Iranian Journal of Mathematical Sciences and Informatics
Vol. 14, No. 1 (2019), pp 159-165
DOI: 10.7508/ijmsi.2019.01.014

Some Generalizations of Locally Closed Sets

Shyamapada Modak[*,a] and Takashi Noiri[b]

[a]Department of Mathematics, University of Gour Banga
P.O. Mokdumpur, Malda 732 103, India.
[b]2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi
Kumomoto-ken, 869-5142 JAPAN.

E-mail: spmodak2000@yahoo.co.in
E-mail: t.noiri@nifty.com

Abstract. Arenas et al. [1] introduced the notion of λ-closed sets as a generalization of locally closed sets. In this paper, we introduce the notions of λ-locally closed sets, Λ_λ-closed sets and λg-closed sets and obtain some decompositions of closed sets and continuity in topological spaces.

Keywords: λ-Open set, λ-Locally closed set, Λ_λ-Closed set, λg-Closed set, Decompositions of continuity.

2000 Mathematics subject classification: 54A05, 54C08.

1. Introduction and Preliminaries

The study of locally closed sets was introduced by Bourbaki [3] in 1966 then the authors Ganster and Reilly [6] have studied it extensively. A subset A of a topological space X is called locally closed if $A = U \cap F$, where U is open and F is closed. It is interesting that a locally closed set is a generalization of both open sets and closed sets. The generalization has also been discussed in completely regular Hausdorff spaces [5] and has also been done on algebra with topology in [12] and [2].

---

*Corresponding Author

Received 23 November 2016; Accepted 29 March 2017

In this paper we consider a new type of sets in the topological space which is called λ-open sets. A set is said to be λ-open if it contains a nonempty open set. This idea is not a new idea. In literature, semi-open sets [7] and α-sets [11] are examples of that type of sets although preopen sets [10] is not an example of it. Because: let $\mathbf{R}$ be the usual real line and Q the rational numbers. Then $\mathrm{Cl}(Q) = \mathbf{R}$ and $Q \subseteq \mathrm{Int}(\mathrm{Cl}(Q)) = \mathbf{R}$ (where 'Cl' and 'Int' denote the closure and interior operators, respectively). But Q does not contain nonempty open set. However Dontechev [4] has introduced an S-space: A topological space X is called an S-space if every subset which contains a non-void open subset is open. But the concept of λ-open sets is different from Dontechev's S-spaces.

Definition 1.1. A subset A of a topological space X is said to be λ-*open* if A contains a nonempty open set. The complement of a λ-open set is said to be λ-*closed*.

For a subset A of a topological space X, $\mathrm{Int}_\lambda(A)$ and $\mathrm{Cl}_\lambda(A)$ are defined as follows:

Definition 1.2. Let X be a topological space and A be a subset of X.
$\mathrm{Int}_\lambda(A) = \cup\{U : U \subseteq A, U$ is λ-open in $X\}$;
$\mathrm{Cl}_\lambda(A) = \cap\{F : A \subseteq F, F$ is λ-closed in $X\}$.

Lemma 1.3. *Let X be a topological space and A, B subsets of X.*
(1) if $A \subseteq B$, then $\mathrm{Int}_\lambda(A) \subseteq \mathrm{Int}_\lambda(B)$ and $\mathrm{Cl}_\lambda(A) \subseteq \mathrm{Cl}_\lambda(B)$,
(2) $X \setminus \mathrm{Int}_\lambda(A) = \mathrm{Cl}_\lambda(X \setminus A)$,
(3) For any index set Δ, if A_α is λ-open (resp. λ-closed), then $\cup\{A_\alpha : \alpha \in \Delta\}$ is λ-open (resp. $\cap\{A_\alpha : \alpha \in \Delta\}$ is λ-closed),
(4) $\mathrm{Int}_\lambda(A)$ is λ-open and $\mathrm{Cl}_\lambda(A)$ is λ-closed.

Remark 1.4. The finite intersection of λ-open sets need not be λ-open. Let $\mathbf{R}$ be the usual real line, $A = (-1, 0]$ and $B = [0, 1)$. The A and B are λ-open but $A \cap B = \{0\}$ is not λ-open.

We generalize the locally closed set by using λ-open sets.

2. λ-Locally Closed Sets

Definition 2.1. A subset A of a topological space X is said to be λ-*locally closed* if $A = U \cap F$, where U is λ-open and F is closed.

Corollary 2.2. *Let $f : X \to Y$ be a continuous function. If L is a λ-locally closed subset of Y, then $f^{-1}(L)$ is λ-locally closed in X.*

From Definition 1.1 it is obvious that every locally closed set is λ-locally closed. But the converse need not hold in general.

Example 2.3. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}\}$. Then $C(X)$(all closed sets in $X) = \{\emptyset, X, \{b, c, d\}\}$. And λ-open sets are: $\emptyset$, X, $\{a\}$, $\{a, b\}$, $\{a, b, c\}$, $\{a, c\}$,

$\{a,d\},\ \{a,b,d\}, \{a,c,d\}$. Therefore, $\{d\} = \{a,d\} \cap \{b,c,d\}$ is a λ-locally closed set but it is not a locally closed set in X.

Remark 2.4. A subset A of a topological space X is λ-locally closed if and only if $X \setminus A$ is the union of a λ-closed set and an open set.

Remark 2.5. For a subset of a topological space, the following hold:

(1) Every λ-open set is λ-locally closed,

(2) Every closed set is λ-locally closed.

Theorem 2.6. *For a subset A of a topological space X, the following are equivalent:*

(1) A is λ-locally closed;

(2) $A = U \cap \mathrm{Cl}(A)$ for some λ-open set U;

(3) $A \cup (X \setminus \mathrm{Cl}(A))$ is λ-open;

(4) $A \subseteq \mathrm{Int}_\lambda[A \cup (X \setminus \mathrm{Cl}(A))]$;

(5) $\mathrm{Cl}(A) \setminus A$ is λ-closed.

Proof. (1) $\Rightarrow$ (2): Suppose A is λ-locally closed. Then $A = U \cap F$ where U is λ-open and F is closed. Then $\mathrm{Cl}(A) = \mathrm{Cl}(U \cap F) \subseteq \mathrm{Cl}(F) = F$. Then $A \subseteq U \cap \mathrm{Cl}(A) \subseteq U \cap F = A$ and hence $A = U \cap \mathrm{Cl}(A)$.

(2) $\Rightarrow$ (3): $X \setminus [A \cup (X \setminus \mathrm{Cl}(A))] = (X \setminus A) \cap \mathrm{Cl}(A) = \mathrm{Cl}(A) \setminus A = \mathrm{Cl}(A) \setminus (U \cap \mathrm{Cl}(A)) = \mathrm{Cl}(A) \setminus U = \mathrm{Cl}(A) \cap (X \setminus U)$. Since U is λ-open, $\mathrm{Cl}(A) \cap (X \setminus U)$ is λ-closed and hence $A \cup (X \setminus \mathrm{Cl}(A))$ is λ-open.

(3) $\Rightarrow$ (4): Since $A \cup (X \setminus \mathrm{Cl}(A))$ is a λ-open set containing A, it is obvious that $A \subset \mathrm{Int}_\lambda[A \cup (X \setminus \mathrm{Cl}(A))]$.

(4) $\Rightarrow$ (1): $A = A \cap \mathrm{Cl}(A) \subseteq \mathrm{Int}_\lambda[A \cup (X \setminus \mathrm{Cl}(A))] \cap \mathrm{Cl}(A) \subseteq [A \cup (X \setminus \mathrm{Cl}(A))] \cap \mathrm{Cl}(A) = A \cap \mathrm{Cl}(A) = A$. Therefore, $A = \mathrm{Int}_\lambda[A \cup (X \setminus \mathrm{Cl}(A))] \cap \mathrm{Cl}(A)$ and A is λ-locally closed.

(3) $\Leftrightarrow$ (5): It is obvious. □

The union of two λ-locally closed sets need not be λ-locally closed.

Example 2.7. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, X, \{a,b\}, \{c,d\}\}$. Then $C(X) = \{\emptyset, X, \{c,d\}, \{a,b\}\}$ and λ-open sets are: $\emptyset$, X, $\{a,b\}$, $\{c,d\}$, $\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$. λ-locally closed sets are: $\emptyset$, X, $\{a,b\}$, $\{c,d\}$, $\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$, $\{c\}$, $\{d\}$, $\{a\}$, $\{b\}$. Therefore, $\{a\}$ and $\{c\}$ are λ-locally closed sets but their union $\{a,c\}$ is not a λ-locally closed set.

3. Λ_λ-Closed Sets

Locally closed sets in a topological space are introduced and investigated in [3] and [6]. As a generalization of locally closed sets, Arenas et al. [1] introduced the notion of λ-closed sets in a topological space. In this section, we introduce the notion of Λ_λ-closed sets which is a generalization of λ-closed sets. We obtain some characterizations of Λ_λ-closed sets and obtain decompositions of closed sets.

Definition 3.1. Let X be a topological space and A a subset of X. The subset $\Lambda_\lambda(A)$ is defined as follows: $\Lambda_\lambda(A) = \cap\{U : A \subseteq U,\ U \text{ is } \lambda\text{-open }\}$.

A subset A is called a Λ_λ-*set* if $A = \Lambda_\lambda(A)$. If U is open in Definition 3.1, then a Λ_λ-set A is called a Λ-*set* [9].

Lemma 3.2. *For any subsets A and B of a topological space X, the following hold:*
(1) $A \subseteq \Lambda_\lambda(A)$,
(2) If $A \subseteq B$, then $\Lambda_\lambda(A) \subseteq \Lambda_\lambda(B)$,
(3) $\Lambda_\lambda(\Lambda_\lambda(A)) = \Lambda_\lambda(A)$,
(4) $\Lambda_\lambda(\cap_{\alpha\in\Delta}A_\alpha) \subseteq \cap_{\alpha\in\Delta}\Lambda_\lambda(A_\alpha)$ for any index set Δ.

Lemma 3.3. *For any subset A of a topological space X, the following hold:*
(1) $\Lambda_\lambda(A)$ is a Λ_λ-set,
(2) If A is λ-open, then A is a Λ_λ-set,
(3) If A_α is a Λ_λ-set for each $\alpha \in \Delta$, then $\cap_{\alpha\in\Delta}A_\alpha$ is a Λ_λ-set.

Remark 3.4. The converse of Lemma 3.3 (2) need not hold as shown by the following example: Let $\mathbf{R}$ be the usual real line and $A = \{0\}$. Then A is a Λ_λ-set but it is not λ-open. Because $\{0\} \subseteq \Lambda_\lambda(\{0\}) \subseteq (-1, 0] \cap [0, 1) = \{0\}$ and hence $\Lambda_\lambda(\{0\}) = \{0\}$. Therefore, $A = \{0\}$ is a Λ_λ-set but it is not λ-open.

Definition 3.5. A subset A of a topological space X is said to be Λ_λ-*closed* (resp. λ-*closed* [1]) if $A = L \cap F$, where L is a Λ_λ-set (resp. Λ-set) and F is a closed set.

Lemma 3.6. *For a subset of a topological space X, the following properties hold:*
(1) Every λ-locally closed set is Λ_λ-closed,
(2) Every λ-closed set is Λ_λ-closed.

Proof. (1) By Lemma 3.3, every λ-open set is a Λ_λ-set and (1) holds.
(2) Let U be a Λ-set. Then,

$$U = \cap\{V : U \subseteq V, V \text{ is open }\} \supseteq \cap\{V : U \subset V, V \text{ is } \lambda\text{-open }\} \supseteq U$$

and hence U is a Λ_λ-set. Therefore, (2) holds. □

Remark 3.7. By Lemma 3.6, we obtain the following diagram.

DIAGRAM I

$$\begin{array}{ccc} \text{locally closed} & \Rightarrow & \lambda\text{-locally closed} \\ \Downarrow & & \Downarrow \\ \lambda\text{-closed} & \Rightarrow & \Lambda_\lambda\text{-closed} \end{array}$$

Theorem 3.8. *For a subset A of a topological space X, the following are equivalent:*

(1) A is Λ_λ-closed;

(2) $A = U \cap \mathrm{Cl}(A)$ for some Λ_λ-set U;

(3) $A = \Lambda_\lambda(A) \cap \mathrm{Cl}(A)$.

Proof. (1) $\Rightarrow$ (2): Let A be a Λ_λ-closed set. Then $A = U \cap F$, where U is a Λ_λ-set and F is a closed set. Thus, we have $A \subseteq U \cap \mathrm{Cl}(A) \subseteq U \cap \mathrm{Cl}(F) = U \cap F = A$. Therefore, $A = U \cap \mathrm{Cl}(A)$.

(2) $\Rightarrow$ (3): Let $A = U \cap \mathrm{Cl}(A)$ for some Λ_λ-set U. Since $A \subseteq U$, by Lemma 3.2 $\Lambda_\lambda(A) \subseteq \Lambda_\lambda(U) = U$ and hence $A \subseteq \Lambda_\lambda(A) \cap \mathrm{Cl}(A) \subseteq U \cap \mathrm{Cl}(A) = A$. Therefore, we obtain $A = \Lambda_\lambda(A) \cap \mathrm{Cl}(A)$.

(3) $\Rightarrow$ (1): Let $A = \Lambda_\lambda(A) \cap \mathrm{Cl}(A)$. By Lemma 3.3, $\Lambda_\lambda(A)$ is a Λ_λ-set and $\mathrm{Cl}(A)$ is closed. Therefore, A is Λ_λ-closed. □

Definition 3.9. Let X be a topological space. A subset A of X is said to be *λg-closed* (resp. *g-closed* [8]) if $\mathrm{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is a λ-open (resp. open) set.

Theorem 3.10. *For a subset A of a topological space X, the following are equivalent:*

(1) A is closed;

(2) A is λ-locally closed and λg-closed;

(3) A is Λ_λ-closed and λg-closed.

Proof. (1) $\Rightarrow$ (2): Let A be closed in X. Since $A = X \cap A$ and X is a Λ_λ-set, A is λ-locally closed. Let U be any λ-open set containing A. Then $\mathrm{Cl}(A) = A \subseteq U$ and hence A is λg-closed.

(2) $\Rightarrow$ (3): By Lemma 3.6, every λ-locally closed set is Λ_λ-closed.

(3) $\Rightarrow$ (1): Let A be Λ_λ-closed and λg-closed. Since A is Λ_λ-closed, $A = P \cap L$, where P is a Λ_λ-set and L is closed in X. Let V be any λ-open set containing A. Since A is λg-closed, $\mathrm{Cl}(A) \subseteq V$ and hence $\mathrm{Cl}(A) \subseteq \cap\{V : A \subseteq V, V \text{ is } \lambda\text{-open}\} = \Lambda_\lambda(A)$. Therefore, $\mathrm{Cl}(A) \subseteq \Lambda_\lambda(A) \subseteq \Lambda_\lambda(P) = P$. On the other hand, $A \subseteq L$ and $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(L) = L$. Therefore, we obtain $\mathrm{Cl}(A) \subseteq P \cap L = A$. Thus A is closed.

□

Theorem 3.11. *Let X be a topological space. If A_α is a Λ_λ-closed set for each $\alpha \in \Delta$, then $\cap_{\alpha \in \Delta} A_\alpha$ is Λ_λ-closed.*

Proof. Let A_α be a Λ_λ-closed set for each $\alpha \in \Delta$. Then $A_\alpha = U_\alpha \cap F_\alpha$, where U_α is a Λ_λ-set and F_α is a closed set for each $\alpha \in \Delta$. By Lemma 3.3, $\cap_{\alpha \in \Delta} U_\alpha$ is a Λ_λ-set, $\cap_{\alpha \in \Delta} F_\alpha$ is closed and $\cap_{\alpha \in \Delta} A_\alpha = (\cap_{\alpha \in \Delta} U_\alpha) \cap (\cap_{\alpha \in \Delta} F_\alpha)$. Therefore, $\cap_{\alpha \in \Delta} A_\alpha$ is Λ_λ-closed.

□

4. Decompositions of Continuity

In this section, we obtain the decompositions of continuity.

Definition 4.1. A function $f : X \to Y$ is said to be

(1) λ-*LC-continuous* if $f^{-1}(V)$ is λ-locally closed in X for any closed set V of Y,

(2) Λ_λ-*continuous* if $f^{-1}(V)$ is Λ_λ-closed in X for any closed set V of Y,

(3) λg-*continuous* if $f^{-1}(V)$ is λg-closed in X for any closed set V of Y.

Theorem 4.2. *For a function $f : X \to Y$, the following are equivalent:*

(1) f is continuous;

(2) f is λ-LC-continuous and λg-continuous;

(3) f is Λ_λ-continuous and λg-continuous.

Proof. This is an immediate consequence of Theorem 3.10 □

Remark 4.3. The following facts are shown by Examples 4.4 and 4.5 and Remark 4.6:

(1) λ-LC-continuity and λg-continuity are independent of each other,

(2) Λ_λ-continuity and λg-continuity are independent of each other.

Example 4.4. Let $X = Y = \{a, b, c, d\}$, $\tau = \sigma = \{\emptyset, X, \{a\}\}$. Then $C(X) = C(Y) = \{\emptyset, \{b, c, d\}\}$ and λ-open sets in X (resp. Y) are: $\emptyset$, X, $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, b, c\}$, $\{a, c, d\}$, $\{a, b, d\}$. λ-locally closed sets in X (resp. Y) are: $\emptyset$, X, $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, b, c\}$, $\{a, c, d\}$, $\{a, b, d\}$, $\{b, c, d\}$, $\{b, c\}$, $\{c, d\}$, $\{b, d\}$, $\{b\}$, $\{c\}$, $\{d\}$. Define a function $f : X \to Y$ by $f(a) = c$, $f(b) = b$, $f(c) = d$, $f(d) = a$. Then we have the following:

(1) Since $f^{-1}(\{b, c, d\}) = \{a, b, c\}$, then f is not continuous.

(2) Since $f^{-1}(\{b, c, d\}) = \{a, b, c\}$, then f is λ-LC-continuous.

(3) Since $Cl(\{a, b, c\}) = X$ (i.e. $\{a, b, c\}$ is not λg-closed), then f is not λg-continuous.

(4) Since $\{a, b, c\} \subseteq \cap\{U : \ \{a, b, c\} \subseteq U, \ U \text{ is } \lambda\text{-open } \} = \{a, b, c\}$ and $\{a, b, c\} = \{a, b, c\} \cap X = \{a, b.c\}$, then $\{a, b, c\}$ is Λ_λ-closed. Thus f is Λ_λ-continuous.

Example 4.5. Let $X = Y = \{a, b, c, d\}$, $\tau = \sigma = \{\emptyset, X, \{a, b\}, \{c, d\}\}$. Then $C(X) = C(Y) = \{\emptyset, \ X, \ \{a, b\}, \ \{c, d\}\}$ and λ-open sets in X (resp. Y) are: $\emptyset$, X, $\{a, b\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$. And λ-locally closed sets in X (resp. Y) are: $\emptyset$, X, $\{a, b\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$. Define $g : X \to Y$ by $g(a) = c$, $g(b) = b$, $g(c) = a$, $g(d) = d$. Then we have the following:

(1) Since $g^{-1}(\{c, d\}) = \{a, d\}$, then g is not a continuous function.

(2) Since $g^{-1}(\{c, d\}) = \{a, d\}$, it is not a λ-locally closed set in X. Then g is not a λ-LC-continuous function.

(3) Since $g^{-1}(\{a, b\}) = \{b, c\} \subseteq \cap\{U : \ \{b, c\} \subseteq U, \ U \text{ is } \lambda \text{ -open in } \ X\} =$

$\{b,c\} \cap X = \{b,c\}$ and $g^{-1}(\{c,d\}) = \{a,d\} = \cap\{U : \{a,d\} \subseteq U,\ U$ is λ-open in $X\}$ $= \{a,d\} \cap X = \{a,d\}$ are Λ_λ-closed, then Λ_λ-continuous.

Remark 4.6. (1) If every λg-continuous function is λ-LC-continuous, then it is continuous from Theorem 4.2 This is not true from Example 4.4(1).

(2) If every λg-continuous function is Λ_λ-continuous, then it is continuous from Theorem 4.2. This not true from Example 4.5(1).

Acknowledgments

The authors wish to thank the referees for their valuable comments.

References

1. F. G. Arenas, J. Dontchev, M. Ganster, On λ-Sets and the Dual of Generalized Continuity, *Questions Answers General Topology*, **15**, (1997), 3–13.
2. R. A. Borzooei, G. R. Rezaei, N. Kouhestani, On (semi) Topological BL-Algebras, *Iranian Journal of Mathematical Sciences and Informatics*, **6**(1), (2011), 59–77.
3. N. Bourbaki, *General Topology*, Chapters 1-4, Springer-Verlag, 1989.
4. J. Dontechev, On Superconnected Spaces, *Serdica-Bulgaricae Mathematicae Publications*, **20**, (1994), 345–350.
5. A. A. Estaji, z-Weak Ideals and Prime Weak Ideals, *Iranian Journal of Mathematical Sciences and Informatics*, **7**(2), (2012), 53–62.
6. M. Ganster, I. L. Reilly, Locally Closed Sets and LC-continuous Functions, *Internat. J. Math. Math. Sci.*, **12**(3), (1989), 417–424.
7. N, Levine, Semi-open Sets and Semi-continuity in Topological Spaces, *Amer. Math. Monthly*, **70**, (1963), 36–41.
8. N. Levine, Generalized Closed Sets in Topology, *Rend. Circ. Mat. Palermo (2)*, **19**, (1970), 89–96.
9. H. Maki, Generalized Λ-Sets and the Associated Closure Operator, *The Special Issue in Commemoration of Prof. Kazusada IKEDA's Retirement*, (1986), 139–146.
10. A. S. Mashhour, M. E. Abd El-Monsef, S. N. El-Deeb, On Precontinuous and Week Precontinuous Mappings, *Proc. Math. Phys. Soc. Egypt.*, **53**, (1982), 47–53.
11. O. Njåstad, On Some Classes of Nearly Open Sets, *Pacific J. Math.*, **15**, (1965), 961–970.
12. T. Roudari, L. Torkzadeh, A Topology on BCK-Algebra via Left and Right Stabilizers, *Iranian Journal of Mathematical Sciences and Informatics*, **4**(2), (2009), 1–8.