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On the Lie-Santilli Admissibility

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Abstract. The largest class of hyperstructures is the one which satisfies the weak properties. We connect the theory of P-hopes, a large class of hyperoperations, with the Lie-Santilli admissibility used in Hardonic Mechanics. This can be achieved by a kind of Rees' sandwich hyperoperation.

Keywords: Hyperstructures, H_v-structures, Hopes.

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1. Introduction

1.1. **Notice.** The largest class of hyperstructures is the one which satisfies the weak properties. These are called H_v-structures introduced in 1990 [13], and they proved to have a lot of applications on several applied science such as linguistic, biology, chemistry, physics, and so on. The H_v-structures satisfy the *weak axioms* where the non-empty intersection replaces the equality. The H_v-structures can be used in models as an organized devise. In this paper we continuous our study on the Lie-Santilli's admissibility needed in applications,

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specially, in producin energy according to R.M. Santilli's iso-theory.

Recall some basic definitions:

Definition 1.1. A set H equipped with at least one hyperoperation (we abbreviate by ***hope*** any *hyperoperation*) $\cdot : H \times H \longrightarrow P(H)$, is called *Hyperstructure*, where $P(H)$ is the set of all subsets of H. We abbreviate by *WASS* the *weak associativity*:$(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H$ and by *COW* the *weak commutativity*: $xy \cap yx \neq \emptyset, \forall x, y \in H$.

The hyperstructure $(H, \cdot)$ is called ***H_v-semigroup*** if it is WASS and is called **H_v-group** if it is reproductive H_v-semigroup, i.e. $xH = Hx = H, \forall x \in H$. The hyperstructure $(R, +, \cdot)$ is called **H_v-ring** if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$, i.e.

$$x(y + z) \cap (xy + xz) \neq \emptyset, (x + y)z \cap (xz + yz) \neq \emptyset, \forall x, y, z \in R.$$

For more definitions and results on H_v-structures one can see in books and papers as [2],[4],[5],[6],[7],[14],[15],[18]. An extreme class of the H_v-structures is the following: An H_v-structure is called ***very thin*** iff all hopes are operations except one, which has all hyperproducts singletons except only one, which has cardinality more than one.

The fundamental relations β^*,γ^* and ε^* are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively [13],[14],[15]. The way to find the fundamental classes is given by analogous theorems to the following one:

Theorem 1.2. *Let $(H, \cdot)$ be an H_v-group and denote by $\boldsymbol{U}$ the set of all finite products of elements of H. We define the relation β in H as follows: $x\beta y$ iff $x, y \subset \mathbf{u}$ where $\mathbf{u} \in \mathbf{U}$. Then the fundamental relation β^* is the transitive closure of the relation β.*

Remark that the main point of the proof is that the β guaranties the validity of the following: Take two elements x, y such that $x, y \subset \mathbf{u} \in \mathbf{U}$ and any hyperproduct where one of the elements x, y, is used. Then, if this element is replaced by the other, the new hyperproduct is inside the same fundamental class where the first hyperproduct is. Therefore, if the 'hyperproducts' of the above β-classes are 'products', then, they are fundamental classes. Analogous remarks for the relations γ and ε, are also applied.

An element is called *single* if its fundamental class is a singleton.

Let $(H,\cdot),(H,\otimes)$ H_v-semigroups defined on the same set H. $(\cdot)$ is called ***smaller*** than $(\otimes)$, and $(\otimes)$ greater than $(\cdot)$, iff there exists automorphism

$$f \in Aut(H,\otimes) \;\; such\ that \;\; xy \subset f(x \otimes y), \forall x, y \in H.$$

Then we write $\cdot \leq \otimes$ and we say that $(H,\otimes)$ *contains* the $(H,\cdot)$. If $(H,\cdot)$ is a structure then it is called *basic structure* and $(H,\otimes)$ is called H_b-*structure*.

The Little Theorem. Greater hopes of hopes which are WASS or COW, are also WASS and COW, respectively.

The fundamental relations are used for general definitions of hyperstructures. Thus, to define the general H_v-field one uses the fundamental relation γ^*: The H_v-ring $(R,+,\cdot)$ is called ***H_v-field*** if the quotient R/γ^* is a field [13],[14]. The *H_v-module* is an H_v-group over an H_v-ring if the weak distributivity and a mixed weak associativity on all hopes, is valid. In an analogous way the *H_v-vector spaces* and the *H_v-algebra* can be defined [14].

The general definition of an H_v-Lie algebra was given in [16],[19] as follows:

Definition 1.3. Let $(\mathbf{L},+)$ be H_v-vector space over the field $(\mathbf{F},+,),\varphi:\mathbf{F}\longrightarrow \mathbf{F}/\gamma^*$, the canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of the fundamental field $\mathbf{F}/\gamma^*$. Similarly, let ω_L be the core of the canonical map $\varphi':\mathbf{L}\longrightarrow \mathbf{L}/\varepsilon^*$ and denote by the same symbol 0 the zero of $\mathbf{L}/\varepsilon^*$. Consider the *bracket (commutator) hope:*

$$[\ ,\ ]: L\times L\longrightarrow P(L):(x,y)\longrightarrow [x,y]$$

then $\mathbf{L}$ is an H_v-Lie algebra over $\mathbf{F}$ if the following axioms are satisfied:
(L1) The bracket hope is bilinear, i.e.
$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y])\neq\emptyset$
$[x,\lambda_1y_1+\lambda_2y_2]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2])\neq\emptyset,\ \forall x,x_1,x_2,y,y_1,y_2\in L\ and\ \lambda_1,\lambda_2\in F$
(L2) $[x,x]\cap\omega_L\neq\emptyset,\ \forall x\in L$
(L3) $([x,[y,z]+[y,[z,x]]+[z,[x,y]])\cap\omega_L\neq\emptyset,\ \forall x,y,z\in L$

This is a general definition thus one can use special cases in order to face problems in applied sciences. Moreover, we see how the weak properties can be defined as the above weak linearity (L1), anti-commutativity (L2) and the Jacobi identity (L3).

The *uniting elements* method was introduced by Corsini-Vougiouklis [14]. With this method one puts in the same class, two or more elements. This leads to structures satisfying additional properties. The 'enlarged' hyperstructures were examined in the sense that an extra element, outside the set, appears in one result. On the other direction one can obtain H_v-vector spaces, by taking out some elements [16].

Definition 1.4. Let $(H, \cdot)$ be hypergroupoid. We say that we *remove* $h \in H$, if we consider the restriction of the hope $(\cdot)$ on the $H - \{h\}$. We say that an $\underline{h} \in H$ absorbs $h \in H$ if we replace h, whenever it appears, by $\underline{h}$. We say that $\underline{h} \in H$ *merges* with $h \in H$, if we take as the product of any $h \in H$ by $\underline{h}$, the union of the results of x with both h and $\underline{h}$, and we consider h and $\underline{h}$ as one class, with representative $\underline{h}$.

The representation problem of H_v-structures by H_v-matrices is the following [14]:

H_v-matrix is a matrix with entries of an H_v-ring or H_v-field. The hyperproduct of two H_v-matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, of type $m \times n$ and $n \times r$, respectively, is defined in the usual manner but it is a set of $m \times r$ H_v-matrices:

$$\mathbf{A}.\mathbf{B} = (a_{ij}).(b_{ij}) = \{\mathbf{C} = (c_{ij}) | c_{ij} \in \oplus \sum a_{ik} b_{kj}\},$$

where $(\oplus)$ denotes the *n-ary circle hope* on the hyperaddition, i.e. the sum of products of elements of the H_v-ring is the union of the sets obtained with all possible parentheses put on them. The hyperproduct is not WASS.

Definition 1.5. Let $(H, \cdot)$ be H_v-group, consider an H_v-ring or H_v-field, $(R, +, \cdot)$ and a set

$$\mathbf{M}_R = \{(a_{ij}) | a_{ij} \in R\},$$

then is called ***H_v-matrix representation***, any map

$$\mathbf{T} : H \longrightarrow M_R, h \longrightarrow \mathbf{T}(h) \textit{ such that } \mathbf{T}(h_1 h_2) \cap \mathbf{T}(h_1)\mathbf{T}(h_2) \neq \emptyset, \forall h_1, h_2 \in H,$$

If $\mathbf{T}(h_1 h_2) \subset \mathbf{T}(h_1)\mathbf{T}(h_2), \forall h_1, h_2 \in H$, then $\mathbf{T}$ is an *inclusion representation*, if

$$\mathbf{T}(h_1 h_2) = \mathbf{T}(h_1)\mathbf{T}(h_2) = \{\mathbf{T}(h) | h \in h_1 h_2\}, \forall h_1, h_2 \in H,$$

then $\mathbf{T}$ is a *good representation*.

The main theorem of the theory of representations is:

Theorem 1.6. *A necessary condition in order to have an inclusion representation $\boldsymbol{T}$ of the H_v-group $(H, \cdot)$ by $n \times n$ H_v-matrices over the H_v-ring $(R, +, \cdot)$ is the following:*
For all classes $\beta^*(a), a \in H$ there must exist elements $a_{ij} \in R, i, j \in \{1, ..., n\}$ such that

$$\boldsymbol{T}(\beta^*(a)) \subset \{\boldsymbol{A} = (a'_{ij}) | a'_{ij} \in \gamma^*(a_{ij}), i, j \in \{1, ..., n\}\}.$$

2. Lie-Santilli Addmisibility

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems [9]. Santilli proposed a 'lifting'of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond to *e-hyperfields* which are the hyperstructures introduced by Santilli and Vougiouklis 1996 [10]. The H_v-fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics or in biology. The main definitions and constructions are presented on the H_v-structures. They are based on the partial order in H_v-structures and the Little Theorem [3],[4],[10],[19].

Definition 2.1. A hyperstructure $(H, \cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we normally assume that for every element x, there exists an inverse element, not necessarily unique, x^{-1}, i.e. $e \in x.x^{-1} \cap x^{-1}.x$.

A hyperstructure $(F, +, \cdot)$, where $(+)$ is operation and $(\cdot)$ is hope, is called *e-hyperfield* if the following axioms are valid:
1. $(F, +)$ is an abelian group with the additive unit 0,
2. $(\cdot)$ is WASS,
3. $(\cdot)$ is weak distributive with respect to $(+)$,
4. 0 is absorbing: $0.x = x.0 = 0, \forall x \in F$,
5. there exists a multiplicative scalar unit 1, i.e. $1.x = x.1 = x, \forall x \in F$,
6. for every non zero $x \in F$ there exists a unique inverse x^{-1}, such that $1 \in x.x^{-1} \cap x^{-1}.x$.

The elements of an e-hyperfield are called *e-hypernumbers*. In the case that the relation: $1 = x.x^{-1} = x^{-1}.x$, is valid, then we say that we have a *strong e-hyperfield.*

***M*ain e-Construction.** Given a group $(G, .)$, where e is the unit, then we define in G, an extremelly large number of hopes $(\otimes)$ as follows:

$$x \otimes y = \{xy, g_1, g_2, ...\}, \forall x, y \in G - \{e\}, \ \ and \ \ g_1, g_2, ... \in G - \{e\},$$

$g_1, g_2, ...$ are not the same for each (x, y). $(G, \otimes)$ becomes H_v-group, in fact an H_b-group containing the $(G, .)$. The $(G, \otimes)$ is an e-hypergroup. Moreover, if for each x, y such that $xy = e$, so we have $x \otimes y = xy$, then $(G, \otimes)$ becomes a strong e-hypergroup.

The proof is immediate. Moreover e is a unique scalar and for each x in G, there exists a unique inverse x^{-1}, such that $1 \in x.x^{-1} \cap x^{-1}.x$ and then we have $1 = x.x^{-1} = x^{-1}.x$. So the $(G, \otimes)$ is a strong e-hypergroup.

The above main e-construction gives an extremely large class of e-hopes but the most useful are the ones where only few products are enlarged.

Example of an e-hypergroup. Consider the non-commutative quaternion group

$$Q = \{1, -1, i, -i, j, -j, k, -k\}$$

One can obtain several hopes which define e-hypergroups. For example, denote

$$\underline{i} = \{i, -i\}, \underline{j} = \{j, -j\}, \underline{k} = \{k, -k\}$$

then we may define the $(*)$ hope by the table:

*	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	$\underline{k}$	k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	$\underline{i}$	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	$\underline{i}$
k	k	$\underline{k}$	$\underline{i}$	-j	-i	i	-1	1
-k	-k	k	$-\underline{i}$	j	$\underline{i}$	-i	1	-1

$(Q, *)$ is strong e-hypergroup since 1 is scalar and $-1, i, -i, j, -j, k$ and $-k$ have unique inverses $-1, -i, i, -j, j, -k$ and k, respectively, which are the inverses in the basic group.

A general way to define hopes, from given operations [12],[14] is the following:

Definition 2.2. Let $(G, .)$ be a groupoid, then for every set $P \subset G, P \neq \emptyset$, we define the following hopes called *P-hopes*:

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy), \forall x, y \in G$$

The $(G, P), (G, \underline{P}_r)$ and $(G, \underline{P}_l)$ are called *P-hyperstructures*. If $(G, .)$ is semigroup, then (G, P) is a semihypergroup but we do not know for $(G, \underline{P}_r), (G, \underline{P}_l)$. In some cases, mainly depending on the choice of P, the $(G, \underline{P}_r), (G, \underline{P}_l)$ can be associative or WASS. If in G, more operations are defined then for each operation several P-hopes can be defined.

In [2],[3] a P-hope was introduced which is appropriate for e-hyperstructures:

Construction. Let $(G, .)$ be abelian group and $P \subset G$, with more than one elements. We define a hope $(\times_P)$ as follows:

$$x \times_P y = \begin{cases} x.P.y = \{x.h.y | h \in P\} & x \neq e \quad and \quad y \neq e \\ x.y & x = e \quad or \quad y = e \end{cases}$$

we call this P_e*-hope*. The hyperstructure $(G, \times_P)$ is an abelian H$_v$-group.

Now we define a hope on non square matrices [13],[14],[17]:

Definition 2.3. Let $\mathbf{M} = \mathbf{M}_{m\times n}$ be a module of $m \times n$ matrices over R and $\mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}$.
We define, a kind of, a P-hope, $\underline{P}$ on $\mathbf{M}$ as follows:

$$\underline{P} : \mathbf{M} \times \mathbf{M} \longrightarrow \mathbf{P(M)} : (A, B) \longrightarrow A\underline{P}B = \{AP_i^t B : i \in I\} \subseteq \mathbf{M}$$

where P^t denotes the transpose of the matrix P.

The hope $\underline{P}$, which is a bilinear map, is a generalization of Rees' operation where, instead of one sandwich matrix, a set of sandwich matrices is used. $\underline{P}$ is associative and the inclusion distributivity law with respect to addition of matrices is valid:

$$A\underline{P}(B + C) \subseteq A\underline{P}B + A\underline{P}C, \forall A, B, C \in \mathbf{M}$$

Therefore, $(\mathbf{M}, +, P)$ defines a multiplicative hyperring, i.e. only the multiplication is a hope, on non-square matrices.

Definition 2.4. Let $\mathbf{M} = \mathbf{M}_{m\times n}$ be module of $m \times n$ matrices over a ring R and let take sets $\mathbf{S} = \{s_k : k \in K\} \subseteq R, \mathbf{Q} = \{Q_i : j \in J\} \subseteq \mathbf{M}$ and $\mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}$. Define three hopes as follows

$$\underline{S} : R \times \mathbf{M} \longrightarrow \mathbf{P(M)} : (r, A) \longrightarrow r\underline{S}A = \{(rs_k)A : k \in K\} \subseteq \mathbf{M}$$

$$\underline{Q}_+ : \mathbf{M} \times \mathbf{M} \longrightarrow \mathbf{P(M)} : (A, B) \longrightarrow A\underline{Q}_+B = \{A + Q_j + B : j \in J\} \subseteq \mathbf{M}$$

$$\underline{P} : \mathbf{M} \times \mathbf{M} \longrightarrow \mathbf{P(M)} : (A, B) \longrightarrow A\underline{P}B = \{AP_i^t B : i \in I\} \subseteq \mathbf{M}$$

Then $(M, S, Q_+, \underline{P})$ is the general ***matrix P-hyperalgebra*** over R.

In a similar way a generalization of this can be defined if one consider an H_v-ring or an H_v-field instead of a ring and using H_v-matrices instead of matrices.

Let $A = (a_{ij}), B = (b_{ij}) \in \mathbf{M}_{m\times n}$, we call (A, B) a *unitize pair* of matrices if $A^t B = I_n$, where I_n denotes the $n \times n$ unit matrix. We prove the following theorem which can be applied in the classical theory as well [19].

Theorem 2.5. *Proof.* If $m < n$, then there is no unitize pair. Suppose that $A^t B = (c_{ij})$, that is $c_{ij} = \sum\limits_{k=1}^{m} a_{ik} b_{kj}$, and we denote by A_m the block of the matrix A such that $A_m = (a_{ij}) \in \mathbf{M}_{m\times m}$, i.e. we consider the matrix of the first m columns. Then we suppose that we have $(A_m)^t B_m = I_m$, thus we must have $det(A_m) \neq 0$.

Now, since $n > m$, we take the homogeneous system with respect to the 'unknowns' $b_{1n}, b_{2n}, ..., b_{mn}$:

$$c_{in} = \sum_{k=1}^{m} a_{ik} b_{kn} = 0 \;\; for \;\; i = 1, 2, ..., m.$$

From which, since $det(A_m) \neq 0$, we obtain that $b_{1n} = b_{2n} = ... = b_{mn} = 0$. Using this fact on the last equation, on the same unknowns, $c_{nn} = \sum_{k=1}^{m} a_{nk}b_{kn} = 1$ we have $0 = 1$, absurd. $\square$

Now we deal with the Lie-Santilli admissibility [8],[9],[11],[19] on the non-square case. This problem can be faced in two ways:

(a) using ordinary numbers, as real or complex numbers, so using ordinary matrices and hopes, instead of operations on non-square matrices,

(b) using hypernumbers (e-hypernumbers) as entries and the ordinary operations on non-square hypermatrices.

The general construction is the following:

Construction Let $(\mathbf{L} = \mathbf{M}_{m\times n}, +)$ be an H$_v$-vector space of $m \times n$ hypermatrices over the H$_v$-field $(F, +, \cdot)$, $\varphi : \mathbf{F} \longrightarrow \mathbf{F}/\gamma^*$, the canonical map and $\omega_F = \{x \in \mathbf{F} : \varphi(x) = 0\}$, where 0 is the zero of the fundamental field $\mathbf{F}/\gamma^*$. Moreover, let ω_L be the core of the canonical map $\varphi' : \mathbf{L} \longrightarrow \mathbf{L}/\varepsilon^*$ and denote by the same symbol 0 the zero of $\mathbf{L}/\varepsilon^*$. Take two subsets $\mathbf{R}, \mathbf{S} \subseteq \mathbf{L}$ then a Santilli's Lie-admissible hyperalgebra is obtained by taking the Lie bracket, which is a hope:

$$[\ ,\ ]_{\mathbf{RS}} : \mathbf{L} \times \mathbf{L} \longrightarrow \mathbf{P(L)} : [x, y]_{\mathbf{RS}} = x\mathbf{R}^t y - y\mathbf{S}^t x.$$

More precisely,

$$[x, y]_{\mathbf{RS}} = x\mathbf{R}^t y - y\mathbf{S}^t x = \{xr^t y - ys^t x | r \in \mathbf{R}, s \in \mathbf{S}\}$$

Special case, but not degenerate, is for $R = \{r_1, r_2\}$ and $S = \{s_1, s_2\}$, even more if $R = S = P = \{P_1, P_2\}$ then we have

$$[x, y]_P = xP^t y - yP^t x = \{xP_1^t y - yP_1^t x, xP_1^t y - yP_2^t x, xP_2^t y - yP_1^t x, xP_2^t y - yP_2^t x\}$$

In the applications the most interesting cases are the ones which have results with small number of elements. Therefore, we need new types of matrices, with more properties especially for the matrices used in the set P of the P-hopes. Thus we introduce the following:

Definition 2.6. An $m \times n$ matrix over an associative ring with identity 1, is called ***monomial-like matrix***, if in each row and column there are at least one non-zero element. We assume that the number of the non-zero entries is the minimum needed. If the non-zero entries of a monomial-like matrix are equal to 1, then the matrix is called a ***permutation-like matrix***.

Remark 2.7. In the following, we restrict ourselves on the type of permutation-like matrices $P \in M_{m\times n}$, with $m < n$, where the first part $m \times m$ block, is the unit matrix I_m. The rest cases are analogous. For example, in the case of

$P \in M_{2\times 3}$, we have only the case $P = \{P_1, P_2\}$, where

$$P_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Property Consider the permutation-like matrix $P \in M_{m\times n}$, with $m < n$, then we could have more than one 1́s only in some lines. Therefore, when we multiply with a matrix $A = (a_{ij}) \in M_{m\times n}$, we have the following cases

1. The matrix PA^t, has some rows the same as in A^t, but maybe in other position, or some rows have as entries sums of elements of two or more corresponding rows of the matrix A^t.

2. In the matrix P^tA, there are repetitions of some rows of the matrix A and maybe in different positions.

3. The matrix AP^t, has some columns the same as in A^t, but maybe in other position, or some columns have as entries sums of elements of two or more corresponding columns of the matrix A^t.

4. In the matrix A^tP, there are repetitions of some columns of the matrix A and maybe in different positions.

An interesting problem is to find the set of matrices which are unitize pair with a given matrix.

EXAMPLE 2.8. The set of the matrices which are unitize pair with the transpose of the permutation-like matrices P_t^1 and P_t^2 , given in the above Remark 2.7, are, respectively,

$$X_1 = \begin{pmatrix} \kappa & 0 & 1-\kappa \\ \lambda & 1 & -\lambda \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & \kappa & -\kappa \\ 0 & \lambda & 1-\lambda \end{pmatrix}$$

since we can see that $X_1P_t^1 = I_2$ and $X_2P_t^2 = I_2$.

EXAMPLE 2.9. In the set $M_{2\times 3}$, we take as $P = \{P_1, P_2\}$, from above Remark 2.7 Consider the four dimensional matrices

$$X = \begin{pmatrix} x_1 & 0 & x_2 \\ 0 & x_3 & x_4 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & 0 & y_2 \\ 0 & y_3 & y_4 \end{pmatrix}$$

Then we obtain, after calculations, that the $P-$Lie bracket of them is

$$[X,Y]_P = XP^tY - YP^tX = \{\begin{pmatrix} x_2y_1 - x_1y_2 & 0 & x_1y_2 - x_2y_1 \\ x_4y_1 - x_1y_4 & 0 & x_4y_2 + x_3y_4 - x_2y_4 - x_4y_3 \end{pmatrix},$$

$$\begin{pmatrix} x_2y_1 & -x_3y_2 & x_1y_2 + x_2y_2 - x_2y_1 - x_4y_2 \\ x_4y_1 & -x_3y_4 & x_4y_2 + x_3y_4 - x_4y_3 - x_4y_4 \end{pmatrix},$$

$$\begin{pmatrix} -x_1y_2 & x_2y_3 & x_1y_2 + x_2y_4 - x_2y_1 - x_2y_2 \\ -x_1y_4 & x_4y_3 & x_3y_4 + x_4y_4 - x_2y_4 - x_4y_3 \end{pmatrix},$$

$$\begin{pmatrix} 0 & x_2y_3 - x_3y_2 & x_1y_2 + x_2y_4 - x_2y_1 - x_4y_2 \\ 0 & x_4y_3 - x_3y_4 & x_3y_4 - x_4y_3 \end{pmatrix}\}$$

Notice that we always have $0_{2\times 3} \in [X,X]_P$, for all $X \in M_{2\times 3}$.

Open problem: Find closed sets of matrices with respect to the P-Lie brackets.

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