

Double Integral Characterization for Bergman Spaces

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ABSTRACT. In this paper we characterize Bergman spaces with respect to double integral of the functions $|f(z) - f(w)|/|z - w|$, $|f(z) - f(w)|/\rho(z, w)$ and $|f(z) - f(w)|/\beta(z, w)$, where ρ and β are the pseudo-hyperbolic and hyperbolic metrics. We prove some necessary and sufficient conditions that implies a function to be in Bergman spaces.

Keywords: Bergman spaces, Pseudo-hyperbolic metric, Hyperbolic metric, Double integral.

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1. INTRODUCTION

For $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in $\mathbb{C}^n$, we define $\langle z, w \rangle = z_1 \overline{w_1} + \dots + z_n \overline{w_n}$, where $\overline{w_k}$ is the complex conjugate of w_k . We also write $|z| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \dots + |z_n|^2}$. Let $\mathbb{B}_n$ denotes the open unit ball of $\mathbb{C}^n$, that is

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}.$$

For any $a \in \mathbb{B}_n - \{0\}$, we define

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a \rangle} \quad z \in \mathbb{B}_n,$$

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where $s_a = \sqrt{1 - |a|^2}$, P_a is the orthogonal projection from $\mathbb{C}^n$ onto the subspace $[a]$ generated by a , and Q_a is the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^n - [a]$. When $a = 0$, we define $\varphi_a(z) = -z$. These functions are called involutions. (see [9] for more information about these functions)

The hyperbolic metric (Bergman metric) is defined by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbb{B}_n.$$

For any $z \in \mathbb{B}_n$ and $r > 0$, we denote Bergman metric ball at z by $D(z, r)$. That is

$$D(z, r) = \{w \in \mathbb{B}_n : \beta(z, w) < r\}.$$

Also, pseudo-hyperbolic metric is defined by $\rho(z, w) = |\varphi_z(w)|$.

For $\alpha > -1$ let

$$dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dv(z)$$

where $dv(z)$ is the Lebesgue volume measure on $\mathbb{B}_n$ and c_α is a positive constant with $v_\alpha(\mathbb{B}_n) = 1$. For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A_α^p consists of all holomorphic functions in $L^p(\mathbb{B}_n, dv_\alpha)$, that is

$$A_\alpha^p = \left\{ f \in H(\mathbb{B}_n) : \|f\|_{\alpha, p}^p = \int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) < \infty \right\}$$

Wulan and Zhu [8], characterized Bergman spaces with standard weights in terms of Lipschitz type conditions in the Euclidean, hyperbolic, and pseudo-hyperbolic metrics. In [4] Li et al. proved that a holomorphic function f belongs to the A_α^p , $p > n + 1 + \alpha$, if and only if the function $|f(z) - f(w)|/|1 - \langle z, w \rangle|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\gamma \times dv_\gamma)$, where $\gamma = (p + \alpha - n - 1)/2$.

Also, it was shown in [5] that for the case $0 < p < n + 1 + \alpha$, $f \in A_\alpha^p$ if and only if the function $|f(z) - f(w)|/|1 - \langle z, w \rangle|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\alpha)$ if and only if the function $|f(z) - f(w)|/|z - w|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\alpha)$.

Our aim in this paper is to prove, for $f \in A_\alpha^p$, $p > n + 1 + \alpha$, the function $|f(z) - f(w)|/|z - w|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_t \times dv_t)$, where $t = (p + \alpha - n - 1)/2$ and if $p = n + 1 + \alpha$, then $|f(z) - f(w)|/|z - w|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\gamma \times dv_\gamma)$, for any $\gamma > \alpha$. Our results are applicable for studying the action of symmetric lifting operator on A_α^p in all cases especially for the case $p = \alpha + 2$.

Also we replace the Euclidean metric with pseudo-hyperbolic metric ρ and Bergman metric β .

2. PRELIMINARIES

Lemma 2.1. [9] *There exists a positive constant C such that*

$$|f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z, r)} |f(w)|^p dv_\alpha(w)$$

for all $f \in H(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$.

Lemma 2.2. [9] Suppose $s > -1$, t is real, and

$$I(z) = \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^s}{|1 - \langle z, w \rangle|^{n+1+s+t}} dv(w), \quad z \in \mathbb{B}_n.$$

Then $I(z)$ is bounded in $\mathbb{B}_n$ whenever $t < 0$, and $I(z)$ is bounded by $(1 - |z|^2)^{-t}$ whenever $t > 0$.

Theorem 2.3. [8] Suppose that $p > 0$, $\alpha > -1$ and f is analytic in $\mathbb{B}_n$. Then the following conditions are equivalent.

- (1) $f \in A_\alpha^p$.
- (2) There exists a continuous function g in $L^p(\mathbb{B}_n, dv_\alpha)$ such that

$$|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w)), \quad z, w \in \mathbb{B}_n.$$

- (3) There exists a continuous function g in $L^p(\mathbb{B}_n, dv_\alpha)$ such that

$$|f(z) - f(w)| \leq \beta(z, w)(g(z) + g(w)), \quad z, w \in \mathbb{B}_n.$$

- (4) There exists a continuous function g in $L^p(\mathbb{B}_n, dv_{p+\alpha})$ such that

$$|f(z) - f(w)| \leq |z - w|(g(z) + g(w)), \quad z, w \in \mathbb{B}_n.$$

Lemma 2.4. [4] Let $r > 0$. Then

$$1 - |z|^2 \sim 1 - |w|^2 \sim |1 - \langle z, w \rangle|$$

for all $z \in \mathbb{B}_n$ and $w \in D(z, r)$. Furthermore, there exists a positive constant C such that

$$(1 - |z|^2)^p |\nabla f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1}} \int_{D(z, r)} |f(w) - f(z)|^p dv(w)$$

for all $z \in \mathbb{B}_n$ and $f \in H(\mathbb{B}_n)$.

3. PSEUDO-HYPERBOLIC METRIC

Lemma 3.1. Suppose $\alpha > -1$ and $f \in H(\mathbb{B}_n)$. Then there exists a positive constant C such that

$$\int_{\mathbb{B}_n} |f(z) - f(0)|^p dv_\alpha(z) \leq C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} dv_\alpha(z) dv_\alpha(w).$$

Proof. Let

$$J(f) = \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} dv_\alpha(z) dv_\alpha(w).$$

By making a change of variable, we have

$$\begin{aligned}
 J(f) &= \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} \frac{|f(z) - f(\varphi_z(w))|^p}{\rho(z, \varphi_z(w))^p} \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} dv_\alpha(w) \\
 &= \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} \frac{|f(z) - f(\varphi_z(w))|^p}{|w|^p} \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} dv_\alpha(w) \\
 &\geq \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} |f(z) - f(\varphi_z(w))|^p \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} dv_\alpha(w) \\
 &\geq \int_{\mathbb{B}_n} dv_\alpha(z) \int_{D(z,r)} |f(z) - f(\varphi_z(w))|^p \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} dv_\alpha(w).
 \end{aligned}$$

From the first part of Lemma 2.4, there exists a positive constant C' such that

$$J(f) \geq C' \int_{\mathbb{B}_n} dv_\alpha(z) \int_{D(z,r)} \frac{|f(z) - f(\varphi_z(w))|^p}{(1 - |z|^2)^{n+1+\alpha}} dv_\alpha(w).$$

Then Lemma 2.1 implies that there exists another positive constant C such that

$$J(f) \geq C \int_{\mathbb{B}_n} |f(z) - f(\varphi_z(z))|^p dv_\alpha(z) = C \int_{\mathbb{B}_n} |f(z) - f(0)|^p dv_\alpha(z).$$

The proof is complete. $\square$

Lemma 3.2. Suppose $\alpha > -1$ and $f \in A_\alpha^p$. Then

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} dv_\alpha(z) dv_\alpha(w) < \infty.$$

Proof. Given $f \in A_\alpha^p$, from Theorem 2.3, there exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that for all $z, w \in \mathbb{B}_n$,

$$|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w)).$$

There exists a positive constant C such that

$$\frac{|f(z) - f(w)|^p}{\rho(z, w)^p} \leq C(g(z)^p + g(w)^p).$$

So,

$$\begin{aligned}
 \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(w) - f(z)|^p}{\rho(z, w)^p} dv_\alpha(z) dv_\alpha(w) &\leq C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} (g(z)^p + g(w)^p) dv_\alpha(z) dv_\alpha(w) \\
 &= 2C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} g(z)^p dv_\alpha(z) dv_\alpha(w) < \infty.
 \end{aligned}$$

$\square$

We can combine these two lemmas and obtain the following theorem.

Theorem 3.3. Suppose that $\alpha > -1$. Then $f \in A_\alpha^p$ if and only if

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} dv_\alpha(z) dv_\alpha(w) < \infty.$$

4. BERGMAN METRIC

Now, we replace metric ρ by Bergman metric β .

Lemma 4.1. Suppose that $\alpha > -1$ and $f \in H(\mathbb{B}_n)$. If

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z, w)^p} dv_\alpha(z) d\tau(w) < \infty,$$

then $f \in A_\alpha^p$, where

$$d\tau(w) = \frac{dv(w)}{(1 - |w|^2)^{n+1}}$$

is the Mobius invariant volume measure on $\mathbb{B}_n$.

Proof. By Lemma 2.4, there exists a positive constant C such that

$$\begin{aligned} (1 - |z|^2)^p |\nabla f(z)|^p &\leq \frac{C}{(1 - |z|^2)^{n+1}} \int_{D(z, r)} |f(z) - f(w)|^p dv(w) \\ &\leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z, r)} |f(z) - f(w)|^p dv_\alpha(w). \end{aligned}$$

Since $D(z, r)$ is open unit ball in metric β , we have

$$(1 - |z|^2)^p |\nabla f(z)|^p \leq \frac{Cr^p}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z, r)} \frac{|f(z) - f(w)|^p}{\beta(z, w)^p} dv_\alpha(w).$$

After integrating

$$\begin{aligned} \int_{\mathbb{B}_n} (1 - |z|^2)^p |\nabla f(z)|^p dv_\alpha(z) &\leq Cr^p \int_{\mathbb{B}_n} \int_{D(z, r)} \frac{|f(z) - f(w)|^p}{\beta(z, w)^p} dv_\alpha(w) d\tau(z) \\ &\leq Cr^p \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z, w)^p} dv_\alpha(w) d\tau(z). \end{aligned}$$

Therefore $(1 - |z|^2) \nabla f(z) \in A_\alpha^p$. It follows from Theorem 2.16 of [9] that $f \in A_\alpha^p$. $\square$

By the same reason as in Lemma 3.2, we can prove the following lemma.

Lemma 4.2. Suppose $\alpha > -1$ and $f \in H(\mathbb{B}_n)$. If $f \in A_\alpha^p$, then

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z, w)^p} dv_\alpha(z) dv_\alpha(w) < \infty.$$

5. EUCLIDEAN METRIC

Theorem 5.1. Suppose $\alpha > -1$, $p = n + 1 + \alpha$ and $f \in A_\alpha^p$, then

$$I(f) = \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|z - w|^p} dv_\gamma(z) dv_\gamma(w) < \infty,$$

for any $\gamma > \alpha$.

Proof. Given $f \in A_\alpha^p$, from Theorem 2.3, there exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that for all $z, w \in \mathbb{B}_n$,

$$|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w)) \leq \frac{|z - w|}{|1 - \langle z, w \rangle|} (g(z) + g(w)).$$

There exists a positive constant C such that

$$\begin{aligned} I(f) &\leq 2C \int_{\mathbb{B}_n} g(z)^p dv_\gamma(z) \int_{\mathbb{B}_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^p} \\ &= 2C \int_{\mathbb{B}_n} g(z)^p dv_\gamma(z) \int_{\mathbb{B}_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha}}. \end{aligned}$$

Since $\alpha - \gamma < 0$, by Lemma 2.2, the last integral is bounded. Then there exists another positive constant M such that

$$\begin{aligned} I(f) &\leq M \int_{\mathbb{B}_n} g(z)^p dv_\gamma(z) \\ &= M c_\gamma \int_{\mathbb{B}_n} g(z)^p (1 - |z|^2)^{\gamma-\alpha} (1 - |z|^2)^\alpha dv(z) \\ &< M \frac{c_\gamma}{c_\alpha} \int_{\mathbb{B}_n} g(z)^p dv_\alpha(z) < \infty. \end{aligned}$$

□

Lemma 5.2. Suppose $\alpha > -1$, $f \in H(\mathbb{B}_n)$ and δ and γ are real parameters such that

$$\delta + \gamma = p + \alpha - (n + 1), \quad -1 < \gamma < p - (n + 1).$$

If $f \in A_\alpha^p$, then

$$I(f) = \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|z - w|^p} dv_\delta(z) dv_\gamma(w) < \infty.$$

Proof. By the proof of the previous lemma, there exists a positive constant C such that

$$\begin{aligned} I(f) &\leq 2C \int_{\mathbb{B}_n} g(z)^p dv_\delta(z) \int_{\mathbb{B}_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^p} \\ &= 2C \int_{\mathbb{B}_n} g(z)^p dv_\delta(z) \int_{\mathbb{B}_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^{n+1+\gamma+(\delta-\alpha)}}. \end{aligned}$$

Since $\delta - \alpha > 0$, by Lemma 2.2, there exists another positive constant M such that

$$I(f) \leq M \int_{\mathbb{B}_n} \frac{g(z)^p}{(1 - |z|^2)^{\delta - \alpha}} dv_\delta(z) = M \int_{\mathbb{B}_n} g(z)^p dv_\alpha(z) < \infty.$$

□

Corollary 5.3. *Suppose that $\alpha > -1$, $p > n + 1 + \alpha$ and $f \in A_\alpha^p$, then*

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|z - w|^p} dv_t(z) dv_t(w) < \infty,$$

where $t = \frac{p + \alpha - (n+1)}{2}$.

If $n = 1$, then we obtain the following corollary.

Corollary 5.4. *Suppose that $\alpha > -1$, $p > \alpha + 2$ and $f \in A_\alpha^p(\mathbb{D})$, then*

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|z - w|^p} dA_t(z) dA_t(w) < \infty,$$

where $t = \frac{p + \alpha - 2}{2}$.

The symmetric lifting operator $L : H(\mathbb{D}) \rightarrow H(\mathbb{D} \times \mathbb{D})$ is defined by

$$L(f)(z, w) = \frac{f(z) - f(w)}{z - w}.$$

The action of symmetric lifting operator on $A_\alpha^p(\mathbb{D})$ in the cases $p > \alpha + 2$ and $p < \alpha + 2$ was studied in [8]. In the case $p = \alpha + 2$, we have the following result.

Corollary 5.5. *Suppose that $\alpha > -1$, $p = \alpha + 2$. Then the symmetric lifting operator maps $A_\alpha^p(\mathbb{D})$ into $A_\gamma^p(\mathbb{D}^2)$, for any $\gamma > \alpha$.*

Proof. The result follows by letting $n = 1$ in Theorem 5.1. □

If $\alpha > -1$, $p > \alpha + 2$ and $f \in A_\alpha^p(\mathbb{D})$, then by Corollary 5.4, $L(f) \in A_t^p(\mathbb{D}^2)$, which means that the symmetric lifting operator maps $f \in A_\alpha^p(\mathbb{D})$ into $A_t^p(\mathbb{D}^2)$, for $t = \frac{p + \alpha - 2}{2}$. This is the Theorem 4.4 in [8].

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