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Frames in 2-inner Product Spaces

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Abstract. In this paper, we introduce the notion of a frame in a 2-inner product space and give some characterizations. These frames can be considered as a usual frame in a Hilbert space, so they share many useful properties with frames.

Keywords: 2-inner product space, 2-norm space, Frame, Frame operator.

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1. Introduction and preliminaries

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [12] in 1952 to study some deep problems in nonharmonic Fourier series. Various generalizations of frames have been proposed; frame of subspaces [2, 6], pseudo-frames [18], oblique frames [10], continuous frames [1, 4, 14] and so on. The concept of frames in Banach spaces have been introduced by Grochenig [16], Casazza, Han and Larson [5] and Christensen and Stoeva [11].

The concept of linear 2-normed spaces has been investigated by S. Gahler in 1965 [15] and has been developed extensively in different subjects by many authors [3, 7, 8, 13, 14, 17]. A concept which is related to a 2-normed space is 2-inner product space which have been intensively studied by many mathematicians in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive

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list of the related references can be found in the book [7]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let $\mathcal{X}$ be a linear space of dimension greater than 1 over the field $\mathbb{K}$, where $\mathbb{K}$ is the real or complex numbers field. Suppose that $(.,.|.)$ is a $\mathbb{K}$-valued function defined on $\mathcal{X}\times\mathcal{X}\times\mathcal{X}$ satisfying the following conditions:
(I1)$(x,x|z)\geq 0$ and $(x,x|z)=0$ if and only if x and z are linearly dependent,
(I2)$(x,x|z)=(z,z|x)$,
(I3)$(y,x|z)=\overline{(x,y|z)}$,
(I4)$(\alpha x,y|z)=\alpha(x,y|z)$ for all $\alpha\in\mathbb{K}$,
(I5)$(x_1+x_2,y|z)=(x_1,y|z)+(x_2,y|z)$.
$(.,.|.)$ is called a 2-*inner product* on $\mathcal{X}$ and $(\mathcal{X},(.,.|.))$ is called a 2-*inner product space* (or 2-*pre Hilbert space*). Some basic properties of 2-inner product $(.,.|.)$ can be immediately obtained as follows ([8, 13]):

- $(0,y|z)=(x,0|z)=(x,y|0)=0$,
- $(x,\alpha y|z)=\overline{\alpha}(x,y|z)$,
- $(x,y|\alpha z)=|\alpha|^2(x,y|z)$, for all $x,y,z\in\mathcal{X}$ and $\alpha\in\mathbb{K}$.

Using the above properties, we can prove the Cauchy-Schwarz inequality

$$|(x,y|z)|^2\leq (x,x|z)(y,y|z). \tag{1.1}$$

Example 1.1. If $(\mathcal{X},\langle .,.\rangle)$ is an inner product space, then the standard 2-inner product $(.,.|.)$ is defined on $\mathcal{X}$ by

$$(x,y|z)=\begin{vmatrix}\langle x,y\rangle & \langle x,z\rangle\\ \langle z,y\rangle & \langle z,z\rangle\end{vmatrix}=\langle x,y\rangle\langle z,z\rangle-\langle x,z\rangle\langle z,y\rangle, \tag{1.2}$$

for all $x,y,z\in\mathcal{X}$.

In any given 2-inner product space $(\mathcal{X},(.,.|.))$, we can define a function $\|.,.\|$ on $\mathcal{X}\times\mathcal{X}$ by

$$\|x,z\|=(x,x|z)^{\frac{1}{2}}, \tag{1.3}$$

for all $x,z\in\mathcal{X}$.

It is easy to see that, this function satisfies the following conditions:
(N1) $\|x,z\|\geq 0$ and $\|x,z\|=0$ if and only if x and z are linearly dependent,
(N2) $\|x,z\|=\|z,x\|$,
(N3) $\|\alpha x,z\|=|\alpha|\|x,z\|$ for all $\alpha\in\mathbb{K}$,
(N4) $\|x_1+x_2,z\|\leq\|x_1,z\|+\|x_2,z\|$.

Any function $\|.,.\|$ defined on $\mathcal{X}\times\mathcal{X}$ and satisfying the conditions (N1)-(N4) is called a 2-*norm* on X and $(\mathcal{X},\|.,.\|)$ is called a *linear* 2-*normed space.* Whenever a 2-inner product space $(\mathcal{X},(.,.|.))$ is given, we consider it as a linear 2-normed space $(\mathcal{X},\|.,.\|)$ with the 2-norm defined by (1.3).

In the present paper, we shall introduce 2-frames for a 2-inner product space and describe some fundamental properties of them. This implies that each element in the underlying 2-inner product space can be written as an unconditionally convergent infinite linear combination of the frame elements.

2. Frames in the standard 2-inner product spaces

Throughout this paper, we assume that $\mathcal{H}$ is a separable Hilbert space, with the inner product $\langle ., . \rangle$ chosen to be linear in the first entry. We first review some basic facts about frames in $\mathcal{H}$, then try to define them in a standard 2-inner product space.

Definition 2.1. A sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \qquad (f \in \mathcal{H}). \tag{2.1}$$

The numbers A, B are called *frame bounds*. The frame is called *tight* if $A = B$. Given a frame $\{f_i\}_{i=1}^{\infty}$, the frame operator is defined by

$$Sf = \sum_{i=1}^{\infty} \langle f, f_i \rangle f_i.$$

The series defining Sf converges unconditionally for all $f \in \mathcal{H}$ and S is a bounded, invertible, and self-adjoint operator. This leads to the frame decomposition:

$$f = S^{-1}Sf = \sum_{i=1}^{\infty} \langle f, S^{-1}f_i \rangle f_i, \qquad (f \in \mathcal{H}).$$

The possibility of representing every $f \in \mathcal{H}$ in this way is the main feature of a frame. The coefficients $\{\langle f, S^{-1}f_i \rangle\}_{i=1}^{\infty}$ are called frame coefficients. A sequence satisfying the upper frame condition is called a *Bessel sequence*. A sequence $\{f_i\}_{i=1}^{\infty}$ is Bessel sequence if and only if the operator $T : \{c_i\} \mapsto \sum_{i=1}^{\infty} c_i f_i$ is a well-defined operator from l^2 into $\mathcal{H}$. In that case T, which is called the *pre-frame operator*, is automatically bounded. When $\{f_i\}_{i=1}^{\infty}$ is a frame, the pre-frame operator T is well-defined and $S = TT^*$. For more details see [9, Section 5.1]. Also see [19] for a class of finite frames.

Let $\mathcal{X}$ be a 2-inner product space. A sequence $\{a_n\}_{n=1}^{\infty}$ of $\mathcal{X}$ is said to be *convergent* if there exists an element $a \in \mathcal{X}$ such that $\lim_{n\to\infty} \|a_n - a, x\| = 0$, for all $x \in \mathcal{X}$. Similarly, we can define a Cauchy sequence in $\mathcal{X}$. A 2-inner product space $\mathcal{X}$ is called a 2-*Hilbert space* if it is complete. That is, every Cauchy sequence in $\mathcal{X}$ is convergent in this space [17]. Clearly, the limit of any convergent sequence is unique and if $(\mathcal{X}, (., .|.))$ is the standard 2-inner product, then this topology is the original topology on $\mathcal{X}$.

Now we are ready to define 2-frames on a 2-Hilbert space.

Definition 2.2. Let $(\mathcal{X}, (.,.|.))$ be a 2-Hilbert space and $\xi \in \mathcal{X}$. A sequence $\{x_i\}_{i=1}^{\infty}$ of elements in $\mathcal{X}$ is called a 2-*frame* (associated to ξ) if there exist $A, B > 0$ such that

$$A\|x,\xi\|^2 \leq \sum_{i=1}^{\infty} |(x, x_i|\xi)|^2 \leq B\|x,\xi\|^2, \qquad (x \in \mathcal{X}). \tag{2.2}$$

A sequence satisfying the upper 2-frame condition is called a 2-*Bessel sequence*. In (2.2) we may assume that every x_i is linearly independent to ξ , by (1.1) and the property (I1).

The following proposition shows that in the standard 2-inner product spaces every frame is a 2-frame.

Proposition 2.3. *Let $(\mathcal{H}, \langle .,. \rangle)$ be a Hilbert space and $\{x_i\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}$. Then it is a 2-frame with the standard 2-inner product on $\mathcal{H}$.*

Proof. Suppose that $\{x_i\}_{i=1}^{\infty}$ is a frame with the bounds A, B and $\xi \in \mathcal{H}$ such that $\|\xi\| = 1$. Then by using (2.1) and (1.2) for every $x \in \mathcal{H}$ we have

$$\begin{aligned} \sum_{i=1}^{\infty} |(x, x_i|\xi)|^2 &= \sum_{i=1}^{\infty} |\langle x - \langle x,\xi\rangle\xi, x_i\rangle|^2 \\ &\leq B\|x - \langle x,\xi\rangle\xi\|^2 \\ &\leq B(\|x\|^2 - |\langle x,\xi\rangle|^2) \\ &= B(x, x|\xi). \end{aligned}$$

The argument for lower bound is similar. □

The converse of the above proposition is not true. In fact, by the following proposition, every 2-frame is a frame for a closed subspace of $\mathcal{H}$ with codimension 1. For each $\xi \in \mathcal{H}$ we denote by L_ξ the subspace generated with ξ.

Proposition 2.4. *Let $(\mathcal{H}, \langle .,. \rangle)$ be a Hilbert space and $\xi \in \mathcal{H}$. Every 2-frame associated to ξ is a frame for $L_\xi^\perp$.*

Proof. If $\{x_i\}_{i=1}^{\infty}$ is a 2-frame with the bounds A, B then (2.2) implies that there exist $A, B > 0$ such that

$$A(\|x\|^2 - |\langle x,\xi\rangle|^2) \leq \sum_{i=1}^{\infty} |\langle x - \langle x,\xi\rangle\xi, x_i\rangle|^2 \leq B(\|x\|^2 - |\langle x,\xi\rangle|^2), \qquad (x \in \mathcal{H}).$$

Therefore, $\{x_i\}_{i=1}^{\infty}$ is a frame for the Hilbert space $L_\xi^\perp$. □

Remark 2.5. Let $\mathcal{H}$ be a Hilbert space and $\{x_i\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}$ with the frame operator S. If $\langle x_j, S^{-1}x_j\rangle = 1$ for some $j \in \mathbb{N}$, then $\{x_i\}_{i\neq j}$ is incomplete and therefore it is not a frame for $\mathcal{H}$ [9, Theorem 5.3.9]. Assume that $\|x_j\| = 1$ and consider the standard 2-inner product on $\mathcal{H}$. It is not difficult to see that

$$\sum_{i=1}^{\infty} |(x, x_i|x_j)|^2 = \sum_{i=1, i\neq j}^{\infty} |(x, x_i|x_j)|^2.$$

Now the proof of Proposition 2.3 shows that $\{x_i\}_{i\neq j}$ is a 2-frame for $\mathcal{H}$ associated to x_j.

3. Some properties of 2-frames

This section is devoted to establishing pre-frame and frame operator for a 2-frame. To extend a well-known result in Hilbert spaces to 2-inner product spaces.

Lemma 3.1. *Let* $(\mathcal{X}, (., .|.))$ *be a 2-inner product space and* $x, z \in \mathcal{X}$. *Then*

$$\|x, z\| = sup\{|(x, y|z)|; \qquad y \in \mathcal{X}, \|y, z\| = 1\}. \tag{3.1}$$

Proof. By the Cauchy-Schwarz inequality (1.1) we observe that

$$(x, y|z) \leq \|x, z\|\|y, z\| = \|x, z\|$$

for every $y \in \mathcal{X}$ such that $\|y, z\| = 1$. Moreover, if $y = \frac{1}{\|x,z\|}x$, then $\|y, z\| = 1$ and therefore $(x, y|z) = \|x, z\|$. □

For the remainder, we assume $(\mathcal{X}, (., .|.))$ is a 2-Hilbert space and L_ξ the subspace generated with ξ for a fix element ξ in $\mathcal{X}$. Denote by $\mathcal{M}_\xi$ the algebraic complement of L_ξ in $\mathcal{X}$. So $L_\xi \oplus \mathcal{M}_\xi = \mathcal{X}$.

We first define the inner product $\langle ., .\rangle_\xi$ on $\mathcal{X}$ as following:

$$\langle x, z\rangle_\xi = (x, z|\xi).$$

A straightforward calculations shows that $\langle ., .\rangle_\xi$ is a semi-inner product on $\mathcal{X}$. It is well-known that this semi-inner product induces an inner product on the quotient space $\mathcal{X}/L_\xi$ as

$$\langle x + L_\xi, z + L_\xi\rangle_\xi = \langle x, z\rangle_\xi, \qquad (x, z \in \mathcal{X}).$$

By identifying $\mathcal{X}/L_\xi$ with $\mathcal{M}_\xi$ in an obvious way, we obtain an inner product on $\mathcal{M}_\xi$. Define

$$\|x\|_\xi = \sqrt{\langle x, x\rangle_\xi} \qquad (x \in \mathcal{M}_\xi). \tag{3.2}$$

Then $(\mathcal{M}_\xi, \|.\|_\xi)$ is a norm space.

Now if $\{x_i\}_{i=1}^{\infty} \subseteq \mathcal{X}$ is a 2-frame associated to ξ with bounds A and B, then we can rewrite (2.2) as

$$A\|x\|_\xi^2 \leq \sum_{i=1}^{\infty} |\langle x, x_i\rangle_\xi|^2 \leq B\|x\|_\xi^2, \qquad (x \in \mathcal{M}_\xi).$$

That is, $\{x_i\}_{i=1}^{\infty}$ is a frame for $\mathcal{M}_\xi$. Let $\mathcal{X}_\xi$ be the completion of the inner product space $\mathcal{M}_\xi$. Due to Lemma 5.1.2 of [9] the sequence $\{x_i\}_{i=1}^{\infty}$ is also a frame for $\mathcal{X}_\xi$ with the same bounds. To summarize, we have the following theorem.

Theorem 3.2. *Let $(\mathcal{X}, (.,.|.))$ be a 2-Hilbert space. Then $\{x_i\}_{i=1}^{\infty} \subseteq \mathcal{X}$ is a 2-frame associated to ξ with bounds A and B if and only if it is a frame for the Hilbert space $\mathcal{X}_\xi$ with bounds A and B.*

By the above theorem, every question about 2-frames in a 2-Hilbert space can be solved as a question about frames in a Hilbert space.

Lemma 3.3. *Let $\{x_i\}_{i=1}^{\infty}$ be a 2-Bessel sequence in $\mathcal{X}$. Then the 2-pre frame operator $T_\xi : l^2 \to \mathcal{X}_\xi$ defined by*

$$T_\xi\{c_i\} = \sum_{i=1}^{\infty} c_i x_i \tag{3.3}$$

is well-defined and bounded.

Proof. Suppose $\{c_i\}_{i=1}^{\infty} \in l^2$, then by using (3.1) and (3.2) we have

$$\begin{aligned}
\|\sum_{i=1}^{m} c_i x_i - \sum_{i=1}^{n} c_i x_i\|_\xi^2 &= \|\sum_{i=1}^{m} c_i x_i - \sum_{i=1}^{n} c_i x_i, \xi\|^2 \\
&= sup\{|(\sum_{i=n}^{m} c_i x_i, y|\xi)|^2, y \in \mathcal{X}, \|y,\xi\| = 1\} \\
&\leq \sum_{i=n}^{m} |c_i|^2 sup\{\sum_{i=n}^{m} |(x_i, y|\xi)|^2, y \in \mathcal{X}, \|y,\xi\| = 1\} \\
&\leq B\sum_{i=n}^{m} |c_i|^2
\end{aligned}$$

where B is the (upper) bound of $\{x_i\}_{i=1}^{\infty}$. This implies that $\sum_{i=1}^{\infty} c_i x_i$ is well-defined as an element of $\mathcal{X}_\xi$. Moreover, if $\{c_i\}_{i=1}^{\infty}$ is a sequence in l^2, then an argument as above shows that $\|T_\xi\{c_i\}\|_\xi \leq \sqrt{B}\|\{c_i\}\|_2$. In particular, $\|T_\xi\| \leq \sqrt{B}$. □

Next, we can compute T_ξ^*, the adjoint of T_ξ as

$$T_\xi^* : \mathcal{X}_\xi \to l^2; \qquad T_\xi^* x = \{(x, x_i|\xi)\}_{i=1}^{\infty}.$$

It is easy to check that T_ξ^* is well-defined. Moreover, it follows by (2.2) that $\|T_\xi^*\| \leq \sqrt{B}$.

Definition 3.4. Let $\{x_i\}_{i=1}^{\infty}$ be a 2-frame associated to ξ with bounds A and B in a 2-Hilbert space $\mathcal{X}$. The operator $S_\xi : \mathcal{X}_\xi \to \mathcal{X}_\xi$ defined by

$$S_\xi x = \sum_{i=1}^{\infty} (x, x_i|\xi) x_i \tag{3.4}$$

is called the 2-frame operator for $\{x_i\}_{i=1}^{\infty}$.

Clearly, $S_\xi = T_\xi T_\xi^*$ and therefore $\|S_\xi\| \leq B$. We can conclude the boundedness of S_ξ directly. Indeed, we see from (I3),(I4),(I5) and (3.1) that

$$\begin{aligned} \|S_\xi x\|_\xi^2 &= \|S_\xi x, \xi\|^2 \\ &= sup\{|(S_\xi x, y|\xi)|^2, \quad y \in \mathcal{X}, \|y, \xi\| = 1\} \\ &\leq sup\{\sum_{i=1}^{\infty} |(x, x_i|\xi)|^2 \sum_{i=1}^{\infty} |(y, x_i|\xi)|^2, \quad y \in \mathcal{X}, \|y, \xi\| = 1\} \\ &\leq B^2 \|x\|_\xi^2. \end{aligned}$$

Now we state some of the important properties of S_ξ.

Theorem 3.5. *Let $\{x_i\}_{i=1}^{\infty}$ be a 2-frame associated to ξ for a 2-Hilbert space $(\mathcal{X}, (.,.|.))$ with 2-frame operator S_ξ and frame bounds A, B. Then S_ξ is invertible, self-adjoint, and positive.*

Proof. Obviously, the operator S_ξ is self-adjoint. The inequality (2.2) means that

$$A\|x\|_\xi^2 \leq \langle S_\xi x, x\rangle_\xi \leq B\|x\|_\xi^2, \qquad (x \in \mathcal{X}_\xi).$$

Hence, S_ξ is a positive element in the set of all bounded operators on the Hilbert space $\mathcal{X}_\xi$. More precisely, with symbols $AI \leq S_\xi \leq BI$ where I is the identity operator on $\mathcal{X}_\xi$. Furthermore,

$$\|I - B^{-1}S_\xi\| = sup_{\|x\|_\xi = 1} |\langle (I - B^{-1}S_\xi)x, x\rangle_\xi| \leq \frac{B - A}{B} < 1.$$

This shows that S_ξ is invertible. □

Corollary 3.6. *Let $\{x_i\}_{i=1}^{\infty}$ be a 2-frame in a 2-Hilbert space $\mathcal{X}$ with frame operator S_ξ. Then each $x \in \mathcal{X}_\xi$ has an expansion of the following*

$$x = S_\xi S_\xi^{-1} x = \sum_{i=1}^{\infty} (S_\xi^{-1} x, x_i|\xi) x_i.$$

Remark 3.7. If $\{x_i\}_{i=1}^{\infty}$ is a 2-frame associated to ξ, then every $x \in \mathcal{X}$ has a representation as

$$x = \alpha\xi + \sum_{i=1}^{\infty} c_i x_i,$$

for some $\alpha \in \mathbb{C}$ and $\{c_i\}_{i=1}^{\infty} \in l^2$. The coefficients $\{c_i\}_{i=1}^{\infty}$ are not unique, but the frame coefficients $\{(S_\xi^{-1}x, x_i|\xi)\}_{i=1}^{\infty}$ introduced in the Corollary 3.6 have minimal l^2-norm among all sequences representing x, see Lemma 5.3.6 of [9].

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