

Lie Ideals and Generalized Derivations in Semiprime Rings

Vincenzo De Filippis^a, Nadeem Ur Rehman^{b*}, Abu Zaid Ansari^c

^aDepartment of Mathematics and Computer Science, University of Messina,
 98166 Messina, Italy.

^bDepartment of Mathemaics, Faculty of Science, Taibah University,
 Al-Madinah, Al-Munawara, KSA.

^cDepartment of Mathematics, Faculty of Science, Islamic University in
 Madinah, KSA.

E-mail: defilippis@unime.it

E-mail: rehman100@gmail.com

E-mail: ansari.abuzaid@gmail.com

ABSTRACT. Let R be a 2-torsion free ring and L a Lie ideal of R . An additive mapping $F : R \rightarrow R$ is called a generalized derivation on R if there exists a derivation $d : R \rightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. In the present paper we describe the action of generalized derivations satisfying several conditions on Lie ideals of semiprime rings.

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1. INTRODUCTION

Let R be an associative ring with center $Z(R)$. A ring R is said to be n -torsion free if $nx = 0$ implies $x = 0$ for all $x \in R$. For any $x, y \in R$, the symbol $[x, y]$ will represent the commutator $xy - yx$. Recall that a ring R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$ and R is semiprime if $aRa = 0$ yields $a = 0$. An additive mapping $d : R \rightarrow R$ is said to be a derivation of R if

*Corresponding Author

$d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, for a fixed $a \in R$ the mapping $I_a : R \rightarrow R$ given by $I_a(x) = [x, a]$ is a derivation which is called an inner derivation determined by a . In 1991 Bresar [5] introduced the concept of generalized derivation: more precisely an additive mapping $F : R \rightarrow R$ is said to be a generalized derivation with associated derivation d if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. For the sake of convenience, we shall denote by (F, d) a generalized derivation F with associated derivation d . A mapping $f : R \rightarrow R$ is known to be centralizing on R if $[f(x), x] \in Z(R)$ for all $x \in R$. In particular, if $[f(x), x] = 0$ for all $x \in R$, then f is said to be commuting on R . We recall that an additive group L of R is said to be a Lie ideal of R if $[L, R] \subseteq L$.

A well known result of Posner [18] states that a prime ring admitting a nonzero centralizing derivation must be commutative. This theorem indicates that the global structure of a ring R is often tightly connected to the behaviour of additive mappings defined on R . Following this line of investigation, several authors studied derivations and generalized derivations acting on appropriate subsets of the ring.

For instance in [19] Quadri et al. prove that if R is a prime ring with a non-zero ideal I and F is a generalized derivation of R such that $F([x, y]) = [x, y]$, for all $x, y \in I$, then R is commutative (Theorem 2.1). Later in [7] Dhara extends all results contained in [19] to semiprime rings.

Further in [10] Gölbaşı and Koç investigate the properties of a prime ring R with a generalized derivation (F, d) acting on a Lie ideal L of R . They prove that if $[F(u), u] \in Z(R)$, for all $u \in L$, then either $d = 0$ or $L \subseteq Z(R)$ (Theorem 3.3). Moreover if $F([u, v]) = [u, v]$, for all $u, v \in L$, then either $d = 0$ or $L \subseteq Z(R)$ (Theorem 3.6).

In this note we will consider a similar situation and extend the cited results to the case of semiprime rings with a generalized derivation (F, d) acting on a Lie ideal. More precisely we prove the following:

Theorem 1. *Let R be a 2-torsion free semiprime ring and L be a non-central Lie ideal of R . Suppose that (F, d) is a generalized derivation of R such that $F[x, y] \in Z(R)$, for all $x, y \in L$. If $d(L) \neq (0)$, then all the following hold simultaneously:*

- (1) $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
- (2) $a[L, R] = (0)$ and $[a, L] = (0)$;
- (3) $aI = (0)$ and $d(I) = (0)$ (that is $F(I) = (0)$), where I denotes the ideal of R generated by $[L, L]$.

Theorem 2. *Let R be a 2-torsion free semiprime ring and L be a non-central Lie ideal of R . Suppose R admits a generalized derivation (F, d) , defined as $F(x) = ax + d(x)$, for all $x \in R$ and fixed element $a \in R$. If $[F(x), x] \in Z(R)$ for all $x \in L$ and $d(L) \neq (0)$, then all the following hold simultaneously:*

- (1) $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
- (2) $[a, L] = a[L, R] = (0)$;
- (3) $aI = d(I) = (0)$ (that is $F(I) = (0)$), where I denotes the ideal of R generated by $[L, L]$.

Theorem 3. Let R be a 2-torsion free semiprime ring and L be a non-central Lie ideal of R . Suppose R admits two generalized derivations (F, d) and (G, g) . Write $F(x) = ax + d(x)$ and $G(x) = bx + g(x)$, for some $a, b \in U$. If $F([x, y]) = [y, G(x)]$ for all $x, y \in L$, then either

- (1) $g(L) = (0)$;
- (2) $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
- (3) $(a + b)[L, R] = (0)$, $[b, L] = (0)$ and $[a, L] = (0)$;
- (4) $(a + b)I = (0)$ and $d(I) = (0)$, where I denotes the ideal of R generated by $[L, L]$.

or

- (1) $d(L) = (0)$;
- (2) $g(R)[L, R] = (0)$ and $[g(R), L] = (0)$;
- (3) $[b, L] = (0)$ and $a[L, L] = (0)$;
- (4) $aI = (0)$ and $g(I) = (0)$, where I denotes the ideal of R generated by $[L, L]$.

or

- (1) $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
- (2) $g(R)[L, R] = (0)$ and $[g(R), L] = (0)$;
- (3) $[a, L] = (0)$, $[b, L] = (0)$, $b[L, R] = a[L, R] = (0)$;
- (4) $d(I) = g(I) = (0)$ and $aI = bI = (0)$ (that is $F(I) = G(I) = (0)$), where I denotes the ideal of R generated by $[L, L]$.

In all that follows let R be a non-commutative semiprime ring, L a non-central Lie ideal of R , U the right Utumi quotient ring of R . We refer the reader to [3] for the definition and the related properties of U .

We begin with the following:

Fact 1.1. Let R be a semiprime ring. Then every generalized derivation F of R is uniquely extended to its right Utumi quotient ring U and assumes the form $F(x) = ax + d(x)$, where $a \in U$ and d is the derivation of U associated with F (see Theorem 4 in [17])

Lemma 1.2. Let R be a prime ring of characteristic different from 2 and L be a Lie ideal of R . Suppose R admits a nonzero generalized derivation (F, d) such that $F(x)[x, y] = 0$ (or $[x, y]F(x) = 0$) for all $x, y \in L$, then $L \subseteq Z(R)$.

Proof. Suppose by contradiction that L is not central in R . By [11] (pages 4-5) there exists a non-central ideal I of R such that $0 \neq [I, R] \subseteq L$. By our

assumption it follows that $F(x)[x, y] = 0$ (or $[x, y]F(x) = 0$) for all $x, y \in [I, R]$. Since I and R satisfy the same differential identities (see the main result in [16]), we also have that $F(x)[x, y] = 0$ (or $[x, y]F(x) = 0$) for all $x, y \in [R, R]$. Let $y_0 \in [R, R]$ be such that $y_0 \notin Z(R)$ and denote by $\delta : R \rightarrow R$ the non-zero inner derivation of R induced by the element y_0 . Therefore $F(x)\delta(x) = 0$ (or $\delta(x)F(x) = 0$) for all $x \in [R, R]$. In light of [6], since $\delta \neq 0$ and $[R, R]$ is not central in R , one has the contradiction that $F = 0$. \square

Lemma 1.3. *Let R be a 2-torsion free semiprime ring and L be a non-central Lie ideal of R . Suppose R admits a nonzero generalized derivation (F, d) , defined as $F(x) = ax + d(x)$, for all $x \in R$ and fixed element $a \in R$. If $F(x)[x, y] = 0$ (or $[x, y]F(x) = 0$) for all $x, y \in L$, then all the following hold simultaneously:*

- (1) $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
- (2) $a[L, R] = (0)$ and $[a, L] = (0)$;
- (3) $aI = (0)$ and $d(I) = (0)$ (that is $F(I) = (0)$), where I denotes the ideal of R generated by $[L, L]$.

Proof. Let P be a prime ideal of R such that $[L, L] \not\subseteq P$.

Assume first that $d(P) \subseteq P$. Then F induces a canonical generalized derivation \bar{F} on $\bar{R} = \frac{R}{P}$. Therefore $\bar{F}(\bar{x})[\bar{x}, \bar{y}] = 0$ for all $\bar{x}, \bar{y} \in \bar{L}$. Moreover \bar{L} is a Lie ideal of \bar{R} , such that $[\bar{L}, \bar{L}] \neq 0$ since $[L, L] \not\subseteq P$. By Lemma 1.2 it follows that $\bar{F}(\bar{R}) = \bar{0}$ that is $aR \subseteq P$, $d(R) \subseteq P$ and $F(R) \subseteq P$.

Assume now that $d(P) \not\subseteq P$, then $\overline{d(P)} \neq \bar{0}$ and $\overline{d(P)}\bar{R} \neq \bar{0}$. Moreover note that, for any $p \in P$ and $r, s \in R$, $d(pr)s = d(p)rs + pd(r)s$ implies that $d(P)R \subseteq d(PR)R + P$, in particular $\overline{d(P)}\bar{R}$ is a non-zero right ideal of \bar{R} .

Starting from our main assumption and linearizing we have that $F(x)[z, y] + F(z)[x, y] = 0$, for all $x, y, z \in L$. For any $p \in P, r \in R, u \in L$, replace x by $[pr, u]$. By computation it follows $[\bar{v}, \bar{u}][\bar{z}, \bar{y}] = \bar{0}$, for all $\bar{v} \in \overline{d(P)}\bar{R}$ and $\bar{u}, \bar{z}, \bar{y} \in \bar{L}$. By using the same argument of Lemma 1.2, since \bar{L} is not central in \bar{R} , one has that $\overline{d(P)}\bar{R}$ is a central right ideal of \bar{R} , which implies that \bar{R} is commutative, a contradiction.

Therefore, for any prime ideal P of R , either $aR \subseteq P$, $d(R) \subseteq P$ and $F(R) \subseteq P$ or $[L, L] \subseteq P$. In this last case, by applying Theorem 3 in [15] in the prime ring \bar{R} , since $\text{char}(\bar{R}) \neq 2$ and $[\bar{L}, \bar{L}] = \bar{0}$, we have that \bar{L} is central in \bar{R} , which means $[L, R] \subseteq P$.

Hence in any case it follows that $d(R)[L, R] = (0)$, $a[R, L] = (0)$ and $[d(R), L] = (0)$.

By $a[R, L] = (0)$ we get $aR[R, L] = (0)$ and so both $aLR[a, L] = (0)$ and $LaR[a, L] = (0)$, that is $[a, L]R[a, L] = (0)$. By the semiprimeness of R it follows $[a, L] = (0)$.

Moreover, if $I = R[L, L]R$ denotes the ideal of R generated by $[L, L]$, it follows that $aI = (0)$ and $d(I) = (0)$, that is $F(I) = (0)$. \square

Corollary 1.4. *Let R be a 2-torsion free semiprime ring and L be a non-central Lie ideal of R . Suppose $a \in R$ be such that $ax[x, y] = 0$ for all $x, y \in L$, then $a[L, R] = (0)$, $[a, L] = (0)$ and $aI = (0)$, where I denotes the ideal of R generated by $[L, L]$.*

Theorem 1.5. *Let R be a 2-torsion free semiprime ring and L be a non-central Lie ideal of R . Suppose that (F, d) is a generalized derivation of R such that $F[x, y] \in Z(R)$, for all $x, y \in L$. If $d(L) \neq (0)$, then all the following hold simultaneously:*

- (1) $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
- (2) $a[L, R] = (0)$ and $[a, L] = (0)$;
- (3) $aI = (0)$ and $d(I) = (0)$ (that is $F(I) = (0)$), where I denotes the ideal of R generated by $[L, L]$.

Proof. Assume first that R is prime and denote $V = [L, L]$. Hence we have $F(V) \subseteq Z(R)$. As a consequence of Lemma 2 in [9] we conclude that either $F = 0$ or $V \subseteq Z(R)$. In the first case we have the contradiction $d = 0$, and in the other case one has $L \subseteq Z(R)$ (see Lemma 2 in [12]), a contradiction again. Let now P be a prime ideal of R such that $[L, L] \not\subseteq P$.

Assume first that $d(P) \subseteq P$. Then F induces a canonical generalized derivation \bar{F} on $\bar{R} = \frac{R}{P}$. Therefore $\bar{F}(\bar{x}, \bar{y}) \in Z(\bar{R})$ for all $\bar{x}, \bar{y} \in \bar{L}$. Moreover \bar{L} is a Lie ideal of \bar{R} , such that $[\bar{L}, \bar{L}] \neq 0$ since $[L, L] \not\subseteq P$. By previous argument it follows that $\bar{F}(\bar{R}) = \bar{0}$ that $d(R) \subseteq P$ and $F(R) \subseteq P$.

Assume now that $d(P) \not\subseteq P$, then $\bar{d}(P) \neq \bar{0}$ and $\bar{d}(P)\bar{R} \neq \bar{0}$. We remark again that $\bar{d}(P)\bar{R}$ is a non-zero right ideal of \bar{R} .

Starting from our main assumption and linearizing we have that

$$F(x)y + F(x)z + xd(y) + xd(z) - F(y)x - F(z)x - yd(x) - zd(x) \in Z(R), \quad \forall x, y, z \in L.$$

For any $p, p', p'' \in P, r, s \in R, u, v \in L$, replace y by $[pr, u]$ and z by $[p's, v], p''$. By computation it follows

$$\bar{x}[\bar{t}, \bar{u}] - [\bar{t}, \bar{u}]\bar{x} \in Z(\bar{R})$$

that is

$$\left[\bar{x}, [\bar{t}, \bar{u}] \right] \in Z(\bar{R})$$

for all $\bar{t} \in \bar{d}(P)\bar{R}$ and $\bar{u}, \bar{x} \in \bar{L}$. As above denote $\bar{V} = [\bar{L}, \bar{L}]$, which is a Lie ideal for \bar{R} , and δ is the derivation of \bar{R} induced by \bar{t} . Hence we have $\delta(\bar{V}) \subseteq Z(\bar{R})$. Again as a consequence of Lemma 2 in [9] it follows that either $\delta = 0$ or $\bar{V} \subseteq Z(\bar{R})$. Since \bar{R} is not commutative, then there exists some $\bar{t} \in \bar{R}$ which is not central. Thus $\bar{V} \subseteq Z(\bar{R})$, and $\bar{L} \subseteq Z(\bar{R})$ follows from Lemma 2 in [12].

Therefore, for any prime ideal P of R , either $d(R) \subseteq P$ and $F(R) \subseteq P$ or $[L, L] \subseteq P$. In this last case, by applying Theorem 3 in [15] in the prime ring \bar{R} , since $\text{char}(\bar{R}) \neq 2$ and $[\bar{L}, \bar{L}] = \bar{0}$, we conclude that \bar{L} is central in \bar{R} , which

means $[L, R] \subseteq P$.

Hence in any case it follows that $d(R)[L, R] = (0)$, $a[R, L] = (0)$ and $[d(R), L] = (0)$. Finally we obtain the required conclusions by following the same argument as in Lemma 1.3. \square

In the sequel we will use the following known result:

Lemma 1.6. *Let R be a 2-torsion free semiprime ring, L a Lie ideal of R such that $L \not\subseteq Z(R)$. Let $a \in L$ be such that $aLa = 0$, then $a = 0$.*

Remark 1.7. If R is a prime ring of characteristic different from 2, $a \in R$ and L is a non-central Lie ideal of R such that $[a, L] \subseteq Z(R)$, then $a \in Z(R)$.

Proof. Denote by $\delta : R \rightarrow R$ the inner derivation of R induced by the element $a \in R$. Since $[[a, x], r] = 0$ for all $x \in L$ and $r \in R$, a fortiori we have $[a, x]_2 = 0$, that is $[\delta(x), x] = 0$, for all $x \in L$. Thus, by [14] it follows $\delta = 0$, that is $a \in Z(R)$. \square

Theorem 1.8. *Let R be a 2-torsion free semiprime ring and L be a non-central Lie ideal of R . Suppose R admits a generalized derivation (F, d) , defined as $F(x) = ax + d(x)$, for all $x \in R$ and fixed element $a \in R$. If*

$$[F(x), x] \in Z(R) \text{ for all } x \in L. \quad (1.1)$$

and $d(L) \neq (0)$, then all the following hold simultaneously:

- (1) $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
- (2) $[a, L] = a[L, R] = (0)$;
- (3) $aI = d(I) = (0)$ (that is $F(I) = (0)$), where I denotes the ideal of R generated by $[L, L]$.

Proof. Let P be a prime ideal of R such that $[L, L] \not\subseteq P$.

Assume first that $d(P) \subseteq P$. Then F induces a canonical generalized derivation \overline{F} on $\overline{R} = \frac{R}{P}$. Therefore $[\overline{F}(\overline{x}), \overline{x}] \in Z(\overline{R})$ for all $\overline{x} \in \overline{L}$. Moreover \overline{L} is a Lie ideal of \overline{R} , such that $[\overline{L}, \overline{L}] \neq 0$ since $[L, L] \not\subseteq P$. Since $[L, L] \not\subseteq P$, a fortiori we get \overline{L} is not central in \overline{R} . Therefore, by Theorem 3.3 in [10], it follows that $\overline{d}(\overline{R}) = \overline{0}$ that is $d(R) \subseteq P$.

Assume now that $d(P) \not\subseteq P$, then $\overline{d(P)} \neq \overline{0}$ and $\overline{d(P)}\overline{R} \neq \overline{0}$. By using similar argument as in Lemma 1.3, $R\overline{d(P)}$ is a non-zero right ideal of \overline{R} .

Linearizing (1.1) and using (1.1), we obtain

$$[F(x), y] + [F(y), x] \in Z(R) \text{ for all } x, y \in L. \quad (1.2)$$

Now, replace y by $[rp, u]$, for $r \in R$, $p \in P$ and $u \in L$ and use (1.2) to get

$$[\overline{F}(\overline{[rp, u]}), \overline{x}] \in Z(\overline{R}). \quad (1.3)$$

Moreover, since $F(r) = ar + d(r)$, for all $r \in R$, by (1.3) it follows

$$[\overline{d(\overline{[rp, u]})}, \overline{L}] \subseteq Z(\overline{R}). \quad (1.4)$$

By the primeness of \overline{R} and Remark 1.7, one has that $\overline{d}(\overline{[rp, u]}) \in Z(\overline{R})$. On the other hand, an easy computation shows that $\overline{d}(\overline{[rp, u]}) = \overline{[rd(p), u]}$, which implies $\overline{[Rd(P), L]} \subseteq Z(\overline{R})$. Once again by Remark 1.7, we have $Rd(P) \subseteq Z(\overline{R})$. Since $\overline{Rd(P)}$ is a non-zero right ideal of \overline{R} , it follows $[\overline{R}, \overline{R}] = (0)$, which contradicts with $[\overline{L}, \overline{L}] \neq (0)$.

The previous argument shows that, for any prime ideal P of R , either $[L, L] \subseteq P$ or $d(R) \subseteq P$. Thus $d(R)[L, L] \subseteq \cap P_i = (0)$. Hence, by Lemma 1.3 and since $L \not\subseteq Z(R)$, we finally get the required conclusions:

- (1) $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
- (2) $a[L, R] = (0)$ and $[a, L] = (0)$;
- (3) $aI = d(I) = (0)$, where I denotes the ideal of R generated by $[L, L]$.

□

Theorem 1.9. *Let R be a 2-torsion free semiprime ring and L be a non-central Lie ideal of R . Suppose R admits two generalized derivations (F, d) and (G, g) . Write $F(x) = ax + d(x)$ and $G(x) = bx + g(x)$, for some $a, b \in U$. If $F([x, y]) = [y, G(x)]$ for all $x, y \in L$, then either*

- (1) $g(L) = (0)$;
- (2) $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
- (3) $(a + b)[L, R] = (0)$, $[b, L] = (0)$ and $[a, L] = (0)$;
- (4) $(a + b)I = (0)$ and $d(I) = (0)$, where I denotes the ideal of R generated by $[L, L]$.

or

- (1) $d(L) = (0)$;
- (2) $g(R)[L, R] = (0)$ and $[g(R), L] = (0)$;
- (3) $[b, L] = (0)$ and $a[L, L] = (0)$;
- (4) $aI = (0)$ and $g(I) = (0)$, where I denotes the ideal of R generated by $[L, L]$.

or

- (1) $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
- (2) $g(R)[L, R] = (0)$ and $[g(R), L] = (0)$;
- (3) $[a, L] = (0)$, $[b, L] = (0)$, $b[L, R] = a[L, R] = (0)$;
- (4) $d(I) = g(I) = (0)$ and $aI = bI = (0)$ (that is $F(I) = G(I) = (0)$), where I denotes the ideal of R generated by $[L, L]$.

Proof. Assume first $g(L) = (0)$, then $F([x, y]) = [y, bx]$ for all $x, y \in L$. Thus

$$a[x, y] + d([x, y]) = b[y, x] \quad (1.5)$$

for all $x, y \in L$, that is $(a + b)[x, y] + d([x, y]) = 0$ for all $x, y \in L$. Therefore, applying Theorem 1.5, one has

- (1) $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
- (2) $(a + b)[L, R] = (0)$ and $[a + b, L] = (0)$;

- (3) $(a+b)I = (0)$ and $d(I) = (0)$, where I denotes the ideal of R generated by $[L, L]$.

In particular $d([L, L]) = (0)$ and $a[x, y] = -b[x, y]$ for all $x, y \in L$, so that (1.5) reduces to $(by - yb)x = 0$, for all $x, y \in L$, that is $[b, L]L = (0)$. Hence by Lemma 1.6, we have $[b, L] = (0)$ and so also $[a, L] = (0)$.

Let now $d(L) = (0)$, then $a[x, y] = [y, G(x)]$ for all $x, y \in L$. In this case, for $x = y$, we have $[G(y), y] = 0$ and by Theorem 1.8 the following hold:

- (1) $g(R)[L, R] = (0)$ and $[g(R), L] = (0)$;
- (2) $[b, L] = (0)$, $b[L, R] = (0)$ and $a[L, L] = (0)$;
- (3) $bI = (0)$ and $g(I) = (0)$, where I denotes the ideal of R generated by $[L, L]$.

Moreover, since $[[L, L], R] \subseteq [L, L]$, we also have $0 = a[[L, L], R] = aR[L, L]$, which implies $aI = (0)$.

Assume finally that both $g(L) \neq (0)$ and $d(L) \neq (0)$. Once again for $x = y \in L$ we have $[G(x), x] = 0$ for any $x \in L$. Thus by Theorem 1.8, we have that all the following hold:

- (1) $g(R)[L, R] = (0)$ and $[g(R), L] = (0)$;
- (2) $[b, L] = (0)$ and $b[L, R] = (0)$;
- (3) $bI = (0)$ and $g(I) = (0)$, where I denotes the ideal of R generated by $[L, L]$.

Hence by the main assumption it follows that $(a+b)[x, y] + d([x, y]) = 0$, for all $x, y \in L$. Denote $H(x) = (a-b)x + d(x)$, then $H(u) = 0$ for all $u \in [L, L]$. Finally, by applying Theorem 1.5, one has

- (1) $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
- (2) $(a+b)[L, R] = (0)$ and $[a, L] = (0)$;
- (3) $(a+b)I = (0)$ and $d(I) = (0)$, where I denotes the ideal of R generated by $[L, L]$.

Note that, since both $bI = (0)$ and $(a+b)I = (0)$, we are done. \square

We conclude our paper with some applications to generalized derivations acting on ideals of semiprime rings:

Theorem 1.10. *Let R be a 2-torsion free semiprime ring and I be a non-central ideal of R . Suppose R admits a generalized derivation (F, d) , defined as $F(x) = ax + d(x)$, for all $x \in R$ and fixed element $a \in R$. If $[F(x), x] = 0$ for all $x \in I$, then either $d(I) = 0$ or R contains a non-zero central ideal.*

Proof. By Theorem 1.8, we have that if $d(I) \neq 0$ then $[d(R), I] = (0)$. Hence, by applying Main Theorem in [13], it follows that R must contain a non-zero central ideal. \square

Corollary 1.11. *Let R be a 2-torsion free semiprime ring F a generalized derivation of R . If $[F(x), x] = 0$ for all $x \in R$, then either R contains a*

non-zero central ideal or there exists $\lambda \in Z(R)$ such that $F(x) = \lambda x$, for all $x \in R$.

Theorem 1.12. *Let R be a 2-torsion free semiprime ring and I be a non-central ideal of R . Suppose R admits two generalized derivations (F, d) and (G, g) . Write $F(x) = ax + d(x)$ and $G(x) = bx + g(x)$, for some $a, b \in U$. If $F([x, y]) = [y, G(x)]$ for all $x, y \in I$, then either $d(I) = g(I) = (0)$ or R contains a non-zero central ideal.*

Proof. Assume either $d(I) \neq 0$ or $g(I) \neq 0$. Thus, by Theorem 1.9 respectively we have that either $[d(R), I] = (0)$ or $[g(R), I] = (0)$. In any case, again by [13], R must contain some non-zero central ideals. \square

Corollary 1.13. *Let R be a 2-torsion free semiprime ring and F, G two generalized derivations of R . If $F([x, y]) = [y, G(x)]$ for all $x, y \in R$, then either R contains a non-zero central ideal or there exist $\lambda \in Z(R)$ such that $F(x) = G(x) = \lambda x$, for all $x \in R$.*

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