

Superminimal fibres in an almost contact metric submersion

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ABSTRACT. The superminimality of the fibres of an almost contact metric submersion is used to study the integrability of the horizontal distribution and the structure of the total space.

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1. INTRODUCTION

Superminimal fibres of a Riemannian submersion have been introduced by M. Falcitelli and A.M. Pastore [9], who examined only the case of almost Kähler submersions. On the other hand, B. Watson [18], studied extensively superminimal fibres of an almost Hermitian submersion. He used this property to derive the structure of the total space according to that of the base space.

In [16], we extended the definition of superminimal submanifolds to the ϕ -invariant fibres of almost contact metric manifolds, considering submersions whose total space is a nearly α -Kenmotsu manifold. There, we showed that if the fibres of an almost contact metric submersion with total space a nearly α -Kenmotsu manifold are superminimal, then the horizontal distribution is completely integrable.

The present paper utilizes superminimality property in almost contact metric geometry to study the integrability of the horizontal distribution and the structure of the total space. This last aspect is the inverse of the problem examined in [15] where we determined the structure of the base space and the fibres according to the structure of the total space.

This text is organized in the following way:

§2 is devoted to the recollection of some basic notions of almost contact metric geometry and define a superminimal submanifold of an almost contact metric manifold.

In §3, which contains a review of those general properties of almost contact metric submersions that will be needed in the sequel, we generalize the Theorem of China [7], concerning the structure equations of a submersion, to the case of almost contact metric submersion of type II.

§4 is concerned with the determination of the classes of almost contact metric submersions whose fibres are, a priori, superminimal. One could expect that the superminimality should imply the minimality of the fibres. We will show that, for this, some conditions are needed.

In §5, devoted to the integrability of the horizontal distribution, we investigate the classes of submersions for which the superminimality of the fibres implies the integrability (or non-integrability) of the horizontal distribution.

At §6, we end our study with an examination of the transference of almost contact metric structures. We prove the following technical result.

Lemma 2. *Let $f : M \rightarrow N$ be an almost contact metric submersion of type I. Suppose that the base space is defined by $d\eta' = 0$. If the fibres are superminimal and $A_X(\phi X) = 0$, then $d\eta = 0$ on the total space.*

This Lemma has many applications in the case of cosymplectic and Kenmotsu geometries

2. PRELIMINARY BACKGROUND.

The differential geometry of almost contact metric manifolds is developed in the fundamental book of D.E. Blair [4] and its recent expansion [5]. Several of the various almost contact metric structures were studied in the articles [2], [3], [6], [8] and [12]. We recall here the definitions of only those objects that are needed for our present study.

Let us recall that an *almost contact metric manifold* is a quintuple (M, ϕ, ξ, η, g) where (M, g) is a Riemannian manifold and

- (1) ξ is a distinguished unit vector field on M ,
- (2) η is the 1 - form on M which is g - dual to ξ ,
- (3) ϕ is a tensor field of type $(1, 1)$ on M satisfying

$$\phi \circ \phi = -Id + \eta \otimes \xi,$$

(4) the Riemannian structure, g , satisfies

$$g(\phi E, \phi F) = g(E, F) - \eta(E)\eta(F).$$

An almost contact metric manifold (M, ϕ, ξ, η, g) is necessarily of odd dimension, $2m + 1$. In fact, we can always find an orthonormal ϕ -basis, $\{E_1, \dots, E_m, \phi E_1, \dots, \phi E_m, \xi\}$, for the local smooth vector fields on $(M^{2m+1}, \phi, \xi, \eta, g)$.

Let (M, ϕ, ξ, η, g) be an almost contact metric manifold. The *fundamental 2-form* on (M, ϕ, ξ, η, g) is

$$\Phi(E, F) = g(E, \phi F).$$

Letting ∇ denote the Riemannian connection of (M, g) , it is immediate that

$$(1.1) \quad (\nabla_E \Phi)(F, G) = g(F, (\nabla_E \phi)G) = g(F, \nabla_E(\phi G) - \phi \nabla_E G).$$

With respect to the local ϕ -basis $\{E_1, \dots, E_m, \phi E_1, \dots, \phi E_m, \xi\}$ on $(M^{2m+1}, \phi, \xi, \eta, g)$, the *codifferential*, $\delta\Phi$, of the *fundamental 2-form*, Φ , on M is given by:

$$(1.2) \quad \delta\Phi(F) = - \sum_{i=1}^m \{(\nabla_{E_i} \Phi)(E_i, F) + (\nabla_{\phi E_i} \Phi)(\phi E_i, F)\} - (\nabla_\xi \Phi)(\xi, F)$$

In analogy with the almost Hermitian situation, we say that the almost contact metric manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ is *normal* if the obvious induced almost complex structure on $M \times \mathbb{R}$ is integrable. One may show that M is normal if and only if the tensor field, $N^{(1)}$, vanishes where: [2]

$$(1.3) \quad N^{(1)}(E, F) = [\phi, \phi](E, F) + 2d\eta(E, F)\xi$$

Recalling that

$$(1.4) \quad (\nabla_E \eta)F = g(E, \nabla_F \xi),$$

it is easy to see that the *differential* of η is

$$(1.5) \quad d\eta(E, F) = \frac{1}{2} \{(\nabla_E \eta)F - (\nabla_F \eta)E\},$$

and that the *codifferential* of η is

$$(1.6) \quad \delta\eta = - \sum_{i=1}^m \{(\nabla_{E_i} \eta)E_i + (\nabla_{\phi E_i} \eta)(\phi E_i)\},$$

with respect to the local ϕ -basis $\{E_1, \dots, E_m, \phi E_1, \dots, \phi E_m, \xi\}$ on the almost contact metric manifold $(M^{2m+1}, \phi, \xi, \eta, g)$.

Through the years, a plethora of almost contact metric structures have been defined. In fact, according to the classification scheme of D. Chinea and C. González [8], there are 4.096 such structures. We recall here only those almost contact metric structures that are relevant to the present work. The interested reader will certainly find many more almost contact metric structures in the mathematical literature.

An almost contact metric manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ is said to be:

- (1) *cosymplectic* if $\nabla\phi = 0$,
- (2) *nearly cosymplectic* if $(\nabla_E\phi)E = 0$,
- (3) *almost cosymplectic* if $d\Phi = 0$ and $d\eta = 0$,
- (4) *semi-cosymplectic* if $\delta\Phi = 0$ and $\delta\eta = 0$,
- (5) *quasi-K-cosymplectic* if $(\nabla_E\phi)F + (\nabla_{\phi E}\phi)(\phi F) - \eta(F)\nabla_{\phi E}\xi = 0$,
- (6) *Kenmotsu* if $(\nabla_E\phi)F = g(\phi E, F)\xi - \eta(E)\phi F$,
- (7) *nearly Kenmotsu* if $(\nabla_E\phi)E = -\eta(E)\phi E$,
- (8) *quasi-Sasakian* if $d\Phi = 0$ and M is normal,
- (9) *nearly Sasakian* if

$$(\nabla_E\phi)F + (\nabla_F\phi)E = 2g(E, F)\xi - \eta(E)F - \eta(F)E,$$

- (10) *Sasakian* if $\Phi = d\eta$ and M is normal,
- (11) C_{11} - manifold if

$$(\nabla_E\Phi)(F, G) = -\eta(E)(\nabla_\xi\Phi)(\phi F, \phi G),$$

- (12) C_{12} - manifold if

$$(\nabla_E\Phi)(F, G) = -\eta(E)\eta(G)(\nabla_\xi\eta)(\phi F) - \eta(E)\eta(F)(\nabla_\xi\eta)(\phi G),$$

Definition 1. Let $(M^{2m+1}, g, \phi, \xi, \eta)$ be an almost contact metric manifold and \hat{M} a ϕ -invariant submanifold of M . Then \hat{M} is said to be *superminimal* if $\nabla_V\phi = 0$ for all vector fields, V , tangent to the submanifold \hat{M} .

3. GENERALITIES ON ALMOST CONTACT METRIC SUBMERSIONS.

We refer the reader to the fundamental articles of B. O'Neill [13] and A. Gray [11] as well as to Chapter 9 of the book [1] for the basic properties of Riemannian submersions. These primarily concern the orthogonal decomposition $T(M) = H(M) \oplus V(M)$ of the local vector fields on the total space of a Riemannian submersion into horizontal and vertical vector fields. Recently, M. Falcitelli, S. Ianus and A.M. Pastore have published an important and interesting book on the main classes of Riemannian submersions [10]. We follow the traditional notation, letting U, V, W, \dots denote *vertical* vector fields and X, Y, Z, \dots denote *horizontal* vector fields on (M, ϕ, ξ, η, g) . In particular, the important properties of the two O'Neill configuration tensors, T and A , of a Riemannian submersion are contained in these references.

We will have occasion to use the following result from B. O'Neill's foundational paper [13]:

Lemma 1. Let $f : M \rightarrow N$ be a Riemannian submersion. Let X be a basic vector field on M . Then,

$$\mathcal{H}\nabla_U X = A_X U.$$

3.1. Almost contact metric submersions of type I.

Definition 2. Let (M, ϕ, ξ, η, g) and $(N, \phi', \xi', \eta', g')$ be almost contact metric manifolds. A Riemannian submersion $f : M \rightarrow N$ that satisfies:

- (1) $f_*\phi E = \phi' f_*E$, and
- (2) $f_*\xi = \xi'$

is called an almost contact metric submersion of type I [17]. If (M, ϕ, ξ, η, g) is in the structure class φ of the classification scheme of D. Chinea and C. González [8], then we say that f is a φ -submersion of type I.

The fibre submanifolds, \hat{M} , of an almost contact metric submersion of type I are almost Hermitian manifolds, $(\hat{M}, \hat{J}, \hat{g})$ in a natural way. The defining relation for a contact manifold is $\Phi = d\eta$. It is easy to see that $d\eta(U, V) = 0$ for all vertical vector fields, U and V , on an almost contact metric submersion of type I. Thus, there are no non-trivial (*trivial* here means having 0-dimensional fibres) almost contact metric submersions of type I with nearly Sasakian, quasi-K-Sasakian, almost - α - Sasakian, α - Sasakian, Sasakian, contact or K-contact total space.

We recall the important properties of an almost contact metric submersion of type I. See [17] and [14] for the proof of the following proposition and related results.

Proposition 1. Let $f : (M, \phi, \xi, \eta, g) \rightarrow (N, \phi', \xi', \eta', g')$ be an almost contact metric submersion of type I. Then,

- a) $\phi\{V(M)\} \subseteq V(M)$,
- b) $\phi\{H(M)\} \subseteq H(M)$,
- c) ξ is horizontal,
- d) $\eta(V) = 0$, for all vertical vector fields, V ,
- e) $f^*\eta' = \eta$,
- f) $\hat{\Phi}(U, V) = \Phi(U, V)$, for all vertical vector fields, U and V ,
- g) $\mathcal{H}(\nabla_X \phi)Y$ is the basic vector field associated to $(\nabla'_{X_*} \phi')Y_*$ on N , for X and Y , basic vector fields on M .

While defining and studying the structure equation of an almost contact metric manifold of type I, D. Chinea defined a tensor $A^*(\cdot, \cdot)$ of type (1, 2) by setting

$$A^*(X, Y) = A_X(\phi Y) - A_{\phi X}Y.$$

Chinea then proved:

Theorem 1. [7] Let $f : (M, \phi, \xi, \eta, g) \rightarrow (N, \phi', \xi', \eta', g')$ be an almost contact metric submersion of type I. Let E be an arbitrary vector field on the total space, M , and let H be the (horizontal) mean curvature vector field of the fibres. Then,

- a) $\delta\Phi(E) = g(H, \phi\mathcal{H}E) + \delta'\Phi'((\mathcal{H}E)_*) + \hat{\delta}\hat{\Phi}(\mathcal{V}E) + \frac{1}{2}g(\text{tr}A^*, \mathcal{V}E)$, and
- b) $\delta\eta = -g(H, \xi) + f^*(\delta'\eta')$.

3.2. Almost contact metric submersions of type II.

Definition 3. Let (M, ϕ, ξ, η, g) be an almost contact metric manifold and let (N, J', g') be an almost Hermitian manifold. A Riemannian submersion $f : M \rightarrow N$ that satisfies:

$$f_*\phi E = J' f_* E$$

is called an almost contact metric submersion of type II [17]. If (M, ϕ, ξ, η, g) is in the structure class \wp of the classification scheme of D. Chinea and C. González [8], then we say that f is a \wp -submersion of type II.

The fibre submanifolds, \hat{M} , of an almost contact metric submersion of type II are almost contact metric manifolds, $(\hat{M}, \hat{\phi}, \hat{\xi}, \hat{\eta}, \hat{g})$ in a natural way.

The analogue of Proposition (1) is:

Proposition 2. Let $f : (M, \phi, \xi, \eta, g) \rightarrow (N, J', g')$ be an almost contact metric submersion of type II. Then,

- a) $\phi\{V(M)\} \subseteq V(M)$,
- b) $\phi\{H(M)\} \subseteq H(M)$,
- c) $f^*\Phi' = \Phi$,
- d) ξ is vertical,
- e) $\eta(X) = 0$, for all horizontal vector fields, X ,
- f) $\mathcal{H}(\nabla_X \phi)Y$ is the basic vector field associated to $(\nabla'_{X_*} J')Y_*$ on N , for X and Y , basic vector fields on M ,
- g) $d\eta(X, Y) = -\frac{1}{2}\eta([X, Y]) = -\eta(A_X Y)$ for all horizontal vector fields, X and Y .

We refer the reader to [17] for a discussion of the restriction of a given almost contact metric structure from the total space to the fibres of an almost contact metric submersion of type II, as well as for the induction of a particular almost Hermitian structure onto the base space

Corollary 1. Let $f : (M, \phi, \xi, \eta, g) \rightarrow (N, J', g')$ be an almost contact metric submersion of type II. If $d\eta = 0$ on M , then $A_X Y$ is g -orthogonal to ξ inside the vertical distribution.

Proof. Obvious from assertion (g) of the previous proposition. \square

Theorem 2. Let $f : (M^{2m+1}, \phi, \xi, \eta, g) \rightarrow (N^{2n}, J', g')$ be an almost contact metric submersion of type II. Then, for an arbitrary vector field, E , on M ,

- (1) $\delta\Phi(E) = g(H, \phi\mathcal{H}E) + \delta'\Phi'(\mathcal{H}E_*) + \hat{\delta}\hat{\Phi}(\mathcal{V}E) + \frac{1}{2}g(\text{tr}A^*, \mathcal{V}E)$,
- (2) $\delta\eta = \hat{\delta}\hat{\eta}$,

Proof. The first relation is proved exactly as in [7] for almost contact metric submersions of type I. To see the second assertion, let $\{E_1, \dots, E_m, \phi E_1, \dots, \phi E_m, F_1, \dots, F_n, \phi F_1, \dots, \phi F_n, \xi\}$ be a basis for the local vector fields on M with the $\{E_i, \phi E_i\} \cup \{\xi\}$, vertical and the $\{F_j, \phi F_j\}$ horizontal. Then,

$$\delta\eta = -(\nabla_\xi\eta)\xi - \sum_{i=1}^{m-n} \{(\nabla_{E_i}\eta)E_i + (\nabla_{\phi E_i}\eta)\phi E_i\} - \sum_{j=1}^n \{(\nabla_{F_j}\eta)F_j + (\nabla_{\phi F_j}\eta)\phi F_j\}$$

The first term together with the first sum obviously comprise $\hat{\delta}\hat{\eta}$. Note that the constancy of $\|\xi\|^2 = g(\xi, \xi)$ implies that $(\nabla_\xi\eta)\xi = g(\xi, \nabla_\xi\xi) = 0$. In the second sum, note that each term of the form $(\nabla_{F_j}\eta)F_j$ is $g(F_j, \nabla_{F_j}\xi)$. This then is $-g(\nabla_{F_j}F_j, \xi) = g(A_{F_j}F_j, \xi)$. The F_j are horizontal vector fields and, therefore, $A_{F_j}F_j$ vanishes for the skew-symmetric tensor A . \square

Letting $\{E_1, \dots, E_m, \phi E_1, \dots, \phi E_m, F_1, \dots, F_n, \phi F_1, \dots, \phi F_n, \xi\}$ be a basis for the local vector fields on M with the $\{E_i, \phi E_i\} \cup \{\xi\}$, vertical and the $\{F_j, \phi F_j\}$, horizontal; we easily calculate that, on an almost contact metric submersion of type I, $f : (M^{2m+1}, \phi, \xi, \eta, g) \longrightarrow (N^{2n+1}, \phi', \xi', \eta', g')$

$$tr A^* = 4 \sum_{j=1}^n A_{F_j}(\phi F_j).$$

4. CLASSES OF SUBMERSIONS WITH

SUPERMINIMAL FIBRES.

We investigate here the classes of almost contact metric submersions whose fibres are, or are not, superminimal.

Let $f : (M, \phi, \xi, \eta, g) \longrightarrow (N, \phi', \xi', \eta', g')$ be an almost contact metric submersion of type I. In order to verify superminimality of the almost Hermitian fibres, $(\hat{M}, \hat{J}, \hat{g})$, there are four components of $g((\nabla_V\phi)E, F)$ to be considered on the total space, M . We easily find:

$$\text{SM-1)} \quad g((\nabla_V\phi)U, W) = g(\hat{\nabla}_V(\hat{J}U) - \hat{J}\hat{\nabla}_V U, W),$$

$$\text{SM-2)} \quad g((\nabla_V\phi)U, X) = g(T_V(\phi U) - \phi(T_V U), X),$$

$$\text{SM-3)} \quad g((\nabla_V\phi)X, U) = -g((\nabla_V\phi)U, X),$$

$$\text{SM-4)} \quad g((\nabla_V\phi)X, Y) = -g(A_{\phi X}Y + A_X(\phi Y), V).$$

Let $f : (M, \phi, \xi, \eta, g) \longrightarrow (N, J', g')$ be an almost contact metric submersion of type II. In order to verify superminimality of the almost contact metric fibres, $(\hat{M}, \hat{\phi}, \hat{\xi}, \hat{\eta}, \hat{g})$ there are four components of $g((\nabla_V\phi)E, F)$ to be considered on the total space, M . We easily find:

$$\text{SM-5)} \quad g((\nabla_V\phi)U, W) = g(\hat{\nabla}_V(\hat{\phi}U) - \hat{\phi}\hat{\nabla}_V U, W),$$

$$\text{SM-6)} \quad g((\nabla_V\phi)U, X) = g(T_V(\phi U) - \phi(T_V U), X),$$

$$\text{SM-7)} \quad g((\nabla_V\phi)X, U) = -g((\nabla_V\phi)U, X),$$

$$\text{SM-8) } g((\nabla_V \phi)X, Y) = -g(A_{\phi X}Y + A_X(\phi Y), V).$$

We note the important fact that if the fibres, $(\hat{M}, \hat{J}, \hat{g})$, of an almost contact metric submersion of type I are superminimal, then the vanishing of calculation SM-4) yields $A_{\phi X}Y = -A_X(\phi Y)$. In this case,

$$A^*(X, Y) = -2A_X(\phi Y).$$

The vanishing of SM-1) implies that if the fibres of a type I almost contact metric submersion are superminimal, then they are Kähler. With this in mind, it is easy to prove the following

Proposition 3. *Let $f : (M^{2m+1}, \phi, \xi, \eta, g) \longrightarrow (N^{2n+1}, \phi', \xi', \eta', g')$ be an almost contact metric submersion of type I such that on the total space, $\delta\eta = 0$. If the fibres are superminimal, then $\delta'\eta' = 0$ on the base space.*

Proposition 4. *Let $f : (M, \phi, \xi, \eta, g) \longrightarrow (N, \phi', \xi', \eta', g')$ be an almost contact metric submersion of type I. If the total space is cosymplectic, a C_{11} or a C_{12} -manifold, then the fibres are superminimal.*

Proof. The case of a cosymplectic submersion is obvious. Let us consider the case of a C_{11} -submersion. Consider a vector field V tangent to the fibres. Since the contact 1-form η vanishes on vertical vector fields, on the light of Proposition 1d), we have

$$(\nabla_V \Phi)(E, F) = 0.$$

It is known that $(\nabla_V \Phi)(E, F) = g(E, (\nabla_V \phi)F)$ which leads to $g(E, (\nabla_V \phi)F) = 0$. According to the non-degeneracy of g , we deduce $(\nabla_V \phi)F = 0$ which shows that the fibres are superminimal. We apply the same procedure for a C_{12} -submersion. \square

Proposition 5. *Let $f : (M, \phi, \xi, \eta, g) \longrightarrow (N, J', g')$ be an almost contact metric submersion of type II. If the total space, M , is cosymplectic, then the fibres are superminimal.*

Proof. If the total space is cosymplectic, we have obviously, that the four calculations SM-5), SM-6), SM-7) and SM-8) vanish. Then the fibres are superminimal. \square

Proposition 6. *Let $f : (M, \phi, \xi, \eta, g) \longrightarrow (N, J', g')$ be an almost contact metric submersion of type II with M , either C_{11} or C_{12} , but not cosymplectic. Then the fibres can not be superminimal.*

Proof. We first note that the base space of an almost contact metric submersion of type II with a C_{11} or C_{12} total space is Kähler. The fundamental 1-form, η , on M vanishes on the horizontal distribution, so the defining relations for a C_{11} or C_{12} - manifold imply that

$$(\nabla_X \Phi)(F, G) = g(F, (\nabla_X \phi)G) = 0.$$

Thus, $\nabla_X \phi = 0$ for all horizontal vector fields, X . Then, if the fibres are taken to be superminimal, we have $\nabla_U \phi = 0$, contradicting the non-cosymplectic nature of the total space, M . \square

5. INTEGRABILITY OF THE HORIZONTAL DISTRIBUTION.

We will see that the superminimality of the fibres plays an important role in the integrability of the horizontal distribution for almost contact metric submersions of both types. Recall that the horizontal distribution of a Riemannian submersion is said to be (*completely*) *integrable* if the O'Neill tensor, A , vanishes identically (i.e., if $A \equiv 0$).

Proposition 7. *Let $f : (M, \phi, \xi, \eta, g) \longrightarrow (N, \phi', \xi', \eta', g')$ be an almost contact metric submersion of type I such that the total space is quasi-K-cosymplectic, (resp. almost cosymplectic; resp., quasi-Sasakian). If the fibres are superminimal, then the horizontal distribution is completely integrable.*

Proof. It is not hard to show that $A_X(\phi Y) = \phi A_X Y$ for the three mentioned almost contact metric submersions. If the fibres are superminimal, we have $g((\nabla_U \phi)X, Y) = -g(A_{\phi X} Y + A_X(\phi Y), U)$ which implies that $A \equiv 0$. \square

Proposition 8. *Let $f : (M^{2m+1}, \phi, \xi, \eta, g) \longrightarrow (N^{2n}, J', g')$ be an almost contact metric submersion of type II with M , almost cosymplectic (resp. closely cosymplectic; resp. nearly cosymplectic; resp. quasi-K-cosymplectic; resp. nearly K-cosymplectic). If the fibres are superminimal, then the horizontal distribution is completely integrable.*

Proof. For each of the manifolds under consideration, it can be quickly shown that $A_X \phi X = A_{\phi X} X$ from which $A \equiv 0$. \square

Proposition 9. [16] *Let $f : (M^{2m+1}, \phi, \xi, \eta, g) \longrightarrow (N^{2n}, J', g')$ be an almost contact metric submersion of type II with M , nearly Kenmotsu. If the fibres are superminimal, then the horizontal distribution is completely integrable.*

Proof. Since η vanishes on horizontal vector fields, the defining relations of a nearly Kenmotsu manifold gives $(\nabla_X \phi)X = 0$. On an almost contact metric submersion of type II with M , nearly Kenmotsu, $(\nabla_X \phi)X = -\eta(X)\phi X$ for any horizontal vector field, X . Therefore, $A_X(\phi X) = 0$. The usual polarization trick implies that $A_X(\phi Y) = A_{\phi X} Y$. Combining this with calculation SM-8) yields $A \equiv 0$. \square

One would think that the superminimality of the fibres of an almost contact metric submersion of type II should be an effective tool in proving the integrability of the horizontal distribution; unfortunately, the following proposition is an obstruction in this matter.

Proposition 10. *Let $f : (M, \phi, \xi, \eta, g) \longrightarrow (N, J', g')$ be an almost contact metric submersion of type II with M , nearly Sasakian. If the fibres are super-minimal, then the horizontal distribution can not be completely integrable.*

Proof. We know from Theorem. 4.12(d) of [17], that

$$A_X(\phi Y) - A_{\phi X}Y = 2g(X, Y)\xi$$

on a nearly Sasakian submersion of type II. The vanishing of calculation SM-8) yields $A_{\phi X}Y = -A_X(\phi Y)$. Hence, $A_X(\phi Y) \equiv g(X, Y)\xi$. If A was zero, then the distinguished vector field, ξ , would vanish which is a contradiction. \square

6. STRUCTURE OF THE TOTAL SPACE.

As in [18], we are able to use the assumed superminimality of the fibres to induce a specific almost contact metric structure onto the total space of an almost contact metric submersion provided that certain necessary structures exist on the base space and the fibres. We begin by proving a technical result.

Lemma 2. *Let $f : (M, \phi, \xi, \eta, g) \longrightarrow (N, \phi', \xi', \eta', g')$ be an almost contact metric submersion of type I. Suppose that $d'\eta' = 0$ on $(N, \phi', \xi', \eta', g')$. If the fibres are superminimal and $A_X(\phi X) = 0$, then $d\eta = 0$ on the total space (M, ϕ, ξ, η, g) .*

Proof. In order to see that $d\eta = 0$, we begin by assuming that X and Y are basic vector fields on the total space, M . Then $d\eta(X, Y) = d'\eta'(X_*, Y_*) = 0$. The vanishing of calculation SM-2) implies, along with $A_X(\phi X) = 0$, that $A \equiv 0$. Now,

$$\begin{aligned} 2d\eta(X, U) &= (\nabla_X \eta)U - (\nabla_U \eta)X \\ &= g(X, \nabla_U \xi) - g(U, \nabla_X \xi) \\ &= g(X, \nabla_U \xi) - g(U, A_X \xi) \\ &= g(X, \nabla_U \xi). \end{aligned}$$

The superminimality of the fibres implies that

$$\begin{aligned} 0 &= g((\nabla_U \phi)\xi, X) \\ &= g(\nabla_U(\phi\xi), X) - g(\phi\nabla_U \xi, X) \\ &= g(\nabla_U \xi, \phi X). \end{aligned}$$

Thus, $\nabla_U \xi$ is g -orthogonal to all horizontal vector fields except, perhaps, ξ . Recall that $\|\xi\|^2 = g(\xi, \xi)$ is constant 1, so that $g(\nabla_U \xi, \xi) = 0$. Hence, $d\eta(X, U) = 0$ and $d\eta(U, X) = 0$.

Recall, too, that the Lie bracket $[U, V]$ is vertical from the complete integrability of the vertical distribution. Then,

$$d\eta(U, V) = \frac{1}{2} \{U\eta(V) - V\eta(U) - \eta([U, V])\}$$

is zero because η vanishes on the vertical distribution. \square

The preceding Lemma applies to the following almost contact metric structures, among others:

- (1) *Closely cosymplectic*,
- (2) *Almost cosymplectic*,
- (3) *Cosymplectic*,
- (4) *Nearly Kenmotsu*.

Theorem 3. *Let $f : (M, \phi, \xi, \eta, g) \longrightarrow (N, \phi', \xi', \eta', g')$ be an almost contact metric submersion of type I. Assume that the base space, N , is nearly cosymplectic (or nearly - K - cosymplectic, or nearly Kenmotsu) and that the fibres are superminimal. If $A_X(\phi X) = 0$, then the total space, M , is respectively nearly cosymplectic (or nearly - K - cosymplectic, or nearly Kenmotsu).*

Proof. There are four calculations that must vanish in order to conclude that the total space, (M, ϕ, ξ, η, g) , is nearly cosymplectic:

$$\text{NC-1) } g((\nabla_U \phi)U, V);$$

$$\text{NC-2) } g((\nabla_U \phi)U, X);$$

$$\text{NC-3) } g((\nabla_X \phi)X, U);$$

$$\text{NC-4) } g((\nabla_X \phi)X, Y).$$

The superminimality of the fibres implies that the first two calculations are 0. We may assume that the horizontal vector fields X and Y are basic for calculation NC-4), in which case that calculation vanishes because the base space is nearly cosymplectic. Finally,

$$g((\nabla_X \phi)X, U) = g(\nabla_X(\phi X), U) - g(\phi \nabla_X X, U) = g(\nabla_X(\phi X), U) = 0$$

yielding the vanishing of calculation NC-3).

Concerning the case of the nearly - K - cosymplectic structure on the base space, we need only establish that $\nabla \eta = 0$ on the total space, M ; that is, we must show that $\nabla_E \xi = 0$ for all vector fields, E , on M . But $\nabla_X \xi = 0$ by projection onto the base space. For $\nabla_U \xi$, we know that $0 = \nabla_U \xi$ by the superminimality of the fibres. Thus,

$$\begin{aligned} 0 &= \nabla_U(\phi \xi) - \phi \nabla_U \xi \\ \text{So, } 0 &= -\phi \nabla_U \xi \\ &= \nabla_U \xi - \eta(\nabla_U \xi)\xi. \end{aligned}$$

But, we established that $\eta(\nabla_U \xi) = g(\nabla_U \xi, \xi) = 0$ during the proof of Lemma 2. Therefore, $\nabla \eta = 0$ and M is nearly- K -cosymplectic.

Now let us consider the case of the nearly Kenmotsu structure. It is known that $d'\eta' = 0$ on the nearly Kenmotsu base space, $(N, \phi', \xi', \eta', g')$. Lemma 2 then implies that $d\eta = 0$ on the total space (M, ϕ, ξ, η, g) . Since η vanishes on

the horizontal distribution, we need only show that $(\nabla_U \phi)U = 0$ and that $0 = (\nabla_X \phi)X + \eta(X)\phi X$. Let X be a basic horizontal vector field. Then,

$$(\nabla_X \phi)X + \eta(X)\phi X = (\nabla'_X \phi')X_* + \eta'(X_*)\phi'X_* = 0.$$

Clearly, $(\nabla_U \phi)U = 0$ because the fibres are superminimal. Therefore, the total space, M , is nearly Kenmotsu. \square

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