^{2}} \right) + \frac{\alpha^{2}}{([n]_{p,q} + \beta)^{2}} \right]^{\frac{1}{2}}.$$

Proof. From lemma (4.1) we have

$$|L_{n,p,q}^{\alpha,\beta}(t-x)^{2};x) = \left(\frac{q^{2}[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q}+\beta)^{2}(p(1-x)+qx)} - \frac{[n]_{p,q}-\beta}{[n]_{p,q}+\beta}\right)x^{2} + \left(\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{([n]_{p,q}+\beta)^{2}}\right)x + \frac{\alpha^{2}}{([n]_{p,q}+\beta)^{2}}.$$
(3.3)

For $x \in [0,1]$, we take

$$|L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)| \le w(f;\delta) \left\{ 1 + \frac{1}{\delta} (L_{n,p,q}^{\alpha,\beta}(t-x)^2 : x)^{\frac{1}{2}} \right\},$$

then we get

$$\begin{split} \|L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)\|_{C[0,1]} &\leq w(f;\delta) \bigg\{ 1 + \frac{1}{\delta} (L_{n,p,q}^{\alpha,\beta}(t-x)^2 : x)^{\frac{1}{2}} \bigg\} \\ &\leq w(f;\delta) \bigg\{ 1 + \frac{1}{\delta} \bigg((\frac{1}{([n]_{p,q} + \beta)^2} \frac{q^2 [n]_{p,q} [n-1]_{p,q}}{(p(1-x) + qx)} \\ &- \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta}) \ + \ (\frac{p^{n-1} [n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2}) \\ &+ \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \bigg\}^{\frac{1}{2}} \bigg\}. \end{split}$$

If we choose

$$\delta_n = \left[\left(\frac{q^2 [n]_{p,q} [n-1]_{p,q}}{([n]_{p,q} + \beta)^2 (p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \left(\frac{p^{n-1} [n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) + \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \right]^{\frac{1}{2}}.$$

Then we have

$$||L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)||_{C[0,1]} \le 2\omega(f;\delta_n).$$

So we have the desired result.

Now we compute the approximation order of operator $L_{n,p,q}^{\alpha,\beta}$ in term of the elements of the usual Lipschitz class.

Let $f \in C[0,1]$ and $0 < \rho \le 1$. We recall that f belongs to $Lip_M(\rho)$ if the inequality

$$|f(x) - f(y)| \le M|x - y|^{\rho}; \text{ for all } x, y \in [0, 1]$$
 (3.4)

holds.

Theorem 3.2. For all $f \in Lip_M(\rho)$

 $||L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)||_{C[0,1]} \leq M\delta_n^{\rho}$

$$\delta_n = \left[\left(\frac{q^2 [n]_{p,q} [n-1]_{p,q}}{([n]_{p,q} + \beta)^2 (p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \left(\frac{p^{n-1} [n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) + \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \right]^{\frac{1}{2}}$$

and M is a positive constant.

Proof. Let $f \in Lip_M(\rho)$ and $0 < \rho \le 1$. by (3.4) and linearity and monotonicity of $L_{n,p,q}^{\alpha,\beta}$ then we have

$$\begin{split} |L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)| &\leq L_{n,p,q}^{\alpha,\beta}(|f(t) - f(x)|;x) \\ &\leq L_{n,p,q}^{\alpha,\beta}(|t - x|^{\rho};x). \end{split}$$

Applying the Holder inequality with $m = \frac{2}{\rho}$ and $n = \frac{2}{2-\rho}$, we get

$$|L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)| \le (L_{n,p,q}^{\alpha,\beta}((t-x)^2;x))^{\frac{\rho}{2}}.$$
 (3.5)

if we choose $\delta = \delta_n$ as above, then proof is completed.

Finally, we will study the rate of convergence of the positive linear operators $L_{n,p,q}^{\alpha,\beta}$ by means of the Peetre's K-functionals.

 $C^2[0,1]$: The space of those functions f for which $f, f', f'' \in C[0,1]$. we recall the following norm in the space $C^2[0,1]$:

$$||f||_{C^2[0,1]} = ||f||_{C[0,1]} + ||f'||_{C[0,1]} + ||f''||_{C[0,1]}.$$

We consider the following Peetre's K-functional

$$K(f,\delta) := \inf_{g \in C^2[0,1]} \bigg\{ \|f - g\|_{C[0,1]} \ + \ \delta \|g\|_{C^2[0,1]} \bigg\}.$$

Theorem 3.3. Let $f \in C[0,1]$. Then we have

$$||L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)||_{C[0,1]} \le 2K(f;\delta_n)$$

Where $K(f; \delta_n)$ is Peetre's functional and

$$\delta_{n} = \frac{1}{4} \left(\frac{q^{2}[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^{2}(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \frac{1}{4} \left(\frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^{2}} \right) + \frac{1}{4} \frac{\alpha^{2}}{([n]_{p,q} + \beta)^{2}} + \frac{1}{2} \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]}$$

Proof. Let $g \in C^2[0,1]$. If we use the Taylor's expansion of the function g at s = x, we have

$$g(s) = g(x) + (s-x)g'(x) + \frac{(s-x)^2}{2}g''(x).$$

Hence we get

$$||L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)||_{C[0,1]} \le ||L_{n,p,q}^{\alpha,\beta}((s-x);x)||_{C[0,1]} ||g(x)||_{C^{2}[0,1]} + \frac{1}{2} ||L_{n,p,q}^{\alpha,\beta}((s-x)^{2};x)||_{C[0,1]} ||g(x)||_{C^{2}[0,1]}.$$

$$(3.6)$$

From the lemma (2.1) we have

$$||L_{n,p,q}^{\alpha,\beta}((s-x);x)||_{C[0,1]} \le \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]}.$$
(3.7)

So if we use (3.3) and (3.7) in (3.6), then we get

$$||L_{n,p,q}^{\alpha,\beta}(g;x) - g(x)||_{C[0,1]} \leq \left[\frac{1}{2} \left(\frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2 (p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \left(\frac{1}{2} \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) + \frac{1}{2} \frac{\alpha^2}{([n]_{p,q} + \beta)^2}$$
(3.8)
$$+ \frac{[\alpha] + [\beta]}{[n] + [\beta]} ||g(x)||_{C[0,1]}.$$
(3.9)

On the other hand, we can write

$$\begin{split} |L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)| & \leq |L_{n,p,q}^{\alpha,\beta}(f-g;x)| + |L_{n,p,q}^{\alpha,\beta}(g;x) - g(x)| \\ & + |f(x) - g(x)|. \end{split}$$

If we take the maximum on [0,1], we have

$$||L_{n,p,q}^{\alpha,\beta}(f;x)-f(x)||_{C[0,1]} \le 2||f-g||_{C[0,1]} + ||L_{n,p,q}^{\alpha,\beta}(g;x)-g(x)||_{C[0,1]}.$$
 (3.10) If we consider (3.8) in (3.10), we obtain

$$\begin{split} \|L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)\|_{C[0,1]} & \leq 2\|f - g\|_{C[0,1]} + \left[\frac{1}{4} \left(\frac{q[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)}\right. \right. \\ & \left. - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta}\right) + \left(\frac{1}{4} \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2}\right) \\ & \left. + \frac{1}{4} \frac{\alpha^2}{([n]_{p,q} + \beta)^2} + \left. \frac{1}{2} \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]} \right] \|g(x)\|_{C^2[0,1]}. \end{split}$$

If we choose

$$\delta_{n} = \frac{1}{4} \left(\frac{q^{2}[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^{2}(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \left(\frac{1}{4} \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^{2}} \right) + \frac{1}{4} \frac{\alpha^{2}}{([n]_{p,q} + \beta)^{2}} + \frac{1}{2} \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]},$$

then we get

$$||L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)||_{C[0,1]} \le 2 \left\{ ||f - g||_{C[0,1]} + \delta_n ||g(x)||_{C^2[0,1]} \right\}.$$
 (3.11)

Finally, one can observe that if we take the infimum of both side of above inequality for the function $g \in C^2[0,1]$, we can find

$$||L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)||_{C[0,1]} \le 2K(f,\delta_n).$$

4. The Rates of Statistical Convergence

At this point, let us recall the concept of statistical convergence. The statistical convergence which was introduced by Fast [41] in 1951, is an important research area in approximation theory. In [41], Gadjiev and Orhan used the concept of statistical convergence in approximation theory. They proved a Bohman-Korovkin type theorem for statistical convergence.

Recently, statistical approximation properties of many operators are investigated in [4, 25, 26, 29, 30].

A sequence $x = (x_k)$ is said to be statistically convergent to a number L if for every $\epsilon > 0$,

$$\delta\{K \in \mathbf{N} : |x_k - L| \ge \varepsilon\} = 0,$$

where $\delta(K)$ is the natural density of the set $K \subseteq \mathbb{N}$.

The density of subset $K \subseteq N$ is defined by

$$\delta(K) := \lim_n \frac{1}{n} \{ \text{the number } k \leq n : k \in K \}$$

whenever the limit exists.

For instance,
$$\delta(\mathbf{N}) = 1$$
, $\delta\{2K : k \in \mathbf{N}\} = \frac{1}{2}$ and $\delta\{k^2 : K \in \mathbf{N}\} = 0$.

To emphasize the importance of the statistical convergence, we have an example: The sequence

$$X_k = \begin{cases} L_1; & if \ k = m^2, \\ L_2; & if \ k \neq m^2. \end{cases} \quad where \ m \in \mathbf{N}$$
 (4.1)

is statistically convergent to L_2 but not convergent in ordinary sense when $L_1 \neq L_2$. We note that any convergent sequence is statistically convergent but not conversley.

Now we consider sequences $q = q_n$ and $p = p_n$ such that:

$$st - \lim_{n} q_n = 1$$
, $st - \lim_{n} p_n = 1$, and $st - \lim_{n} q_n^n = 1$. (4.2)

Gadjiev and Orhan [41] gave the following theorem for linear positive operators which is about statistically Korovkin type theorem. Now, we recall this theorem.

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Theorem 4.1. If A_n be the sequence of linear positive operators from C[a,b] to C[a,b] satisfies the conditions

$$st - \lim_{n} ||A_n((t^{\nu}; x)) - (x)^{\nu}||_C[0, 1] = 0 \text{ for } \nu = 0, 1, 2.$$

then for any function $f \in C[a, b]$,

$$st - \lim_{n} ||A_n(f;.) - f||_C[a,b] = 0.$$

Now we will discuss the rates of statistical convergence of $L_{n,p,q}^{\alpha,\beta}$ operators.

Remark 4.2. For $q \in (0,1)$ and $p \in (q,1]$, it is obvious that

$$\lim_{n\to\infty}[n]_{p,q}=\left\{\begin{array}{ll} 0,\ when\ p,q\in(0,1)\\\\ \frac{1}{1-q},\ when\ p=1\ and\ q\in(0,1). \end{array}\right.$$

In order to reach to convergence results of the operator $L_{p,q}^n(f;x)$, we take a sequence $q_n \in (0,1)$ and $p_n \in (q_n,1]$ such that $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$. So we get $\lim_{n\to\infty} [n]_{p_n,q_n} = \infty$.

Theorem 4.3. Let $L_{n,p,q}^{\alpha,\beta}$ be the sequence of operators and the sequences $p = p_n$ and $q = q_n$ satisfies Remark 4.2 then for any function $f \in C[0,1]$

$$st - \lim_{n} || L_{n,p_n,q_n}^{\alpha,\beta}(f,.) - f|| = 0.$$
 (4.3)

Proof. Clearly for $\nu = 0$,

$$L_{n,p_n,q_n}^{\alpha,\beta}(1,x) = 1,$$

which implies

$$st - \lim_{n} \|L_{n,p_n,q_n}^{\alpha,\beta}(1;x) - 1\| = 0.$$

For $\nu = 1$

$$\begin{aligned} \|L_{n,p_{n},q_{n}}^{\alpha,\beta}(t;x) - x \| &\leq \left| \frac{[n]_{p_{n},q_{n}}}{[n]_{p_{n},q_{n}} + \beta} x + \frac{\alpha}{[n]_{p_{n},q_{n}} + \beta} - x \right| \\ &= \left| \left(\frac{[n]_{p_{n},q_{n}}}{[n]_{p_{n},q_{n}} + \beta} - 1 \right) x + \frac{\alpha}{[n]_{p_{n},q_{n}} + \beta} \right| \\ &\leq \left| \frac{[n]_{p_{n},q_{n}}}{[n]_{p_{n},q_{n}} + \beta} - 1 \right| + \left| \frac{\alpha}{[n]_{p_{n},q_{n}} + \beta} \right|. \end{aligned}$$

For a given $\epsilon > 0$, let us define the following sets.

$$\begin{split} U &= \{n: \|L_{n,p_n,q_n}^{\alpha,\beta}(t;x) - x\| \geq \epsilon\} \\ U' &= \{n: 1 - \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta}\} \geq \epsilon \\ U'' &= \{n: \frac{\alpha}{[n]_{p_n,q_n} + \beta} \geq \epsilon\}. \end{split}$$

It is obvious that $U \subseteq U'' \cup U'$,

So using

$$\delta\{k \le n : 1 - \frac{[n]_{p_n, q_n}}{[n]_{p_n, q_n} + \beta} \ge \epsilon\},$$

then we get

$$st - \lim_{n} ||L_{n,p_n,q_n}^{\alpha,\beta}(t;x) - x|| = 0.$$
 (4.4)

Lastly for $\nu = 2$, we have

$$||L_{n,p_n,q_n}^{\alpha,\beta}(t^2:x) - x^2|| \le \left| \frac{q^2[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{p(1-x) + qx} \frac{1}{([n]_{p_n,q_n} + \beta)^2} - 1 \right| + \left| \frac{[n]_{p_n,q_n}(2\alpha + p^{n-1})^2}{[n]_{p_n,q_n} + \beta} x \right| + \left| \frac{\alpha^2}{([n]_{p_n,q_n} + \beta)^2} \right|.$$

If we choose

$$\alpha_n = \frac{q^2 [n]_{p_n, q_n} [n-1]_{p_n, q_n}}{p(1-x) + qx} \frac{1}{([n]_{p_n, q_n} + \beta)^2} - 1$$
$$\beta_n = \frac{[n]_{p_n, q_n} (2\alpha + p^{n-1})^2}{[n]_{p_n, q_n} + \beta}$$
$$\gamma_n = \frac{\alpha^2}{([n]_{p_n, q_n} + \beta)^2}.$$

Then

$$st - \lim_{n} \alpha_n = st - \lim_{n} \beta_n = st - \lim_{n} \gamma_n = 0.$$

Now given $\epsilon > 0$, we define the following four sets:

$$U = ||L_{n,p_n,q_n}^{\alpha,\beta}(t^2:x) - x^2|| \ge \epsilon,$$

$$U_1 = \{n : \alpha_n \ge \frac{\epsilon}{3}\},$$

$$U_2 = \{n : \beta_n \ge \frac{\epsilon}{3}\},$$

$$U_3 = \{n : \gamma_n \ge \frac{\epsilon}{3}\}.$$

It is obvious that $U \subseteq U_1 \bigcup U_2 \bigcup U_3$. Thus we obtain

$$\begin{split} &\delta\{K \leq n: \|L_{n,p,q}^{\alpha,\beta}(t^2:x) - x^2\| \geq \epsilon\} \\ &\leq \delta\{K \leq n: \alpha_n \geq \frac{\epsilon}{3}\} \ + \ \delta\{K \leq n: \beta_n \geq \frac{\epsilon}{3}\} + \delta\{K \leq n: \gamma_n \geq \frac{\epsilon}{3}\}. \end{split}$$

So the right hand side of the inequalities is zero.

Then

$$st - \lim_{n} ||L_{n,p_n,q_n}^{\alpha,\beta}(t;x) - x|| = 0$$

holds and thus the proof is completed.

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