

## On Tensor Product of Graphs, Girth and Triangles

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ABSTRACT. The purpose of this paper is to obtain a necessary and sufficient condition for the tensor product of two or more graphs to be connected, bipartite or eulerian. Also, we present a characterization of the duplicate graph  $G \oplus K_2$  to be unicyclic. Finally, the girth and the formula for computing the number of triangles in the tensor product of graphs are worked out.

**Keywords:** Tensor product, Bipartite graph, Connected graph, Eulerian graph, Girth, Cycle, Path.

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### 1. INTRODUCTION

We shall consider only finite, undirected graphs without loops or multiple edges. We follow the terminology of [1]. For a graph  $G$ ,  $V(G)$  and  $E(G)$  denote the vertex set and edge set of  $G$ , respectively. For a connected graph  $G$ ,  $nG$  is the graph with  $n$  components, each being isomorphic to  $G$ . It is well-known that a graph is *bipartite* if and only if it contains no odd cycle. We now define the tensor product of two graphs [8] as follows: The *tensor product* of two graphs  $G_1$  and  $G_2$  is the graph, denoted by  $G_1 \oplus G_2$ , with vertex set  $V(G_1 \oplus G_2) = V(G_1) \times V(G_2)$ , and any two of its vertices  $(u_1, v_1)$  and  $(u_2, v_2)$  are adjacent, whenever  $u_1$  is adjacent to  $u_2$  in  $G_1$  and  $v_1$  is adjacent to  $v_2$  in  $G_2$ .

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The graphs  $G_1$  and  $G_2$  are called *factors* of the product  $G_1 \oplus G_2$ . Other popular names for tensor product that have appeared in the literature are *Kronecker product*, *cross product*, *direct product*, *conjunction product*. Sampathkumar [6] defines the tensor product of a graph  $G$  by  $K_2$  as the *duplicate graph* of  $G$ , and studied its properties and a characterization in great detail. This product is also studied in [5]. Now, we define the two special type of tensor products:  $G \oplus nK_2$  and  $G \oplus [\oplus_{i=1}^n K_2]$  as the *generalized duplicate graphs of a graph  $G$* , for any integer  $n \geq 2$ , and study their structural properties for our later use.

## 2. STRUCTURAL PROPERTIES OF THE GENERALIZED DUPLICATE GRAPHS

The following theorem of Weichsel [8] will be useful in the proof of our results.

**Theorem 2.1.** *If the connected graphs  $G$  and  $H$  are bipartite, then  $G \oplus H$  has exactly two components.*

Next, we present some elementary results of the generalized duplicate graphs for our immediate use.

**Theorem 2.2.** *For any connected, bipartite graph  $G$ ,  $G \oplus nK_2 = 2nG$  for  $n \geq 1$ .*

*Proof.* For  $n = 1$ , Theorem 2.1 implies that  $G \oplus K_2$  has exactly two components. Furthermore, by using the definition of tensor product, we see that each component of  $G \oplus K_2$  is isomorphic to  $G$ . Therefore,  $G \oplus K_2 = 2G$ . Moreover, corresponding to  $n \geq 1$  copies of  $K_2$ ,  $G \oplus nK_2$  certainly contains exactly  $2n$  copies of  $G$ . Thus,  $G \oplus nK_2 = 2nG$ .  $\square$

**Theorem 2.3.** *For any connected graph  $G$ ,  $G \oplus nK_2$  for  $n \geq 1$ , is bipartite.*

*Proof.* We discuss two cases depending on  $G$ .

**Case 1.** Suppose  $G$  is bipartite. By Theorem 2.2,  $G \oplus nK_2 = 2nG$ . Since  $G$  is bipartite, it follows immediately that  $G \oplus nK_2$  is bipartite.

**Case 2.** Suppose  $G$  is non-bipartite. Certainly,  $G$  contains a cycle  $C_m$  for odd  $m \geq 3$ . Corresponding to each copy of  $K_2$  in  $G \oplus nK_2$ , there are exactly  $n$  distinct subgraphs in  $G \oplus nK_2$ , each is isomorphic to  $C_m \oplus K_2$ . It is shown in [2] that  $C_m \oplus K_2$  is isomorphic to  $C_{2m}$ . For even  $m \geq 4$ , it is also shown in [2] that  $C_m \oplus K_2 = C_m \cup C_m$ . This proves that  $G \oplus nK_2$  has no odd cycles. Hence,  $G \oplus nK_2$  is bipartite.  $\square$

**Theorem 2.4.** *Let  $G$  be a connected, bipartite graph and let  $H = \oplus_{i=1}^n K_2$ . Then  $G \oplus H = 2^n G$  for  $n \geq 1$ .*

*Proof.* We proceed by induction on  $n$ . If  $n = 1$ , then by Theorem 2.2,  $G \oplus H = 2G$ . Assume the result holds with at most  $n-1$ . Consider  $G \oplus H = G \oplus [\oplus_{i=1}^n K_2]$

$= G \oplus [\oplus_{i=1}^{n-1} K_2 \oplus K_2] = [G \oplus (\oplus_{i=1}^{n-1} K_2)] \oplus K_2$ . By induction hypothesis, we have  $G \oplus [\oplus_{i=1}^{n-1} K_2] = 2^{n-1}G$ . Hence,

$$G \oplus [\oplus_{i=1}^n K_2] = 2^{n-1}G \oplus K_2. \dots \dots (2.1)$$

In view of Theorem 2.2 (with  $n = 1$ ),  $G \oplus K_2 = 2G$ . Using this in (2.1), we get  $G \oplus H = 2^n G$ . □

### 3. CHARACTERIZATION OF CONNECTED TENSOR PRODUCT OF GRAPHS

Now, we obtain a characterization of connected tensor product of arbitrarily many graphs. We see that Weichsel [6] studied the connectedness of the tensor product of two graphs as follows:

**Theorem 3.1.** *Let  $G$  and  $H$  be connected graphs. Then  $G \oplus H$  is connected if and only if either  $G$  or  $H$  contains an odd cycle.*

Now, we present the natural finite extension of Weichsel’s Theorem as follows:

**Theorem 3.2.** *Let  $G_k$  ( $1 \leq k \leq n ; n \geq 2$ ) be connected graph, and let  $G = \oplus_{k=1}^n G_k$ . Then  $G$  is connected if and only if at most one of  $G_k$ ’s is bipartite.*

*Proof.* Assume that  $G$  is connected. We prove by contradiction. If possible, assume that there are at least two distinct graphs  $G_i$  and  $G_j$  ( $1 \leq i, j \leq n$ ), which are bipartite. By Theorem 2.1,  $G_i \oplus G_j$  contains exactly two components say,  $F$  and  $H$ . Now, we have  $G = \oplus_{k=1}^n G_k = (F \oplus M) \cup (H \oplus M)$ , where  $M = \oplus_{k=1}^n G_k$  ( $k \neq i, j$ ). This shows that  $G$  is certainly disconnected, and hence we immediately arrive at a contradiction. Thus, it proves that at most one of  $G_k$ ’s is bipartite.

Conversely, assume that at most one of  $G_k$ ’s is bipartite.

We discuss two cases.

**Case 1.** None of  $G_k$ ’s is bipartite. Immediately, it follows that each  $G_k$  contains an odd cycle.

**Case 2.** Exactly one of  $G_k$ ’s is bipartite. Without loss of generality, we assume that  $G_1$  is bipartite. The remaining  $G_i$  ( $2 \leq i \leq n$ ) is non-bipartite, and hence each such  $G_i$  contains an odd cycle.

In either case, by applying Theorem 3.1 and the mathematical induction on the number of factors, the result follows. □

### 4. CHARACTERIZATION OF BIPARTITE TENSOR PRODUCT OF GRAPHS

Now, we shall obtain a necessary and sufficient condition for the tensor product of two or more graphs to be bipartite, (which is proposed in [3]).

**Theorem 4.1.** *Let  $G_1$  and  $G_2$  be two connected graphs. Then  $G_1 \oplus G_2$  is bipartite if and only if at least one of  $G_1$  and  $G_2$  is bipartite.*

*Proof.* Suppose  $G_1 \oplus G_2$  is bipartite. We claim that at least one of  $G_1$  and  $G_2$  is bipartite. If this is not so, then both  $G_1$  and  $G_2$  are non-bipartite. Consequently, there exist two odd cycles  $C_m$  (for  $m \geq 3$ ) and  $C_n$  (for  $n \geq 3$ ) in  $G_1$  and  $G_2$ , respectively. Without loss of generality, we consider  $m \leq n$ . Let  $C_m : u_1, u_2, \dots, u_m, u_1$  and let  $C_n : v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_n, v_1$ . Then  $C_m \oplus C_n$  contains the cycle  $Z$  of length  $n$  as follows:

$$Z : (u_1, v_1), (u_2, v_2), \dots, (u_m, v_m), (u_{m-1}, v_{m+1}), (u_m, v_{m+2}), (u_{m-1}, v_{m+3}), \\ (u_m, v_{m+4}), \dots, (u_{m-1}, v_{n-1}), (u_m, v_n), (u_1, v_1).$$

So,  $G_1 \oplus G_2$  contains the odd cycle  $Z$ . Hence,  $G_1 \oplus G_2$  is non-bipartite. This is a contradiction.

Conversely, assume that at least one of  $G_1$  and  $G_2$  is bipartite.

We discuss two cases.

**Case 1.** Suppose both  $G_1$  and  $G_2$  are bipartite. Then by Theorem 2.1,  $G_1 \oplus G_2$  contains exactly two components, say  $H_1$  and  $H_2$ . Now, we claim that both  $H_1$  and  $H_2$  are bipartite. On contrary, if one of  $H_i$ 's is non-bipartite. Without loss of generality, we assume that  $H_1$  is non-bipartite. Then  $H_1$  contains an odd cycle, say

$C : (u_1, v_1), (u_2, v_2), \dots, (u_n, v_n), (u_1, v_1)$  in  $G_1 \oplus G_2$ , where  $u_i \in V(G_1)$  ( $1 \leq i \leq n$ ),  $v_j \in V(G_2)$  ( $1 \leq j \leq n$ ). Certainly, the first co-ordinate vertices  $u_1, u_2, \dots, u_n, u_1$  of the cycle  $C$  forms a closed odd walk  $W$  in  $G_1$ . Since every closed odd walk in a graph contains an odd cycle, it follows that the walk  $W$  contains an odd cycle in  $G_1$ . This shows that  $G_1$  is not bipartite. But this contradicts the hypothesis that  $G_1$  is bipartite. Since each  $H_i$  is bipartite, it follows that  $G_1 \oplus G_2 = H_1 \cup H_2$  is also bipartite.

**Case 2.** Suppose one of  $G_1$  and  $G_2$  is bipartite, and the other is non-bipartite. Assume that  $G_1$  is bipartite. Since  $G_2$  is non-bipartite,  $G_2$  contains an odd cycle. From Theorem 3.1,  $G_1 \oplus G_2$  is connected. Next, we claim that  $G_1 \oplus G_2$  is bipartite. If this is not so, then  $G_1 \oplus G_2$  is non-bipartite, and hence it contains an odd cycle  $Z$ . By repeating the same argument as in Case 1, we obtain  $G_1 \oplus G_2$  is bipartite.

In either case, we see that  $G_1 \oplus G_2$  is bipartite.  $\square$

Next, we obtain the finite extension of the above theorem, and its proof follows by the mathematical induction on the number of factors.

**Corollary 4.2.** Let  $G_k$  ( $1 \leq k \leq n$ ;  $n \geq 2$ ) be a connected graph, and let  $G = \bigoplus_{k=1}^n G_k$ . Then  $G$  is bipartite if and only if at least one of  $G_k$ 's is bipartite.

## 5. CHARACTERIZATION OF EULERIAN TENSOR PRODUCT OF GRAPHS

An *Euler tour* of a graph  $G$  is a closed walk in  $G$  that traverses each edge of  $G$  exactly once. A graph is eulerian if it contains an Euler tour. It is well-known that a connected graph  $G$  is eulerian if and only if every vertex in  $G$  has an even degree. For any vertex  $(u, v)$  in a tensor product  $(G \oplus H)$  of two graphs

$G$  and  $H$ ,  $\deg(u, v) = \deg(u) \bullet \deg(v)$ . Now, we present a characterization of eulerian tensor product of two graphs.

**Theorem 5.1.** *Let  $G$  and  $H$  be connected graphs such that at most one of them is bipartite. Then  $G \oplus H$  is eulerian if and only if at least one of  $G$  and  $H$  is eulerian.*

*Proof.* Suppose  $G$  is eulerian, and it contains an odd cycle. Then by Theorem 4.1,  $G \oplus H$  is connected. Since  $G$  is eulerian,  $\deg(u)$  is even for all vertices  $u$  in  $G$ . Consequently, for any vertex  $v$  of  $H$ , the pair  $(u, v)$  is a vertex in  $G \oplus H$ , and  $\deg(u, v) = \deg(u) \bullet \deg(v)$ , which is an even degree, because  $\deg(u)$  is even, and  $\deg(v) \geq 1$ . This implies that  $G \oplus H$  is eulerian.

Conversely, assume that  $G \oplus H$  is eulerian. By definition,  $G \oplus H$  is certainly connected. Again by Theorem 4.1, one of  $G$  and  $H$  contains an odd cycle. To complete the proof, we claim that at least one of  $G$  and  $H$  is eulerian. On contrary, if possible assume that both  $G$  and  $H$  are not eulerian graphs. Immediately, there exist at least two odd degree vertices  $x$  and  $y$  in  $G$  and  $H$ , respectively. Thus,  $(x, y)$  is a vertex in  $G \oplus H$ , and also  $\deg(x, y) = \deg(x) \bullet \deg(y)$ , which is odd, because both  $\deg(x)$  and  $\deg(y)$  are odd. This shows that  $G \oplus H$  is not eulerian, and it contradicts the hypothesis that  $G \oplus H$  is eulerian.  $\square$

The finite extension of Theorem 5.1 is the following result, and its proof directly follows by the induction on the number of factors.

**Corollary 5.2.** *Let  $G_k$  ( $1 \leq k \leq n$ ;  $n \geq 2$ ) be a connected graph such that at most one of  $G_k$ 's is bipartite, and let  $G = \bigoplus_{k=1}^n G_k$ . Then  $G$  is eulerian if and only if at least one of  $G_k$ 's is eulerian.*

## 6. CHARACTERIZATION OF UNICYCLIC DUPLICATE GRAPH

A *unicyclic graph* is a connected graph which contains exactly one cycle. Next, we obtain a characterization of unicyclic duplicate graph  $G \oplus K_2$ .

**Theorem 6.1.** *A non-bipartite graph  $G$  is unicyclic if and only if the duplicate graph  $G \oplus K_2$  is unicyclic.*

*Proof.* Suppose a non-bipartite graph  $G$  is unicyclic. Then  $G$  contains exactly one odd cycle  $C$ . Hence by Theorem 4.1,  $G \oplus K_2$  is connected. Let  $C : u_1, u_2, \dots, u_{2k+1}, u_1$  for  $k \geq 1$ . Next, we show that  $G \oplus K_2$  is unicyclic. For this, let us consider  $V(K_2) = \{v_1, v_2\}$ . It is easy to see that the subgraph induced by  $C \oplus K_2$  in  $G \oplus K_2$  is certainly isomorphic to an even cycle  $C_{2(2k+1)}$ , where  $C_{2(2k+1)} : (u_1, v_1), (u_2, v_2), (u_3, v_1), \dots, (u_{2k-1}, v_1), (u_{2k}, v_2), (u_{2k+1}, v_1), (u_{2k}, v_2), (u_{2k+1}, v_1), (u_1, v_2), (u_2, v_1), (u_3, v_2), \dots, (u_{2k-1}, v_2), (u_{2k}, v_1), (u_{2k+1}, v_2), (u_1, v_1)$ . Since  $G$  is unicyclic, it follows that  $G \oplus K_2$  has no cycles other than  $C_{2(2k+1)}$ . If this is not so, then there exists another cycle  $J$  in  $G \oplus K_2$ , which is different from  $C_{2(2k+1)}$ . Consequently, the first co-ordinates

of the vertices of the cycle  $J$ , which are in pairs, will form another cycle  $C'$  in  $G$ . Since  $J \neq C_{2(2k+1)}$  in  $G \oplus K_2$ , it follows  $C \neq C'$  in  $G$ . This is a contradiction to the fact that  $G$  is unicyclic. Therefore,  $G \oplus K_2$  is unicyclic.

Conversely, suppose that  $G \oplus K_2$  is unicyclic. Let  $Z$  be the only one cycle in  $G \oplus K_2$ . By Theorem 2.3 (with  $n = 1$ ),  $G \oplus K_2$  is bipartite. Hence,  $Z$  is a unique even cycle. Clearly, we notice that the first co-ordinate vertices of  $G$  in  $Z$  forms an odd cycle  $C$  in  $G$ . Since  $G \oplus K_2$  is unicyclic, it follows that  $C$  is the unique cycle in  $G$ . Moreover, since  $G \oplus K_2$  is connected, it implies that  $G$  is connected. Therefore,  $G$  is unicyclic.  $\square$

## 7. THE GIRTH AND TRIANGLES IN TENSOR PRODUCT GRAPHS

The *girth* of a graph  $G$ , denoted by  $g(G)$ , is the length of a shortest cycle in  $G$ , if any. Otherwise, it is undefined if  $G$  is a forest. It is clear that the girth of a graph  $G$  is the minimum of the girths of its components. Firstly, we determine the girth of the generalized duplicate graphs.

**Theorem 7.1.** *Let  $G$  be a connected graph with  $g(G) = k$ . For any positive integer  $n \geq 1$ , we have*

$$g(G \oplus n K_2) = g(G \oplus [\oplus_{i=1}^n K_2]) = \begin{cases} k & \text{if } G \text{ is bipartite,} \\ \min\{2p, q\} & \text{otherwise,} \end{cases}$$

where  $C_p$  and  $C_q$  are the minimal odd and even cycles in a non-bipartite graph  $G$ , respectively.

*Proof.* First, we discuss the result when  $n = 1$ .

**Case 1.** Assume  $G$  is bipartite. From Theorem 2.2 (with  $n = 1$ ), we have for the duplicate graph  $G \oplus K_2 = 2G$ . Consequently,  $g(G \oplus K_2) = k$ .

**Case 2.** Suppose  $G$  is not bipartite. Then  $G$  contains an odd cycle. Let  $C_p$  for  $p \geq 3$ , be a minimal odd cycle in  $G$ .

Now, there are two possibilities to discuss:

**2.1.** If  $G$  is free from even cycles, then  $C_p \oplus K_2$  contains an even cycle  $C_{2p}$  in  $G \oplus K_2$ .

**2.2.** If  $G$  contains a minimal even cycle  $C_q$ ,  $q \geq 4$ , then  $C_q \oplus K_2 = 2C_q$  appears in  $G \oplus K_2$ .

From the above possibilities, it follows that  $g(G \oplus K_2)$  is the minimum of  $2p$  and  $q$ . Thus,  $g(G \oplus K_2) = \min\{2p, q\}$ .

Finally, consider the result when  $n \geq 2$ . The result follows immediately if we proceed as above by applying Theorem 2.2 or 2.4 repeatedly.  $\square$

Next, we derive a formula (which is proposed in [4]) for computing the number of triangles in the tensor product of two graphs. For this, firstly we establish the following lemma.

**Lemma 7.2.** *Let  $G_k$  ( $1 \leq k \leq n$  ;  $n \geq 2$ ) be a connected graph. Then the product  $\oplus_{k=1}^n G_k$  contains a triangle if and only if each  $G_k$  contains a triangle.*

*Proof.* Now, we discuss the case when  $n = 2$ . Suppose  $G_1 \oplus G_2$  contains a triangle  $T$ , and let  $(a_1, b_1), (a_2, b_2)$  and  $(a_3, b_3)$  be any three vertices of  $T$ . By definition,  $(a_1, b_1)(a_2, b_2), (a_2, b_2)(a_3, b_3)$  and  $(a_3, b_3)(a_1, b_1)$  are the edges of  $T$  in  $G_1 \oplus G_2$  if and only if the edges:  $a_1a_2, a_2a_3$  and  $a_3a_1$  constitute a triangle  $T_1$  in  $G_1$  and also the edges :  $b_1b_2, b_2b_3$  and  $b_3b_1$  constitute a triangle  $T_2$  in  $G_2$ . But this is so if and only if both  $G_1$  and  $G_2$  have triangles  $T_1$  and  $T_2$ , respectively. Finally, we discuss the case when  $n \geq 3$ . The result follows immediately if we proceed by applying induction on the number of factors.  $\square$

**Theorem 7.3.** *Let  $G_i$  ( $1 \leq i \leq 2$ ) be a connected graph having the number of triangles  $n_i$ . Then the product  $G_1 \oplus G_2$  contains  $6n_1n_2$  triangles.*

*Proof.* First, let us compute the actual number of triangles in the product  $T_1 \oplus T_2$ , when  $T_i$  is any triangle in  $G_i$  (for  $i = 1, 2$ ). It is easy to see that there are exactly 6 distinct triangles in  $T_1 \oplus T_2$ . But each  $G_i$  contains  $n_i$  triangles. Consequently, the product  $G_1 \oplus G_2$  contains  $6n_1n_2$  triangles, and there are no more other triangles because of Lemma 7.2.  $\square$

The immediate consequence of the above theorem is the following corollary.

**Corollary 7.4.** *Let  $G_k$  ( $1 \leq k \leq n$  ;  $n \geq 2$ ) be a connected graph having the number of triangles  $n_k$ . Then the product  $\oplus_{k=1}^n G_k$  contains  $6^{n-1}(\prod_{k=1}^n n_k)$  triangles.*

**Corollary 7.5.** *The number of triangles in  $K_m \oplus K_n$  is  $\frac{1}{6}[mn(m-1)(n-1)(m-2)(n-2)]$ .*

*Proof.* We know that the number of triangles in  $K_p = pC_3$ . Therefore from Theorem 7.3, the number of triangles in  $K_m \oplus K_n$  is  $6(mC_3)(nC_3) = \frac{1}{6}[mn(m-1)(n-1)(m-2)(n-2)]$ .  $\square$

Finally to determine the girth of the tensor product of graphs, we need to establish the following lemma.

**Lemma 7.6.** *Let  $G_k$  ( $1 \leq k \leq n$  ;  $n \geq 2$ ) be a connected, triangle-free graph such that each  $G_k$  contains an induced subgraph isomorphic to  $P_3$ . Then  $g(\oplus_{k=1}^n G_k) = 4$ .*

*Proof.* We discuss the case when  $n = 2$ . Let  $a_i$  ( $1 \leq i \leq 3$ ) and  $b_i$  ( $1 \leq i \leq 3$ ) be the vertices of a subgraph isomorphic to  $P_3$  in  $G_1$  and  $G_2$ , respectively. Then the subgraph  $\langle \{a_1, a_2, a_3\} \rangle \oplus \langle \{b_1, b_2, b_3\} \rangle$  is isomorphic to  $P_3 \oplus P_3$  in  $G_1 \oplus G_2$ . It is easy to see that  $P_3 \oplus P_3 = K_{1,4} \cup C_4$ . Immediately, a 4-cycle  $C_4$  appears as a subgraph in  $G_1 \oplus G_2$ . However from Lemma 7.2, there is no triangle in  $G_1 \oplus G_2$ . Consequently,  $C_4$  is the smallest cycle in  $G_1 \oplus G_2$ .

Therefore,  $g(G_1 \oplus G_2) = 4$ .

When  $n \geq 3$ , the result follows easily if we proceed by induction on the number of factors.  $\square$

The following result gives the girth of tensor product of arbitrarily many graphs.

**Theorem 7.7.** *Let  $G_k$  ( $1 \leq k \leq n$ ;  $n \geq 2$ ) be a connected graph of order  $\geq 3$ . Then  $g(\bigoplus_{k=1}^n G_k)$  is either 3 or 4.*

*Proof.* We discuss three cases when  $n = 2$ .

**Case 1.** Suppose both  $G_1$  and  $G_2$  have triangles. By Lemma 7.2,  $G_1 \oplus G_2$  contains a triangle. Hence,  $g(G_1 \oplus G_2) = 3$ .

**Case 2.** Suppose one of  $G_1$  and  $G_2$  is triangle-free. Without loss of generality, we assume that  $G_1$  contains a triangle, and  $G_2$  has an induced subgraph isomorphic to  $P_3$ . It is easy to check that  $K_3 \oplus P_3$  contains a 4-cycle  $C_4$ . Consequently, this  $C_4$  appears in  $(G_1 \oplus G_2)$ . However again by Lemma 7.2,  $G_1 \oplus G_2$  is triangle-free. This implies that  $C_4$  is the smallest cycle in  $G_1 \oplus G_2$ . Therefore,  $g(G_1 \oplus G_2) = 4$ .

**Case 3.** Suppose  $G_1$  and  $G_2$  are triangle-free. Then each  $G_1$  and  $G_2$  contains an induced subgraph isomorphic to  $P_3$ . From Lemma 7.6,  $g(G_1 \oplus G_2) = 4$ .

From the above cases, it follows that  $g(G_1 \oplus G_2) = 3$  or 4.

When  $n \geq 3$ , It is not difficult to prove the result if we proceed by induction on the number of factors.  $\square$

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