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BCK-Algebras and Hyper BCK-Algebras Induced by a Deterministic Finite Automaton

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Abstract. In this note first we define a BCK-algebra on the states of a deterministic finite automaton. Then we show that it is a BCK-algebra with condition (S) and also it is a positive implicative BCK-algebra. Then we find some quotient BCK-algebras of it. After that we introduce a hyper BCK-algebra on the set of all equivalence classes of an equivalence relation on the states of a deterministic finite automaton and we prove that this hyper BCK-algebra is simple, strong normal and implicative. Finally we define a semi continuous deterministic finite automaton. Then we introduce a hyper BCK-algebra S on the states of this automaton and we show that S is a weak normal hyper BCK-algebra.

Keywords: Deterministic finite automaton, BCK-algebra , hyper BCK-algebra, quotient BCK-algebra.

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1. Introduction

The hyper algebraic structure theory was introduced by F. Marty [9] in 1934. Imai and Iseki [6] in 1966 introduced the notion of BCK-algebra. Meng and

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Jun [10] defined the quotient hyper BCK-algebras in 1994. Torkzadeh, Roodbari and Zahedi [12] introduced the hyper stabilizers and normal hyper BCK-algebras. Corsini and Leoreanu [4] found some connections between a deterministic finite automaton and the hyper algebraic structure theory. Now in this note first we introduce a BCK-algebra on the states of a deterministic finite automaton and we prove some theorems and obtain some related results. Also we define a hyper BCK-algebra on the set of all equivalence classes of an equivalence relation on the states of a deterministic finite automaton. Finally we introduce a hyper BCK-algebra on the states of a semi continuous deterministic finite automaton.

2. Preliminaries

Definition 2.1. [10] Let X be a set with a binary operation " $*$ " and a constant "0". Then $(X, *, 0)$ is called a BCK-algebra if it satisfies the following condition:

(i) $((x * y) * (x * z)) \ * \ (z * y) = 0$,

(ii) $(x * (x * y)) * y = 0$,

(iii) $x * x = 0$,

(iv) $0 * x = 0$,

(v) $x * y = 0$ and $y * x = 0$ imply $x = y$.

For all $x, y, z \in X$.

For brevity we also call X a BCK-algebra. If in X we define a binary relation"$\leq$ " by $x \leq y$ if and only if $x * y = 0$, then $(X, *, 0)$ is a BCK-algebra if and only if it satisfies the following axioms for all $x, y, z \in X$; ;

(I) $(x * y) * (x * z) \leq \ z * y$,

(II) $x * (x * y) \leq y$,

(III) $x \leq x$,

(IV) $0 \leq x$,

(V) $x \leq y$ and $y \leq x$ imply $x = y$.

Definition 2.2. [10] Given a BCK-algebra $(X, *, 0)$ and given elements a, b of X, we define

$$A\,(a, b) = \{x \in X| \ x * a \leq b\}.$$

If for all x, y in X, $A(x, y)$ has a greatest element then the BCK-algebra is called to be with condition (S).

Definition 2.3. [10] Let $(X, *, 0)$ be a BCK-algebra and let I be a nonempty subset of X. Then I is called to be an ideal of X if, for all x, y in X,

(i) $0 \in I$,

(ii) $x * y \in I$ and $y \in I$ imply $x \in I$.

Theorem 2.4. [10] Let I be an ideal of BCK-algebra X. if we define the relation $\sim_I$ on X as follows:

$$x \sim_I y \text{ if and only if } x\ o\ y \in I \text{ and } y\ o\ x \in I.$$

Then $\sim_I$ is a congruence relation on H.

Definition 2.5. [10] Let $(X, *, 0)$ be a BCK-algebra, I be an ideal of X and $\sim_I$ be an equivalence relation on X. we denote the equivalence class containing x by C_x and we denote X/I by $\{C_x : x \in H\}$. Also we define the operation $*\ : X/I \times X/I \to X/I$ as follows:

$$C_x *\ C_y \longrightarrow C_{x*y}.$$

Theorem 2.6. [10] Let I be an ideal of BCK-algebra X. Then I=C_0.

Theorem 2.7. [10] Let $(X, *, 0)$ be a BCK-algebra and I be an ideal of X . Then $(X/I, *,\ C_0)$ is a BCK-algebra.

Definition 2.8. [10] A BCK-algebra $(X, *, 0)$ is called positive implicative if it satisfies the following axiom:

$$(x * z) * (y * z) = (x * y) * z$$

for all $x, y, z \in X$.

Definition 2.9. [10] A nonempty subset I of a BCK-algebra X is called a varlet ideal of X if

(VI1) $x \in I$ and $y \leq x$ imply $y \in I$,

(VI2) $x \in I$ and $y \in I$ imply that there exists $z \in I$ such that $x \leq z$ and $y \leq z$.

Definition 2.10. [8] Let H be a nonempty set and "o" be a hyper operation on H, that is "o" is a function from $H \times H\ \ to\ \ \mathcal{P}^*(H) = \mathcal{P}(H) - \{\emptyset\}$. Then H is called a hyper BCK-algebra if it contains a constant "0" and satisfies the following axioms:

(HK1) $(x\ o\ z)\ \ o\ \ (y\ o\ z) \ll\ \ x\ o\ y$,

(HK2) $(x \text{ o } y) \text{ o } z$ =$(x \text{ o } z) \text{ o } y$,

(HK3) $x\ o\ H \ll \{x\}$,

(HK4) $x \ll y,\ y \ll x \Longrightarrow x = y$.

For all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x\ o\ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a\ \in A,\ \exists b\ \in B$ Such that $a \ll b$.

Theorem 2.11. [2] In a hyper BCK-algebra $(H, o, 0)$, the condition (HK3) is equivalent to the condition:

$x\ o\ y \ll \{x\}$ for all $x, y \in H$.

Definition 2.12. [7] Let I be a non-empty subset of a hyper BCK-algebra H and $0 \in I$. Then,

(1) I is called a weak hyper BCK-ideal of H if $x\ o\ y\ \subseteq I$ and $y \in I$ imply that $x \in I,$ for all $x, y \in H$.

(2)I is called a hyper BCK-ideal of H if $x\ o\ y\ \ll I$ and $y \in I$ imply that $x \in I,$ for all $x, y \in H$.

(3) I is called a strong hyper BCK-ideal of H if $(x\ o\ y) \cap\ I \neq \emptyset$ and $y \in I$ imply that $x \in I,$ for all $x, y \in H$.

Theorem 2.13. [7] Any strong hyper BCK-ideal of a hyper BCK-algebra H is a hyper BCK-ideal and a weak hyper BCK-ideal. Also any hyper BCK-ideal of a hyper BCK-algebra H is a weak hyper BCK-ideal.

Definition 2.14. [12] Let H be a hyper BCK-algebra and A be a nonempty subset of H. Then the sets$_lA = \{x \in H|\ \ a \in a\ o\ x\ \ \ \forall a \in A\}$ and $A_r = \{x \in H|\ \ x \in x\ o\ a\ \ \forall a \in A\}$ are called left hyper BCK-stabilizer of A and right hyper BCK-stabilizer of A, respectively.

Definition 2.15. [12] A hyper BCK-algebra H is called:
(i) Weak normal, if a_r is a weak hyper BCK-ideal of H for any element $a \in H$.
(ii) Normal, if a_r is a hyper BCK-ideal of H for any element $a \in H$.
(iii) Strong normal, if a_r is a strong hyper BCK-ideal of H for any element $a \in H$.

Definition 2.16. [11] A hyper BCK-algebra (H, o, 0) is called simple if for all distinct elements $a, b \in H - \{0\}\,,\ \ a \nleq b$ and $b \nleq a$.

Definition 2.17. [2] A hyper BCK-algebra $(H, o, 0)$ is called:
(i) Weak positive implicative (resp. positive implicative), if it satisfies the axiom

$$(x\ o\ z)\ o\ (y\ o\ z) \ll ((x\ o\ y)\ o\ z)\ \text{(resp.}\ \ (x\ o\ z)\,o\ (y\ o\ z) = (x\ o\ y)\,o\ z)$$

for all $x, y, z \in H$.
(ii) Implicative. if $x \ll x\ o\ (y\ o\ x)$, for all $x, y, z \in H$.

Definition 2.18. [5] A deterministic finite automaton consists of:
(i) A finite set of states, often denoted by S.
(ii) A finite set of input symbols, often denoted by M.
(iii) A transition function that takes as arguments a state and an input symbol and returns a state. The transition function will commonly be denoted by t, and in fact $t : S \times M \to S$ is a function.
(iv) A start state, one of the states in S such as s_0.
(v) A set of final or accepting states F. The set F is a subset of S.
For simplicity of notation we write $(S,\ M, s_0,\ F,\ t)$ for a deterministic finite automaton.

Remark 2.19. [5] Let (S, M, s_0, F, t) be a deterministic finite automaton. A word of M is the product of a finite sequence of elements in M, λ is empty word and M^* is the set of all words on M. We define recursively the extended transition function, $t^* : S \times M^* \longrightarrow S$, as follows:

$$\forall\ s\ \in S,\ \forall\ a\ \in M,\ t^*(s, a) = t(s, a)\,,$$

$$\forall\ s\ \in S,\ t^*(s, \lambda) = s,$$

$$\forall\ s\ \in S,\ \forall\ x \in M^*,\ \forall\ a\ \in M,\ t^*(s, ax) = t^*(t(s,a), x).$$

Note that the length $\ell(x)$ of a word $x \in M^*$ is the number of its letters. so $\ell(\lambda) = 0$ and $\ell(a_1 a_2) = 2$, where $a_1,\ a_2 \in M$.

Definition 2.20. [4] The state s of $S - \{s_0\}$ will be called connected to the state s_0 of S if there exists $x \in M^*$, such that $s = t^*(s_0, x)$.

3. *BCK*-algebras induced by a deterministic finite automaton

In this section we present some relationships between *BCK*-algebras and deterministic finite automata.

Definition 3.1. Let (S, M, s_0, F, t) be a deterministic finite automaton. If $s \in S - \{s_0\}$ is connected to s_0 , then the order of a state s is the natural number $l + 1$, where $l = \min\ \{\ell(x) \mid t^*(s_0, x) = s, \quad x \in M^*\}$, and if $s \in S - \{s_0\}$ is not connected to s_0 we suppose that the order of s is 1. Also we suppose that the order of s_0 is 0.

We denote the order of a state s by $ord\ s$.

Now, we define the relation $\sim$ on the set of states S, as follows:

$$s_1 \sim s_2 \Leftrightarrow ord\ s_1 = ord\ s_2$$

It is obvious that this relation is an equivalence relation on S.

Note that we denote the equivalence class of s by $\overline{s}$. Also we denote the set of all these classes by $\overline{S}$.

Theorem 3.2. Let (S, M, s_0, F, t) be a deterministic finite automaton. We define the following operation on S:

$$\forall\,(s_1, s_2) \in S^2,\ \ s_1 o s_2 = \begin{cases} s_0, & \text{if } ord\ s_1 < ord\ s_2, \quad s_1, s_2 \neq s_0, \quad s_1 \neq s_2 \\ s_1, & \text{if } ord\ s_1 \geq ord\ s_2, \quad s_1, s_2 \neq s_0, \quad s_1 \neq s_2 \\ s_0, & \text{if } s_1 = s_2 \\ s_0, & \text{if } s_1 = s_0\ , \quad s_2 \neq s_0 \\ s_1, & \text{if } s_2 = s_0\ , \quad s_1 \neq s_0 \end{cases}$$

Then $(S,\ o,\ s_0)$ is a *BCK*-algebra and s_0 is the zero element of S.

Proof. By definition of the operation 'o', we know that $t\ o\ t = s_0$ and $s_0\ o\ t = s_0$ for all $t \in S$. So $(S,\ o,\ s_0)$ satisfies (III) and (IV).

Now we consider the following situations to show that $(S,\ o,\ s_0)$ satisfies (I) and (II).

(i) Let $s_1, s_2, s_3 \neq s_0$ and $ord\ s_1 < ord\ s_2 < ord\ s_3$. Then $(s_1\ o\ s_2)\ o\ (s_1\ o\ s_3) = s_0\ o\ s_0 = s_0$ and $s_3\ o\ s_2 = s_3$. Since $s_0 \leq s_3$ we obtain that in this case (I) holds.

On the other hand, $s_1 \; o \; (s_1 \; o \; s_2) = s_1 \; o \; s_0 = s_1$ and $s_1 \; o \; s_2 = s_0$. Hence, in this case (II) holds.
(ii) Let $s_1, s_2, s_3 \neq s_0$ and $ord\, s_2 < ord\, s_1 < ord\, s_3$. Then $(s_1 \; o \; s_2) \; o \; (s_1 \; o \; s_3) = s_1 \; o \; s_0 = s_1$ and $s_3 \; o \; s_2 = s_3$. Since $s_1 \; o \; s_3 = s_0$ we get that $s_1 \leq s_3$. Thus in this case (I) holds.
Also $s_1 \; o \; (s_1 \; o \; s_2) = s_1 \; o \; s_1 = s_0$ and $s_0 \; o \; s_2 = s_0$. Therefore in this case (II) holds.
(iii) Let $s_1, s_2, s_3 \neq s_0$ and $ord\, s_2 < ord\, s_3 < ord\, s_1$. Then $(s_1 \; o \; s_2) \, o \, (s_1 \; o \; s_3) = s_1 \; o \; s_1 = s_0$ and $s_3 \; o \; s_2 = s_3$. Since $s_0 \leq s_3$ we obtain that in this case (I) holds.
On the other hand, $s_1 \; o \; (s_1 \; o \; s_2) = s_1 \; o \; s_1 = s_0$ and $s_0 \; o \; s_2 = s_0$. So in this case (II) holds.
(iv) Let $s_1, s_2, s_3 \neq s_0$ and $ord\, s_1 < ord\, s_3 < ord\, s_2$. Then $(s_1 \; o \; s_2) \, o \, (s_1 \; o \; s_3) = s_0 \; o \; s_0 = s_0$ and $s_3 \; o \; s_2 = s_0$. Since $s_0 \leq s_0$ we get that in this case (I) holds.
Also $s_1 \; o \; (s_1 \; o \; s_2) = s_1 \; o \; s_0 = s_1$ and $s_1 \; o \; s_2 = s_0$. Hence, in this case (II) holds.
(v) Let $s_1, s_2, s_3 \neq s_0$ and $ord\, s_3 < ord\, s_1 < ord\, s_2$. Then $(s_1 \; o \; s_2) \, o \, (s_1 \; o \; s_3) = s_0 \; o \; s_1 = s_0$ and $s_3 \; o \; s_2 = s_0$. Since $s_0 \leq s_0$ we obtain that in this case (I) holds.
On the other hand, $s_1 \; o \; (s_1 \; o \; s_2) = s_1 \; o \; s_0 = s_1$ and $s_1 \; o \; s_2 = s_0$. Thus in this case (II) holds.
(vi) Let $s_1, s_2, s_3 \neq s_0$ and $ord\, s_3 < ord\, s_2 < ord\, s_1$. Then $(s_1 \; o \; s_2) \, o \, (s_1 \; o \; s_3) = s_1 \; o \; s_1 = s_0$ and $s_3 \; o \; s_2 = s_0$. Since $s_0 \leq s_0$ we get that in this case (I) holds.
Also $s_1 \; o \; (s_1 \; o \; s_2) = s_1 \; o \; s_1 = s_0$ and $s_0 \; o \; s_2 = s_0$. So in this case (II) holds.
(vii) Let $s_1, s_2, s_3 \neq s_0$, $ord\; s_1 = ord\; s_2 < ord\; s_3$ and $s_1 \neq s_2$. Then $(s_1 \; o \; s_2) \, o \, (s_1 \; o \; s_3) = s_1 \; o \; s_0 = s_1$ and $s_3 \; o \; s_2 = s_3$. Since $s_1 \; o \; s_3 = s_0$ we get that $s_1 \leq s_3$. So in this case (I) holds.
On the other hand, $s_1 \; o \; (s_1 \; o \; s_2) = s_1 \; o \; s_1 = s_0$ and $s_0 \; o \; s_2 = s_0$. Therefore in this case (II) holds.
(viii) Let $s_1, s_2, s_3 \neq s_0$, $ord\; s_1 = ord\; s_2 > ord\; s_3$ and $s_1 \neq s_2$. Then $(s_1 \; o \; s_2) \, o \, (s_1 \; o \; s_3) = s_1 \; o \; s_1 = s_0$ and $s_3 \; o \; s_2 = s_0$. Since $s_0 \leq s_0$ we get that in this case (I) holds.
Also $s_1 \; o \; (s_1 \; o \; s_2) = s_1 \; o \; s_1 = s_0$ and $s_0 \; o \; s_2 = s_0$. Hence, in this case (II) holds.
(ix) Let $s_1, s_2, s_3 \neq s_0$, $ord\; s_1 = ord\; s_3 < ord\; s_2$ and $s_1 \neq s_3$. Then $(s_1 \; o \; s_2) \, o \, (s_1 \; o \; s_3) = s_0 \; o \; s_1 = s_0$ and $s_3 \; o \; s_2 = s_0$. Since $s_0 \leq s_0$ we obtain that in this case (I) holds.
On the other hand, $s_1 \; o \; (s_1 \; o \; s_2) = s_1 \; o \; s_0 = s_1$ and $s_1 \; o \; s_2 = s_0$. Thus in this case (II) holds.
(x) $s_1, s_2, s_3 \neq s_0$, $ord\; s_1 = ord\; s_3 > ord\; s_2$ and $s_1 \neq s_3$.
Then $(s_1 \; o \; s_2) \, o \, (s_1 \; o \; s_3) = s_1 \; o \; s_1 = s_0$ and $s_3 \; o \; s_2 = s_3$. Since $s_0 \leq s_3$ we get that in this case (I) holds.

Also $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. So in this case (II) holds.
(xi) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_2 = ord\ s_3 > ord\ s_1$ and $s_2 \neq s_3$. Then $(s_1\ o\ s_2)\,o\,(s_1\ o\ s_3) = s_0\ o\ s_0 = s_0$ and $s_3\ o\ s_2 = s_3$. Since $s_0 \leq s_3$ we obtain that in this case (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. Therefore in this case (II) holds.
(xii) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_2 = ord\ s_3 < ord\ s_1$ and $s_2 \neq s_3$. Then $(s_1\ o\ s_2)\,o\,(s_1\ o\ s_3) = s_1\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_3$. Since $s_0 \leq s_3$ we get that in this case (I) holds.
Also $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. Hence, in this case (II) holds.
(xiii) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_1 = ord\ s_2 = ord\ s_3$ and $s_1 \neq s_2 \neq s_3 \neq s_1$. Then $(s_1\ o\ s_2)\,o\,(s_1\ o\ s_3) = s_1\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_3$. Since $s_0 \leq s_3$ we obtain that in this case (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. Thus in this case (II) holds.
(xiv) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_1 = ord\ s_3$, $s_1 \neq s_3$ and $s_1 = s_2$. Then $(s_1\ o\ s_2)\,o\,(s_1\ o\ s_3) = s_0\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_3$. Since $s_0 \leq s_3$ we get that in this case (I) holds.
Also $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. So in this case (II) holds.
(xv) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_1 = ord\ s_2$, $s_1 \neq s_2$ and $s_1 = s_3$. Then $(s_1\ o\ s_2)\,o\,(s_1\ o\ s_3) = s_1\ o\ s_0 = s_1$ and $s_3\ o\ s_2 = s_3$. Since $s_1\ o\ s_3 = s_0$ we get that $s_1 \leq s_3$. So in this case (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. Therefore in this case (II) holds.
(xvi) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_1 = ord\ s_2$, $s_1 \neq s_2$ and $s_2 = s_3$. Then $(s_1\ o\ s_2)\,o\,(s_1\ o\ s_3) = s_1\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we get that in this case (I) holds.
Also $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. Hence, in this case (II) holds.
(xvii) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_1 < ord\ s_3$ and $s_1 = s_2$. Then $(s_1\ o\ s_2)\,o\,(s_1\ o\ s_3) = s_0\ o\ s_0 = s_0$ and $s_3\ o\ s_2 = s_3$. Since $s_0 \leq s_3$ we obtain that in this case (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. Thus in this case (II) holds.
(xviii) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_1 > ord\ s_3$ and $s_1 = s_2$. Then $(s_1\ o\ s_2)\,o\,(s_1\ o\ s_3) = s_0\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we get that in this case (I) holds.
Also $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. So in this case (II) holds.
(xix) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_1 < ord\ s_2$ and $s_1 = s_3$. Then $(s_1\ o\ s_2)\,o\,(s_1\ o\ s_3) = s_0\ o\ s_0 = s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we obtain that in this case (I) holds.

On the other hand, $s_1\ o\ (s_1\ o\ s_2)\ = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. Therefore in this case (II) holds.

(xx) Let $s_1, s_2, s_3 \neq s_0$, $ord\, s_1 > ord\, s_2$ and $s_1 = s_3$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_0 =\ s_1$ and $s_3\ o\ s_2 = s_3 =\ s_1$. Since $s_1 \leq s_1$ we get that in this case (I) holds.

Also $s_1\ o\ (s_1\ o\ s_2)\ = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. Hence, in this case (II) holds.

(xxi) Let $s_1, s_2, s_3 \neq s_0$, $ord\, s_1 < ord\, s_2$ and $s_2 = s_3$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_0 =\ s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we obtain that in this case (I) holds.

On the other hand, $s_1\ o\ (s_1\ o\ s_2)\ = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. Thus in this case (II) holds.

(xxii) Let $s_1, s_2, s_3 \neq s_0$, $ord\, s_1 > ord\, s_2$ and $s_2 = s_3$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_1 =\ s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we get that in this case (I) holds.

Also $s_1\ o\ (s_1\ o\ s_2)\ = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. So in this case (II) holds.

(xxiii) Let $s_1 = s_2 = s_3$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_0 =\ s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we obtain that in this case (I) holds.

On the other hand, $s_1\ o\ (s_1\ o\ s_2)\ = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. Therefore in this case (II) holds.

(xxiv) Let $s_1 = s_0$ and $s_2, s_3 \neq s_0$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_0 =\ s_0$. Let $s_3\ o\ s_2 = t$ and $t \in S$. Since $s_0 \leq t$ we get that in this case (I) holds.

Also $s_1\ o\ (s_1\ o\ s_2)\ = s_0\ o\ s_0 = s_0$ and $s_0\ o\ s_2 = s_0$. Hence, in this case (II) holds.

(xxv) Let $s_2 = s_0$, $s_1, s_3 \neq s_0$. Since $s_1\ o\ s_3 = s_1$ or $s_1\ o\ s_3 = s_0$, we have two cases:

(6) $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_1 =\ s_0$. We know that $s_3\ o\ s_2 = s_3$. Since $s_0 \leq s_3$ we conclude that in this case (I) holds.

(7) $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_0 =\ s_1$. We know that $s_3\ o\ s_2 = s_3$ and in this case $s_1\ o\ s_3 = s_0$. So $s_1 \leq s_3$ and (I) holds.

On the other hand, $s_1\ o\ (s_1\ o\ s_2)\ = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. Thus in this case (II) holds.

(xxvi) Let $s_3 = s_0$ and $s_1, s_2 \neq s_0$. Since $s_1\ o\ s_2 = s_1$ or $s_1\ o\ s_2 = s_0$, we obtain that $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_1 =\ s_0$ or $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_1 =\ s_0$. Also $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we conclude that in this case (I) holds.

The proof of (II) is studied in other cases.

(xxvii) Let $s_1 \neq s_0$ and $s_2 = s_3 = s_0$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_1 =\ s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we obtain that in this case (I) holds.

On the other hand, $s_1\ o\ (s_1\ o\ s_2)\ = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. Therefore in this case (II) holds.

(xxviii) Let $s_3 \neq s_0$ and $s_1 = s_2 = s_0$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_0 =\ s_0$. and $s_1\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we get that in this case (I) holds.

Also $s_1\ o\ (s_1\ o\ s_2)\ = s_0\ o\ s_0 = s_0$ and $s_0\ o\ s_2 = s_0$. Hence, in this case (II) holds.
(xxix) Let $s_2 \neq s_0$ and $s_1 = s_3 = s_0$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_0 =\ s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we obtain that in this case (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2)\ = s_0\ o\ s_0 = s_0$ and $s_0\ o\ s_2 = s_0$. Thus in this case (II) holds.
So we conclude that $(S,\ o,\ s_0)$ satisfies (I) and (II).
To prove (V), Let $s_1 \leq s_2$ and $s_2 \leq s_1$. If $s_1 = s_2$, then we are done. Otherwise, since $s_1 \leq s_2$, there exist two cases:
(i) $ord\ s_1 < ord\ s_2,\quad s_1, s_2 \neq s_0,\quad s_1 \neq s_2$. Then $s_2 o\ s_1 = s_2$. Therefore $s_2 \nleq s_1$, which is a contradiction.
(ii) $s_1 = s_0$, $s_2 \neq s_0$. Then $s_2 o\ s_1 = s_2 o\ s_0 = s_2$. Thus $s_2 \nleq s_1$, which is a contradiction.
So we show that $(S,\ o,\ s_0)$ is a BCK-algebra.
Example 3.3. Let $A = (S, M, s_0, F, t)$ be a deterministic finite automaton such that $S = \{q_0, q_1, q_2, q_3\}$, $M = \{a, b\}$, $s_0 = q_0$, $F = \{q_1, q_3\}$ and t is defiend by

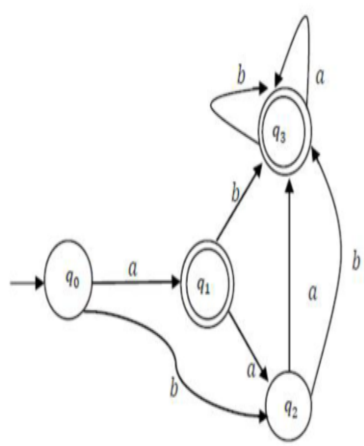

FIGURE 1

$$t\,(q_0, a) = q_1,\ t\,(q_0, b) = q_2,\ t\,(q_1, a) = q_2, t\,(q_1, b) = q_3,$$

$$t\,(q_2, a) = q_3,\ t\,(q_2, b) = q_3,\ t\,(q_3, a) = q_3,\ t\,(q_3, b) = q_3.$$

It is easy to see that $ord\ q_1 = ord\ q_2 = 2, ord\ q_3 = 3$ and $ord\ q_0 = 0$. According to the definition of operation "o" which is defined in Theorem 3.2, we have the following table:

Table 1.

O	q_0	q_1	q_2	q_3
q_0	q_0	q_0	q_0	q_0
q_1	q_1	q_0	q_1	q_0
q_2	q_2	q_2	q_0	q_0
q_3	q_3	q_3	q_3	q_0

In this section we suppose that $(S,\ o,\ s_0)$ is the BCK-algebra, which is defined in Theorem 3.2.

Notation. We denote the class of all states which their order is n by $\overline{s_n}$.

Theorem 3.4. $(S,\ o, s_0)$ is a BCK-algebra with condition (S).

Proof: Let $s_1, s_2 \in S, ord\ s_1 = n$ and $ord\ s_2 = m$. Then we should consider following situations:

(1) Let $ord\ s_1 < ord\ s_2, \quad s_1, s_2 \neq s_0, \quad s_1 \neq s_2$. Then $A(s_1, s_2) = \bigcup_{i=0}^{m-1} \overline{s_i} \cup \{s_2\}$ and the greatest element of $A(s_1, s_2)$ is s_2.

(2) Let $ord\ s_1 \geq ord\ s_2, \quad s_1, s_2 \neq s_0, \quad s_1 \neq s_2$. Then $A(s_1, s_2) = \bigcup_{i=0}^{n-1} \overline{s_i} \cup \{s_1\}$ and the greatest element of $A(s_1, s_2)$ is s_1.

(3) $s_1 = s_2$. Then $A(s_1, s_2) = \bigcup_{i=0}^{n-1} \overline{s_i} \cup \{s_1\}$ and the greatest element of $A(s_1, s_2)$ is s_1.

(4) Let $s_1 = s_0\ , \quad s_2 \neq s_0$. Then $A(s_1, s_2) = \bigcup_{i=0}^{m-1} \overline{s_i} \cup \{s_2\}$ and the greatest element of $A(s_1, s_2)$ is s_2.

(5) Let $s_1 \neq s_0\ , \quad s_2 = s_0$. Then $A(s_1, s_2) = \bigcup_{i=0}^{n-1} \overline{s_i} \cup \{s_1\}$ and the greatest element of $A(s_1, s_2)$ is s_1.

Theorem 3.5. Let $I_n = \{s \in S \mid s \in \bigcup_{i=0}^{n} \overline{s_i}\}$ for any $n \in N$. Then I_n is an ideal of $(S,\ o, s_0)$.

Proof. Suppose that $s_1\ o\ s_2 \in I_n$ and $s_2 \in I_n$, then we have the following situations:

(1) $$s_1 \neq s_2,\ s_2 \neq s_0 \text{ and } ords_2 < ords_1.$$

By definition of the operation "o", we know that $s_1 o\ s_2 = s_1$. So $s_1 \in I_n$.

(2) $$s_1 \neq s_2,\ s_2 \neq s_0 \text{ and } ords_2 = ords_1.$$

Since $s_2 \in I_n$and $\overline{s_2} \subseteq I_n$, we obtain that $s_1 \in I_n$.

(3) $$s_1 \neq s_2,\ s_1 \neq s_0 \text{ and } ords_1 < ords_2.$$

By definition of I_n, it is easy to see that $s_1 \in I_n$.

(4) $$s_1 = s_2.$$

It is clear that $s_1 \in I_n$.

(5) $$s_2 = s_0.$$

By definition of the operation "o", we know that $s_1 o\ s_2 = s_1$. So $s_1 \in I_n$.
(6) $s_1 = s_0$.
Since $s_0 \in I_n$, we get that $s_1 \in I_n$.
Also by definition of I_n, we know that $s_0 \in I_n$. So I_n is an ideal of S.
Theorem 3.6. Let I_n be a set, which is defined in Theorem 3.5. Then $C_x = \{x\}$ for all $x \notin I_n$.
Proof. Let $x \notin I_n$. By Theorem 2.6, we know that $I_n = C_{s_0}$. So $s_0 \notin C_x$. Now we suppose that $y \in C_x$ and $y \neq x$. By definition of the equivalence relation $\sim_{I_n}$, we know that $x\ o\ y \in I_n$ and $y\ o\ x \in I_n$. Since $x \notin I_n$ and $x\ o\ y \in I_n$, we obtain that $ord\ x \ngeq ord\ y$. So $ord\ y > ord\ x$ and $y\ o\ x = y \in I_n = C_{s_0}$, which is a contradiction. Hence, $y = x$.
Theorem 3.7. Let $I_n$be the ideal of S which is defined in Theorem 3.5. Then $(S/I_n, *,\ C_{s_0})$ is a BCK-algebra.
Proof. By Theorem 2.7, it is obvious that $(S/I_n, *,\ C_{s_0})$ is a BCK-algebra.
Theorem 3.8. $(S,\ o,\ s_0)$ is a positive implicative BCK-algebra.
Proof. By considering 29 situations which have been stated in the proof of Theorem 3.2, we get that in all cases $(s_1\ o\ s_3)\,o\ (s_2\ o\ s_3) = (s_1\ o\ s_2)\,o\ s_3$, for all $s_1, s_2, s_3 \in S$. So $(S,\ o,\ s_0)$ is a positive implicative BCK-algebra.
Theorem 3.9. Let $n = \max\ \{ord\ s \mid s \in S\}$. Then $I = \bigcup_{i=0}^{m-1} \overline{s_i} \cup \{z\}$ for $1 \leq m \leq n$ and $z \in s_m$, is a varlet ideal of $(S,\ o,\ s_0)$.
Proof. To prove (VI1), we suppose that $x \in I$ and $y \leq x$. Then $s_0 = y\ o\ x$ and we have three cases:
(6) Let $ord\ y < ord\ x$, $x, y \neq s_0$ and $x \neq y$. Then by definition of I, it is obvious that $y \in I$.
(7) Let $x = y$. Then it is clear that $y \in I$.
(3) Let $y = s_0$, $x \neq s_0$. Then by definition of I, it is easy to see that $s_0 = y \in I$.
Therefore (VI1) holds.
Now we show that I satisfies (VI2). let $x \in I,\ y \in I$ and $x, y \neq z$. Since $ord\ x < ord\ z$ and $ord\ y < ord\ z$, we get that $x\ o\ z = s_0$ and $y\ o\ z = s_0$. So $x \leq z$ and $y \leq z$. Also if $x \in I,\ y \in I$, $x = z$ and $y \neq z$, then $x\ o\ z = z\ o\ z = s_0$ and $y\ o\ z = s_0$. Thus $x \leq z$ and $y \leq z$. Similarly we can prove that $x \leq z$ and $y \leq z$ for the following cases:

$$(6) \qquad x \in I,\ y \in I, x \neq z \text{ and } y = z,$$

$$(7) \qquad x \in I,\ y \in I, x = z \text{ and } y = z.$$

So (VI2) holds.

4. Hyper BCK-algebras induced by a deterministic finite automaton

Theorem 4.1. Let (S, M, s_0, F, t) be a deterministic finite automata. We define the following hyper operation on $\overline{S}$:

$$\forall\ (\overline{s_1}, \overline{s_2}) \in \overline{S}^2,\ \ \overline{s_1} o\, \overline{s_2} = \begin{cases} \overline{s_1}\ , & \text{if } \ \overline{s_1} \neq \overline{s_2},\ \overline{s_2} \neq \overline{s_0} \neq \overline{s_1} \\ \{\overline{s_0},\ \overline{s_1}\}\ , & \text{if } \ \overline{s_1} = \overline{s_2} \\ \overline{s_0}\ , & \text{if } \ \overline{s_1} = \overline{s_0},\ \overline{s_2} \neq \overline{s_0} \\ \overline{s_1}\ , & \text{if } \ \overline{s_1} \neq \overline{s_0},\ \overline{s_2} = \overline{s_0}. \end{cases}$$

Then $(\overline{S},\ o,\ \overline{s_0})$ is a hyper BCK-algebra and $\overline{s_0}$ is the zero element of $\overline{S}$.
Proof. First we have to consider the following situations to show that $(\overline{S},\ o,\ \overline{s_0})$satisfies (HK1) and (HK2).
(i) Let $\overline{s_1}, \overline{s_2}, \overline{s_3} \neq \overline{s_0}$ and $\overline{s_3} \neq\ \overline{s_2} \neq \overline{s_1} \neq \overline{s_3}$. Then $(\overline{s_1}\ o\, \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\, \overline{s_3}) = \overline{s_1}\ \ o\ \ \overline{s_2}$. Since $\overline{s}\ \ o\ \ \overline{s}\ =\ \{\overline{s_0},\ \overline{s}\}$ we obtain that $\overline{s} \ll \overline{s}$ for any $\overline{s} \in \overline{S}$. So $(\overline{s_1}\ o\, \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\, \overline{s_3}) \ll \overline{s_1}\ o\, \overline{s_2}$ and in this case (HK1) holds.
Also $(\overline{s_1}\ o\, \overline{s_2})\, o\, \overline{s_3} = \overline{s_1}\ o\, \overline{s_3} = \overline{s_1}$ and $(\overline{s_1}\ o\, \overline{s_3})\, o\, \overline{s_2} = \overline{s_1}\ o\, \overline{s_2} = \overline{s_1}$. Thus in this case (HK2) holds.
(ii) Let $\overline{s_1}, \overline{s_2}, \overline{s_3} \neq \overline{s_0}$ and $\overline{s_1} = \overline{s_2} \neq \overline{s_3}$. Then$(\overline{s_1}\ o\, \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\, \overline{s_3}) = \overline{s_1}\ o\, \overline{s_2}$. So $(\overline{s_1}\ o\, \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\, \overline{s_3}) \ll \overline{s_1}\ o\, \overline{s_2}$ and in this case (HK1) holds.
On the other hand, $(\overline{s_1}\ o\, \overline{s_2})\, o\, \overline{s_3} = \{\overline{s_0},\ \overline{s_1}\}\ \ o\, \overline{s_3} = \{\overline{s_0},\ \overline{s_1}\}$ and $(\overline{s_1}\ o\, \overline{s_3})\, o\, \overline{s_2} = \overline{s_1}\ o\, \overline{s_2} = \{\overline{s_0},\ \overline{s_1}\}$. Therefore in this case (HK2) holds.
(iii) Let $\overline{s_1}, \overline{s_2}, \overline{s_3} \neq \overline{s_0}$ and $\overline{s_1} = \overline{s_3} \neq \overline{s_2}$.
Then $(\overline{s_1}\ o\, \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\, \overline{s_3}) = \{\overline{s_0},\ \overline{s_1}\}\ \ o\, \overline{s_2} = \{\overline{s_0},\ \overline{s_1}\}$ and $\overline{s_1}\ o\, \overline{s_2} = \overline{s_1}$. Since $\overline{s_0}\ o\, \overline{s_1} = \overline{s_0}$ we obtain that $\overline{s_0} \ll\ \overline{s_1}$ and also we know that $\overline{s_1} \ll \overline{s_1}$. Hence, $(\overline{s_1}\ o\, \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\, \overline{s_3}) \ll \overline{s_1}\ o\, \overline{s_2}$ and in this case (HK1) holds.
Also $(\overline{s_1}\ o\, \overline{s_2})\, o\, \overline{s_3} = \overline{s_1}\ o\, \overline{s_3} = \{\overline{s_0},\ \overline{s_1}\}$ and $(\overline{s_1}\ o\, \overline{s_3})\, o\, \overline{s_2} = \{\overline{s_0},\ \overline{s_1}\}\ \ o\, \overline{s_2} = \{\overline{s_0},\ \overline{s_1}\}$. So in this case (HK2) holds.
(iv) Let $\overline{s_1}, \overline{s_2}, \overline{s_3} \neq \overline{s_0}$ and $\overline{s_2} = \overline{s_3} \neq \overline{s_1}$.
Then$(\overline{s_1}\ o\, \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\, \overline{s_3}) = \overline{s_1}\ \ o\ \ \{\overline{s_0},\ \overline{s_2}\} =\ \overline{s_1}$ and $\overline{s_1}\ \ o\ \ \overline{s_2} = \overline{s_1}$. Thus $(\overline{s_1}\ o\, \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\, \overline{s_3}) \ll \overline{s_1}\ o\, \overline{s_2}$ and in this case (HK1) holds.

On the other hand, $(\overline{s_1}\ o\ \overline{s_2})\ o\ \overline{s_3} = \overline{s_1}\ o\ \overline{s_3} = \overline{s_1}$ and $(\overline{s_1}\ o\ \overline{s_3})\ o\ \overline{s_2} = \overline{s_1}\ o\ \overline{s_2} = \overline{s_1}$. Therefore in this case (HK2) holds.

(v) Let $\overline{s_1} = \overline{s_2} = \overline{s_3}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) = \{\overline{s_0},\ \overline{s_1}\}\ o\ \{\overline{s_0},\ \overline{s_1}\} = \{\overline{s_0},\ \overline{s_1}\}$ and $\overline{s_1}\ o\ \overline{s_2} = \{\overline{s_0},\ \overline{s_1}\}$. So $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case$(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK1).

Also $(\overline{s_1}\ o\ \overline{s_2})\ o\ \overline{s_3} = (\overline{s_1}\ o\ \overline{s_1})\ o\ \overline{s_1} = (\overline{s_1}\ o\ \overline{s_3})\ o\ \overline{s_2}$. Hence, in this case $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK2).

(vi) Let $\overline{s_2}, \overline{s_3} \neq \overline{s_0}$, $\overline{s_1} = \overline{s_0}$ and $\overline{s_2} \neq \overline{s_3}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) = \overline{s_0}\ o\ \overline{s_2} = \overline{s_0}$ and $\overline{s_1}\ o\ \overline{s_2} = \overline{s_0}\ o\ \overline{s_2} = \overline{s_0}$. Thus $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case (HK1) holds.

On the other hand, $(\overline{s_1}\ o\ \overline{s_2})\ o\ \overline{s_3} = \overline{s_0}\ o\ \overline{s_3} = \overline{s_0}$ and $(\overline{s_1}\ o\ \overline{s_3})\ o\ \overline{s_2} = \overline{s_0}\ o\ \overline{s_2} = \overline{s_0}$. So in this case (HK2) holds.

(vii) Let $\overline{s_2}, \overline{s_3} \neq \overline{s_0}$, $\overline{s_1} = \overline{s_0}$ and $\overline{s_2} = \overline{s_3}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) = \overline{s_0}\ o\ \{\overline{s_0},\ \overline{s_2}\} = \overline{s_0}$ and $\overline{s_1}\ o\ \overline{s_2} = \overline{s_0}\ o\ \overline{s_2} = \overline{s_0}$. Therefore $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case$(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK1).

Also$(\overline{s_1}\ o\ \overline{s_2})\ o\ \overline{s_3} = \overline{s_0}\ o\ \overline{s_3} = \overline{s_0}$ and $(\overline{s_1}\ o\ \overline{s_3})\ o\ \overline{s_2} = \overline{s_0}\ o\ \overline{s_2} = \overline{s_0}$. So in this case $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK2).

(viii) Let $\overline{s_1}, \overline{s_3} \neq \overline{s_0}$, $\overline{s_2} = \overline{s_0}$ and $\overline{s_1} \neq \overline{s_3}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) = \overline{s_1}\ o\ \overline{s_0} = \overline{s_1}$ and $\overline{s_1}\ o\ \overline{s_2} = \overline{s_1}\ o\ \overline{s_0} = \overline{s_1}$. Hence, $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case (HK1) holds.

On the other hand, $(\overline{s_1}\ o\ \overline{s_2})\ o\ \overline{s_3} = \overline{s_1}\ o\ \overline{s_3} = \overline{s_1}$ and $(\overline{s_1}\ o\ \overline{s_3})\ o\ \overline{s_2} = \overline{s_1}\ o\ \overline{s_0} = \overline{s_1}$. Thus in this case (HK2) holds.

(ix) Let $\overline{s_1}, \overline{s_3} \neq \overline{s_0}$, $\overline{s_2} = \overline{s_0}$ and $\overline{s_1} = \overline{s_3}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) = \{\overline{s_0},\ \overline{s_1}\}\ o\ \overline{s_0} = \{\overline{s_0},\ \overline{s_1}\}$ and $\overline{s_1}\ o\ \overline{s_2} = \overline{s_1}\ o\ \overline{s_0} = \overline{s_1}$. Since $\overline{s_0} \ll \overline{s_1}$ and $\overline{s_1} \ll \overline{s_1}$ we obtain that $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK1).

Also $(\overline{s_1}\ o\ \overline{s_2})\ o\ \overline{s_3} = \overline{s_1}\ o\ \overline{s_3} = \{\overline{s_0},\ \overline{s_1}\}$ and $(\overline{s_1}\ o\ \overline{s_3})\ o\ \overline{s_2} = \{\overline{s_0},\ \overline{s_1}\}\ o\ \overline{s_0} = \{\overline{s_0},\ \overline{s_1}\}$. Hence, in this case $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK2).

(x) Let $\overline{s_1}, \overline{s_2} \neq \overline{s_0}$, $\overline{s_3} = \overline{s_0}$ and $\overline{s_1} \neq \overline{s_2}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) = \overline{s_1}\ o\ \overline{s_2} = \overline{s_1}$ and $\overline{s_1}\ o\ \overline{s_2} = \overline{s_1}$. Therefore $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case (HK1) holds.

On the other hand, $(\overline{s_1}\ o\ \overline{s_2})\ o\ \overline{s_3} = \overline{s_1}\ o\ \overline{s_0} = \overline{s_1}$ and $(\overline{s_1}\ o\ \overline{s_3})\ o\ \overline{s_2} = \overline{s_1}\ o\ \overline{s_2} = \overline{s_1}$. So in this case (HK2) holds.

(xi) Let $\overline{s_1}, \overline{s_2} \neq \overline{s_0}$, $\overline{s_3} = \overline{s_0}$ and $\overline{s_1} = \overline{s_2}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) = \overline{s_1}\ o\ \overline{s_2} = \{\overline{s_0},\ \overline{s_1}\}$ and $\overline{s_1}\ o\ \overline{s_2} = \{\overline{s_0},\ \overline{s_1}\}$. Since $\overline{s_0} \ll \overline{s_0}$ and $\overline{s_1} \ll \overline{s_1}$ we get that $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK1).

Also $(\overline{s_1}\ o\ \overline{s_2})\ o\ \overline{s_3} = \{\overline{s_0},\ \overline{s_1}\}\ o\ \overline{s_0} = \{\overline{s_0},\ \overline{s_1}\}$ and $(\overline{s_1}\ o\ \overline{s_3})\ o\ \overline{s_2} = \overline{s_1} o\ \overline{s_2} = \{\overline{s_0},\ \overline{s_1}\}$. Thus in this case $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK2).

(xii) Let $\overline{s_1} = \overline{s_2} = \overline{s_0}$ and $\overline{s_3} \neq \overline{s_0}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) = \overline{s_0}\ o\ \overline{s_0} = \overline{s_0}$ and $\overline{s_1}\ o\ \overline{s_2} = \overline{s_0}$. Therefore $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case (HK1) holds.

On the other hand, $(\overline{s_1}\ o\ \overline{s_2})\ o\ \overline{s_3} = \overline{s_0}\ o\ \overline{s_3} = \overline{s_0}$ and $(\overline{s_1}\ o\ \overline{s_3})\ o\ \overline{s_2} = \overline{s_0}\ o\ \overline{s_0} = \overline{s_0}$. Hence, in this case (HK2) holds.

(xiii) Let $\overline{s_1} = \overline{s_3} = \overline{s_0}$ and $\overline{s_2} \neq \overline{s_0}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) = \overline{s_0}\ o\ \overline{s_2} = \overline{s_0}$ and $\overline{s_1}\ o\ \overline{s_2} = \overline{s_0}$. So $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK1).

On the other hand, $(\overline{s_1}\ o\ \overline{s_2})\ o\ \overline{s_3} = \overline{s_0}\ o\ \overline{s_0} = \overline{s_0}$ and $(\overline{s_1}\ o\ \overline{s_3})\ o\ \overline{s_2} = \overline{s_0}\ o\ \overline{s_2} = \overline{s_0}$. Thus this case $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK2).

(xiv) Let $\overline{s_2} = \overline{s_3} = \overline{s_0}$ and $\overline{s_1} \neq \overline{s_0}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) = \overline{s_1}\ o\ \overline{s_0} = \overline{s_1}$ and $\overline{s_1}\ o\ \overline{s_2} = \overline{s_1}$. Therefore $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case (HK1) holds.

On the other hand, $(\overline{s_1}\ o\ \overline{s_2})\ o\ \overline{s_3} = \overline{s_1}\ o\ \overline{s_0} = \overline{s_1}$ and $(\overline{s_1}\ o\ \overline{s_3})\ o\ \overline{s_2} = \overline{s_1}\ o\ \overline{s_0} = \overline{s_1}$. Hence, in this case (HK2) holds.

So we show that $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK1) and (HK2).

Now we should prove that $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK3). By Theorem 2.11, it is enough to show that $\overline{s_1}\ o\ \overline{s_2} \ll \overline{s_1}$ for all $\overline{s_1}$, $\overline{s_2} \in \overline{S}$. By definition of the hyper operation "o" we know that $\overline{s_1}\ o\ \overline{s_2}$ is equal to $\overline{s_1}$ or $\{\overline{s_0},\ \overline{s_1}\}$ or $\overline{s_0}$ for any $\overline{s_1}$, $\overline{s_2} \in \overline{S}$. Also we know that $\overline{s_1} \ll \overline{s_1}$ and $\overline{s_0} \ll \overline{s_1}$.

Hence $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK3).

To prove (HK4), Let $\overline{s_1} \ll \overline{s_2}$ and $\overline{s_2} \ll \overline{s_1}$. If $\overline{s_1} = \overline{s_2}$, then we are done. Otherwise, since $\overline{s_1} \ll \overline{s_2}$, we obtain that $\overline{s_1} = \overline{s_0}$, $\overline{s_2} \neq \overline{s_0}$. So $\overline{s_2}\ o\ \overline{s_1} = \overline{s_2}\ o\ \overline{s_0} = \overline{s_2}$. Therefore $\overline{s_2} \not\ll \overline{s_1}$, which is a contradiction.

Example 4.2. Consider the deterministic finite automaton $A = (S, M, s_0, F, t)$ in Example 3.3. Then the structure of the hyper BCK-algebra $(\overline{S},\ o,\ \overline{s_0})$ induced on $\overline{S}$ according to Theorem 4.1 is as follows:

Table 2.

O	$\overline{q_0}$	$\overline{q_1}$	$\overline{q_3}$
$\overline{q_0}$	$\overline{q_0}$	$\overline{q_0}$	$\overline{q_0}$
$\overline{q_1}$	$\overline{q_1}$	$\{\overline{q_0}, \overline{q_1}\}$	$\overline{q_1}$
$\overline{q_3}$	$\overline{q_3}$	$\overline{q_3}$	$\{\overline{q_0}, \overline{q_3}\}$

Theorem 4.3. Let $(\overline{S},\ o,\ \overline{s_0})$ be the hyper BCK-algebra, which is defined in Theorem 4.1. Then $(\overline{S},\ o,\ \overline{s_0})$ is a strong normal hyper BCK-algebra.

Proof. By definition of the hyper operation "o", we obtain that $\overline{a} \in \overline{a}\ o\ \overline{t}$, for any $\overline{a}$ and $\overline{t}$ in $\overline{S}$. So we have:

$$_l\overline{a} = \{\overline{t} \in \overline{S} \mid \overline{a} \in \overline{a}\ o\ \overline{t}\} = \overline{S}, \quad \overline{a}_r = \{\overline{t} \in \overline{S} \mid \overline{t} \in \overline{t}\ o\ \overline{a}\} = \overline{S}, \quad \forall\ \overline{a} \in \overline{S}.$$

It is clear that $\overline{S}$ is a strong hyper BCK-ideal. So $(\overline{S},\ o,\ \overline{s_0})$ is a strong normal hyper BCK-algebra.

Theorem 4.4. Let $(\overline{S},\ o,\ \overline{s_0})$ be the hyper BCK-algebra, which is defined in Theorem 4.1. Then $(\overline{S},\ o,\ \overline{s_0})$ is a simple hyper BCK-algebra.
Proof. Let $\overline{s_1} \neq \overline{s_2}$ and $\overline{s_1}, \overline{s_2} \neq \overline{s_0}$. Then $\overline{s_1} o\ \overline{s_2} = \overline{s_1}$ and $\overline{s_2} o\ \overline{s_1} = \overline{s_2}$. Hence, $\overline{s_1} \nleq\ \overline{s_2}$ and $\overline{s_2} \nleq\ \overline{s_1}$. So $(\overline{S},\ o,\ \overline{s_0})$ is a simple hyper BCK-algebra.
Theorem 4.5. Let $(\overline{S},\ o,\ \overline{s_0})$ be the hyper BCK-algebra, which is defined in Theorem 4.1. Then $(\overline{S},\ o,\ \overline{s_0})$ is an implicative hyper BCK-algebra.
Proof. Since $\overline{s_1} \in \overline{s_1}\ o\ \overline{s_2}$ and $\overline{s_1}\ o\ \overline{s_2} \neq \emptyset$ for all $\overline{s_1}\ ,\ \overline{s_2} \in \overline{S}$, we obtain that $\overline{s_1} \in \overline{s_1}\ o(\overline{s_2}\ o\ \overline{s_1}\ )$. So $\overline{s_1} \ll \overline{s_1}\ o(\overline{s_2}\ o\ \overline{s_1}\ )$ and $(\overline{S},\ o,\ \overline{s_0})$ is an implicative hyper BCK-algebra.
Definition 4.6. A deterministic finite automaton (S, M, s_0, F, t) is called semi continuous if for all distinct elements $s, s' \in S$, the following implication holds: If $\exists x \in M^*,$ such that $s' = t^*(s, x) \ \Rightarrow \nexists x'^{\in M^*,}$ such that $s = t^*\left(s', x'\right)$.
Theorem 4.7. Let (S, M, s_0, F, t) be a semi continuous deterministic finite automata. We define the following hyper operation on S:

$$\forall\ (s_1, s_2) \in S^2,\quad s_1 o s_2 = \begin{cases} \{s_1, s_0\}, & \text{if } s_2 \text{ is connected to } s_1,\quad s_1, s_2 \neq s_0 \text{ and } s_1 \neq s_2 \\ s_1, & \text{if } s_2 \text{ is not connected to } s_1,\quad s_1, s_2 \neq s_0 \text{ and } s_1 \neq s_2 \\ s_0, & \text{if } s_1 = s_2 \\ s_0, & \text{if } s_1 = s_0,\quad s_2 \neq s_0 \\ s_1, & \text{if } s_2 = s_0,\quad s_1 \neq s_0. \end{cases}$$

Then $(S,\ o,\ s_0)$ is a hyper BCK-algebra and s_0 is the zero element of S.
Proof. First we consider the following situations to prove (HK1) and (HK2).
(i) Let $s_1, s_2, s_3 \neq s_0,\ \ s_3 \neq s_1 \neq s_2 \neq s_3,\ \ s_2$ is connected to s_1 , s_3 is connected to s_1 and s_3 is connected to s_2 .
Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = \{s_1, s_0\}\ \ o\ \ \{s_2, s_0\} = \ \{s_1, s_0\}$ and $s_1\ o\ s_2 = \{s_1, s_0\}$. Since $s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_1 = s_0$, we obtain that $s_1 \ll s_1$and $s_0 \ll s_1$. So in this case (HK1) holds.
On the other hand, $(s_1\ o\ s_2)\ o\, s_3 = \{s_1, s_0\}\ o\ s_3 = \{s_1, s_0\}$ and $(s_1\ o\ s_3)\ o\, s_2 = \{s_1, s_0\}\, o\ s_2 = \{s_1, s_0\}$. Thus in this case (HK2) holds.
(ii) Let $s_1, s_2, s_3 \neq s_0,\ \ s_3 \neq s_1 \neq s_2 \neq s_3,\ \ s_2$ is not connected to s_1 , s_3 is connected to s_1 and s_3 is connected to s_2 .
Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = \{s_1, s_0\}\ \ o\ \ \{s_2, s_0\} = \ \{s_1, s_0\}$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$and $s_0 \ll s_1$, we conclude that in this case (HK1) holds.
Also $(s_1\ o\ s_2)\ \ o\ s_3 = \{s_1\}\ \ o\ s_3 = \{s_1, s_0\}$ and $(s_1\ o\ s_3)\ \ o\ s_2 = \{s_1, s_0\}\, o\ s_2 = \{s_1, s_0\}$. Therefore in this case (HK2) holds.
(iii) Let $s_1, s_2, s_3 \neq s_0,\ \ s_3 \neq s_1 \neq s_2 \neq s_3,\ \ s_2$ is connected to s_1 , s_3 is not connected to s_1 and s_3 is connected to s_2 . Since s_2 is connected to s_1 and s_3 is connected to s_2 , we get that s_3 is connected to s_1 . So this case does not happen.
(iv) Let $s_1, s_2, s_3 \neq s_0,\ \ s_3 \neq s_1 \neq s_2 \neq s_3,\ \ s_2$ is connected to s_1 , s_3 is connected to s_1 and s_3 is not connected to s_2 .
Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = \{s_1, s_0\}\ \ \ o\ s_2 = \ \{s_1, s_0\}$ and $s_1\ o\ s_2 = \{s_1, s_0\}$. Since $s_1 \ll s_1$and $s_0 \ll s_1$, we obtain that in this case (HK1) holds.

Also $(s_1\ o\ s_2)\ o\, s_3 = \{s_1, s_0\}\ o\ s_3 = \{s_1, s_0\}$ and $(s_1\ o\ s_3)\ o\, s_2 = \{s_1, s_0\}\ o\, s_2 = \{s_1, s_0\}$. Hence, in this case (HK2) holds.

(v) Let $s_1, s_2, s_3 \neq s_0$, $s_3 \neq s_1 \neq s_2 \neq s_3$, s_2 is not connected to s_1, s_3 is not connected to s_1 and s_3 is connected to s_2.

Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_1\ o\ \{s_2, s_0\} = s_1$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$ we conclude that in this case (HK1) holds.

On the other hand, $(s_1\ o\ s_2)\ o\, s_3 = s_1\ o\ s_3 = s_1$ and $(s_1\ o\ s_3)\ o\, s_2 = s_1 o\, s_2 = s_1$. Thus in this case (HK2) holds.

(vi) Let $s_1, s_2, s_3 \neq s_0$, $s_3 \neq s_1 \neq s_2 \neq s_3$, s_2 is not connected to s_1, s_3 is connected to s_1 and s_3 is not connected to s_2.

Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = \{s_1, s_0\}\ o\ s_2 = \{s_1, s_0\}$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$ and $s_0 \ll s_1$, we get that in this case (HK1) holds.

Also $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_3 = \{s_1, s_0\}$ and $(s_1\ o\ s_3)\ o\ s_2 = \{s_1, s_0\}\ o\ s_2 = \{s_1, s_0\}$. So in this case (HK2) holds.

(vii) Let $s_1, s_2, s_3 \neq s_0$, $s_3 \neq s_1 \neq s_2 \neq s_3$, s_2 is connected to s_1, s_3 is not connected to s_1 and s_3 is not connected to s_2.

Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_1\ o\ s_2 = \{s_1, s_0\}$ and $s_1\ o\ s_2 = \{s_1, s_0\}$. Since $s_1 \ll s_1$ and $s_0 \ll s_1$, we obtain that in this case (HK1) holds.

On the other hand, $(s_1\ o\ s_2)\ o\, s_3 = \{s_1, s_0\}\ o\ s_3 = \{s_1, s_0\}$ and $(s_1\ o\ s_3)\ o\, s_2 = s_1 o\ s_2 = \{s_1, s_0\}$. Therefore in this case (HK2) holds.

(viii) Let $s_1, s_2, s_3 \neq s_0$, $s_3 \neq s_1 \neq s_2 \neq s_3$, s_2 is not connected to s_1, s_3 is not connected to s_1 and s_3 is not connected to s_2.

Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_1\ o\ s_2 = s_1$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$ we conclude that in this case (HK1) holds.

Also $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\, s_3 = s_1$ and $(s_1\ o\ s_3)\ o\ s_2 = s_1 o\ s_2 = \mathrm{s}_1$. Hence, in this case (HK2) holds.

(ix) Let $s_1, s_2, s_3 \neq s_0$, $s_1 = s_2 \neq s_3$ and s_3 is connected to s_1.

Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = \{s_1, s_0\}\ o\ \{s_2, s_0\}$

$= s_0$ and $s_1\ o\ s_2 = s_0$. Since $s_0 \ll s_0$ we get that in this case (HK1) holds.

On the other hand, $(s_1\ o\ s_2)\ o\ s_3 = s_0\ o\, s_3 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = \{s_1, s_0\}\ o\ s_1 = s_0$. Thus in this case (HK2) holds.

(x) Let $s_1, s_2, s_3 \neq s_0$, $s_1 = s_2 \neq s_3$ and s_3 is not connected to s_1. Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_1\ o\ s_2 = s_0$ and $s_1\ o\ s_2 = s_0$. Since $s_0 \ll s_0$ we obtain that in this case (HK1) holds.

Also $(s_1\ o\ s_2)\ o\ s_3 = s_0\ o\ s_3 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = s_1 o\ s_1 = \mathrm{s}_0$. So in this case (HK2) holds.

(xi) Let $s_1, s_2, s_3 \neq s_0$, $s_1 = s_3 \neq s_2$ and s_3 is connected to s_2. By definition of semi continuous automaton we know that when s_3 is connected to s_2 then s_2 is not connected to s_3 or s_1.

So $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_0\ o\ \{s_2, s_0\} = s_0$ and $s_1\ o\ s_2 = s_1$. Since $s_0 \ll s_1$ we conclude that in this case (HK1) holds.

On the other hand, $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_1 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = s_0 o\ s_2 = s_0$. Hence, in this case (HK2) holds.

(xii) Let $s_1, s_2, s_3 \neq s_0$, $s_1 = s_3 \neq s_2$, s_3 is not connected to s_2 and s_2 is connected to s_3 . Then we have

$(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_0\ o\ s_2 = s_0$ and $s_1\ o\ s_2 = \{s_1, s_0\}$. Since $s_0 \ll s_1$ we get that in this case (HK1) holds.

Also $(s_1\ o\ s_2)\ o\ s_3 = \{s_1, s_0\}\ o\ s_1 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = s_0 o\ s_2 = s_0$. Therefore in this case (HK2) holds.

(xiii) Let $s_1, s_2, s_3 \neq s_0$, $s_1 = s_3 \neq s_2$, s_3 is not connected to s_2 and s_2 is not connected to s_3 . Then we have $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_0\ o\ s_2 = s_0$ and $s_1\ o\ s_2 = s_1$. Since $s_0 \ll s_1$ we obtain that in this case (HK1) holds.

Also $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_1 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = s_0 o\ s_2 = s_0$. Thus in this case (HK2) holds.

(xiv) Let $s_1, s_2, s_3 \neq s_0$, $s_1 \neq s_2 = s_3$ and s_3 is connected to s_1 . Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = \{s_1, s_0\}\ o\ s_0 = \{s_1, s_0\}$ and $s_1\ o\ s_2 = \{s_1, s_0\}$. Since $s_1 \ll s_1$ and $s_0 \ll s_0$ we conclude that in this case (HK1) holds.

On the other hand, $(s_1\ o\ s_2)\ o\, s_3 = \{s_1, s_0\}\ o\ s_3 = \{s_1, s_0\}$ and $(s_1\ o\ s_3)\ o\, s_2 = \{s_1, s_0\}\, o\ s_2 = \{s_1, s_0\}$. So in this case (HK2) holds.

(xv) Let $s_1, s_2, s_3 \neq s_0$, $s_1 \neq s_2 = s_3$ and s_3 is not connected to s_1 . Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$ we get that in this case (HK1) holds.

Also $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_3 = s_1$ and $(s_1\ o\ s_3)\ o\ s_2 = s_1 o\ s_2 = s_1$. Hence, in this case (HK2) holds.

(xvi) Let $s_1 = s_2 = s_3$. Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_0\ o\ s_0 = s_0$ and $s_1\ o\ s_2 = s_0$. Since $s_0 \ll s_0$ we obtain that in this case (HK1) holds.

Also $(s_1\ o\ s_2)\ o\ s_3 = s_0\ o\ s_3 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = s_0 o\ s_2 = s_0$. Therefore in this case (HK2) holds.

(xvii) Let $s_1 = s_0$. Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_0\ o\ (s_2\ o\ s_3) = s_0$ and $s_1\ o\ s_2 = s_0$. Since $s_0 \ll s_0$ we conclude that in this case (HK1) holds.

On the other hand, $(s_1\ o\ s_2)\ o\ s_3 = s_0\ o\ s_3 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = s_0 o\ s_2 = s_0$. Thus in this case (HK2) holds.

(xviii) Let $s_2 = s_0$, $s_3 \neq s_1, s_1 \neq s_0 \neq s_3$ and s_3 is connected to s_1 . Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = \{s_1, s_0\}\ o\ s_0$ $= \{s_1, s_0\}$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$ and $s_0 \ll s_1$, we get that in this case (HK1) holds.

Also $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_3 = \{s_1, s_0\}$ and $(s_1\ o\ s_3)\ o\ s_2 = s_1 o\ s_3 = \{s_1, s_0\}$. So in this case (HK2) holds.

(xix) Let $s_2 = s_0$, $s_3 \neq s_1, s_1 \neq s_0 \neq s_3$ and s_3 is not connected to s_1 . Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$ we obtain that in this case (HK1) holds.

On the other hand, $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_3 = s_1$ and $(s_1\ o\ s_3)\ o\ s_2 = s_1 o\ s_2 = s_1$. Hence, in this case (HK2) holds.

(xx) Let $s_2 = s_0$, $s_3 = s_1$ and $s_1 \neq s_0 \neq s_3$. Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_0\ o\ s_0 = s_0$ and $s_1\ o\ s_2 = s_1$. Since $s_0 \ll s_1$ we conclude that in this case (HK1) holds.
Also $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_3 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = s_0 o\ s_0 = s_0$. Therefore in this case (HK2) holds.
(xxi) Let $s_3 = s_0$, $s_2 \neq s_1, s_1 \neq s_0 \neq s_2$ and s_2 is connected to s_1 . Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_1\ o\ s_2 = \{s_1, s_0\}$ and $s_1\ o\ s_2 = \{s_1, s_0\}$. Since $s_1 \ll s_1$ and $s_0 \ll s_0$, we get that in this case (HK1) holds.
On the other hand, $(s_1\ o\ s_2)\ o\ s_3 = \{s_1, s_0\}\ o\ s_3 = \{s_1, s_0\}$ and $(s_1\ o\ s_3)\ o\, s_2 = s_1 o\ s_2 = \{s_1, s_0\}$. So in this case (HK2) holds.
(xxii) Let $s_3 = s_0$, $s_2 \neq s_1, s_1 \neq s_0 \neq s_2$ and s_2 is not connected to s_1 . Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_1\ o\ s_2 = s_1$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$ we obtain that in this case (HK1) holds.
Also $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_3 = s_1$ and $(s_1\ o\ s_3)\ o\ s_2 = s_1 o\ s_2 = s_1$. Hence, in this case (HK2) holds.
(xxiii) Let $s_3 = s_0$, $s_2 = s_1$ and $s_1 \neq s_0 \neq s_2$. Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_1\ o\ s_2 = s_0$ and $s_1\ o\ s_2 = s_0$. Since $s_0 \ll s_0$ we conclude that in this case (HK1) holds.
On the other hand, $(s_1\ o\ s_2)\ o\ s_3 = s_0\ o\ s_0 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = s_1 o\ s_2 = s_0$. Therefore in this case (HK2) holds.
(xxiv) Let $s_2 = s_3 = s_0$ and $s_1 \neq s_0$. Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$ we get that in this case (HK1) holds.
Also $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_0 = s_1$ and $(s_1\ o\ s_3)\ o\ s_2 = s_1 o\ s_0 = s_1$. Thus in this case (HK2) holds.
So we obtain that $(S,\ o,\ s_0)$ satisfies (HK1)and (HK2).
Now we should prove that $(S,\ o,\ s_0)$ satisfies (HK3). By Theorem 2.11, it is enough to show that $s_1 o\ s_2 \ll \{s_1\}$ for all $s_1, s_2 \in S$. By definition of the hyper operation "o" we know that $s_1 o\ s_2$ is equal to s_1 or $\{s_1, s_0\}$ or s_0 for any $s_1, s_2 \in S$. Also we know that $\mathrm{s}_1 \ll \mathrm{s}_1$ and $\mathrm{s}_0 \ll \mathrm{s}_1$.
Hence $(S,\ o,\ s_0)$ satisfies (HK3).
To prove (HK4), Let $s_1 \ll s_2$ and $s_2 \ll s_1$. If $s_1 = s_2$, then we are done. Otherwise, since $s_1 \ll s_2$, there exist two cases:
(i) s_2 is connected to s_1 , $s_1, s_2 \neq s_0$ and $s_1 \neq s_2$. Then by definition of semi continuous automaton we know that s_2 is not connected to s_1 and we have $s_2 o\ s_1 = s_2$. Therefore $s_2 \not\ll s_1$, which is a contradiction.
(ii) $s_1 = s_0$, $s_2 \neq s_0$. Then $s_2 o\ s_1 = s_2 o\ s_0 = s_2$. Thus $s_2 \not\ll s_1$, which is a contradiction.
So we show that $(S,\ o,\ s_0)$ is a hyper BCK-algebra.
Theorem 4.8. Let $(S,\ o, s_0)$ be a hyper BCK-algebra which is defined in Theorem 4.7. Then $(S,\ o, s_0)$ is a weak normal hyper BCK-algebra.
Proof. By definition of the hyper operation "o", we know that $a_r = \{t \in S \mid t \in t\ o\ a\} = S - \{a\}$ for all $a \neq s_0$ and $a \in S$. Also $a_r = S$ for $a = s_0$.

It is clear that S is a weak hyper BCK-ideal. So it is enough to show that $S-\{s\}$ for all $s \neq s_0$ and $s \in S$, is a weak hyper BCK-ideal.
It is easy to see that $s_0 \in S-\{s\}$. Let $s_1\ o\ s_2 \subseteq S-\{s\}$ and $s_2 \in S-\{s\}$. Then we have to consider the following situations:

(1) s_2 is connected to s_1 , $s_1, s_2 \neq s_0$ and $s_1 \neq s_2$.
$Since s_1 o\ s_2 = \{s_1, s_0\}$ and $s_1\ o\ s_2 \subseteq S-\{s\}$, we obtain that $s_1 \in S-\{s\}$.
(2) s_2 is not connected to s_1 , $s_1, s_2 \neq s_0$ and $s_1 \neq s_2$.
$Since s_1 o\ s_2 = s_1$ and $s_1\ o\ s_2 \subseteq S-\{s\}$, we get that $s_1 \in S-\{s\}$.
(3) $s_1 = s_2$.
$Since$ $s_2 \in S-\{s\}$, it is clear that $s_1 \in S-\{s\}$.
(4) $s_1 = s_0$, $s_2 \neq s_0$.
$Since s_1 o\ s_2 = s_0$ and $s_0 \in S-\{s\}$, we obtain that $s_1 \in S-\{s\}$.
(5) $s_2 = s_0$, $s_1 \neq s_0$.
$Since s_1 o\ s_2 = s_1$ and $s_1\ o\ s_2 \subseteq S-\{s\}$, we conclude that $s_1 \in S-\{s\}$.
So $(S,\ o, s_0)$ is a weak normal hyper BCK-algebra.

Example 4.9. Consider the deterministic finite automaton $A=(S,M,s_0,F,t)$ in Example 3.3. Then the structure of the hyper BCK-algebra $(S,\ o,\ s_0)$ induced on the states of this automaton according to Theorem 4.7 is as follows:

Table 3.

O	q_0	q_1	q_2	q_3
q_0	q_0	q_0	q_0	q_0
q_1	q_1	q_0	$\{q_0,q_1\}$	$\{q_0,q_1\}$
q_2	q_2	q_2	q_0	$\{q_0,q_2\}$
q_3	q_3	q_3	q_3	q_0

Thus $(S,\ o,\ s_0)$ is a hyper BCK-algebra.

Remark 4.10. Let $(S,\ o,\ s_0)$ be the hyper BCK-algebra which is defined in Theorem 4.7. In example 4.9, we saw that $q_0 \in q_1 o\ q_3$ and $q_0 \notin q_3 o\ q_1$. So $q_1 \ll q_3$ and $q_3 \not\leq q_1$. Hence, $(S,\ o, s_0)$ may not be simple.

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