Common Fixed Points and Invariant Approximations for C_q-commuting Generalized nonexpansive mappings

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ABSTRACT. Some common fixed point theorems for C_q-commuting generalized nonexpansive mappings have been proved in metric spaces. As applications, invariant approximation results are also obtained. The results proved in the paper extend and generalize several known results including those of M. Abbas and J.K. Kim [Bull. Korean Math. Soc. 44(2007) 537-545], I. Beg, N. Shahzad and M. Iqbal [Approx. Theory Appl. 8(1992) 97-105], W.G. Dotson [J. London Math. Soc. 4(1972) 408-410; Proc. Amer. Math. Soc. 38(1973) 155-156.], M. D. Guay, K. L. Singh and J. H. M. Whitfield [Proc. Conference on nonlinear analysis (Ed. S.P.Singh and J. H. Bury) Marcel Dekker 80(1982) 179-189], L. Habiniak [J. Approx. Theory 56(1989) 241-244], N. Hussain and B.E. Rhoades [Fixed Point Theory Appl. Vol. 2006, Article ID 24543, 1-9], T.D. Narang and S. Chandok [Indian J. Math. 51(2009) 293-303; Ukrainian Math. J. 62 (2010) 1367-1376], N. Shahzad [Fixed point Theory Appl. 1(2005) 79-86] and of Y. Song [Comm. Math. Anal. 2(2007) 17-26].

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1. Introduction and Preliminaries

Fixed point theory is one of the famous and traditional theories in mathematics and has a large number of applications. Fixed point and common fixed point theorems for different types of mappings have been investigated extensively by various researchers (see e.g. [1]-[23] and references cited therein). In this paper, we obtain some new common fixed point theorems for C_q-commuting generalized nonexpansive mappings in the setting of metric spaces, and as applications various best approximation results for such mappings are obtained. The proved results generalize and extend several known results including those of Abbas and Kim [3], Beg et al. [4], Dotson [8],[9], Guay et al. [10], Habiniak [11], Hussain and Rhoades [12], Narang and Chandok [16] [17], Shahzad [21] and Song [22].

First, we recall some basic definitions and related results.

For a metric space (X, d), a continuous mapping $W : X \times X \times [0, 1] \to X$ is said to be (s.t.b.) a **convex structure** on X if for all $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y)$$

holds for all $u \in X$. The metric space (X, d) together with a convex structure is called a **convex metric space**[23].

A subset K of a convex metric space (X, d) is s.t.b. **convex** [23] if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$. The set K is said to be p-**starshaped** [10] if there exists some $p \in K$ such that $W(x, p, \lambda) \in K$ for all $x \in K$ and $\lambda \in [0, 1]$ i.e. the segment $[p, x] = \{W(x, p, \lambda) : \lambda \in [0, 1]\}$ joining p to x is contained in K for all $x \in K$.

Clearly, each convex set is starshaped but converse is not true.

A convex metric space (X, d) is said to satisfy **Property (I)** [10] if for all $x, y, q \in X$ and $\lambda \in [0, 1]$,

$$d(W(x, q, \lambda), W(y, q, \lambda)) \leq \lambda d(x, y)$$

holds.

A normed linear space X and each of its convex subsets are simple examples of convex metric spaces with W given by $W(x, y, \lambda) = \lambda x + (1 - \lambda) y$, $0 \leq \lambda \leq 1$. There are many convex metric spaces which are not normed linear spaces (see e.g. [10], [23]). Property (I) is always satisfied in a normed linear space.

For a non-empty subset M of a metric space (X, d) and $x \in X$, an element $y \in M$ is s.t.b. a **best approximation** of x to M or a **best M-approximant** if $d(x, y) = dist(x, M) \equiv \inf\{d(x, y) : y \in M\}$. The set of all such $y \in M$ is denoted by $P_M(x)$.

For a convex subset M of a convex metric space (X, d), a mapping $g : M \to X$ is s.t.b. **affine** if for all $x, y \in M$ and $\lambda \in [0, 1]$, $g(W(x, y, \lambda)) = W(gx, gy, \lambda)$. The mapping g is s.t.b. **affine with respect to** $p \in M$ if $g(W(x, p, \lambda)) = W(gx, gp, \lambda)$ for all $x \in M$ and $\lambda \in [0, 1]$.

Suppose (X, d) is a metric space, M is a non-empty subset of X, and S,T are self mappings of M. Then T is s.t.b. an **S-contraction** on M if there exists a $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(Sx, Sy)$, (*S*-**nonexpansive** if $d(Tx, Ty) \leq d(Sx, Sy)$) for all $x, y \in M$.

T is s.t.b. **demicompact** if every bounded sequence $\{x_n\}$ of points of M satisfying $d(x_n, Tx_n) \to 0$ has a convergent subsequence. T is s.t.b. **hemicompact** (see [12]) if every sequence $\{x_n\}$ in M satisfying $d(x_n, Tx_n) \to 0$ has a convergent subsequence.

A point $x \in M$ is a **common fixed point** (respectively, **coincidence point**) of S and T if $x = Sx = Tx$ $(Sx = Tx)$. The set of fixed points (respectively, coincidence points) of S and T is denoted by $F(S,T)$ (respectively, $C(S,T)$).

The pair (S,T) is s.t.b.

(a) **commuting** on M if $STx = TSx$ for all $x \in M$,

(b) R-**weakly commuting** [19] on M if there exists an $R > 0$ such that $d(TSx, STx) \leq R\, d(Tx, Sx)$ for all $x \in M$,

(c) **compatible** [13] if $\lim d(TSx_n, STx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim Tx_n = \lim Sx_n = t$ for some t in M,

(d) **weakly compatible** [14] if S and T commute at their coincidence points, i.e., if $STx = TSx$ whenever $Sx = Tx$.

Suppose (X,d) is a convex metric space and M a q-starshaped set with $q \in F(S) \cap M$ and is both T- and S- invariant. Then T and S are called

(e) R-**subweakly commuting** [20] on M if for all $x \in M$, there exists a real number $R > 0$ such that $d(TSx, STx) \leq R\, dist(Sx, [q, Tx])$ for all $x \in M$

(f) C_q-**commuting** [2] if $STx = TSx$ for all $x \in C_q(S,T)$, where $C_q(S,T) = \cup\{C(S, T_k) : 0 \leq k \leq 1\}$ and $T_k x = \{W(Tx, q, k) : 0 < k < 1\}$.

Clearly, C_q-commuting mappings are weakly compatible. However, converse is not true.

Example 1.1. [2] *Let $X = \mathbb{R}$ be endowed with the usual metric and $M = [0, \infty)$. Define $T, S : M \to M$ by $Tx = x^2$ for all $x \neq 2$ and $T2 = 1$; and $Sx = 2x$ for all $x \in M$. Then M is q-starshaped with $q = 0$, $C(T,S) = \{0\}$ and $C_q(T,S) = \{0\} \cup [2, \infty)$. Moreover, T and S are weakly compatible but not C_q-commuting.*

R-subweakly commuting mappings are C_q-commuting but the converse is not true.

Example 1.2. [2] *Let $X = \mathbb{R}$ be endowed with the usual metric and $M = [0, \infty)$. Define $T, S : M \to M$ by $Tx = \frac{1}{2}$ if $0 \leq x < 1$ and $Tx = x^2$ if $x \geq 1$; and $Sx = \frac{x}{2}$ if $0 \leq x < 1$ and $Sx = x$ if $x \geq 1$. Then M is q-starshaped with $q = 1$, and $C_q(T,S) = [1, \infty)$. Moreover S and T are C_q-commuting but not R-weakly commuting for all $R > 0$. Hence, T and S are not R-subweakly commuting mapping.*

Let M be a subset of a metric space (X,d) and $\mathfrak{F} = \{f_\alpha : \alpha \in M\}$ a family of functions from $[0,1]$ into M, having the property $f_\alpha(1) = \alpha$, for each $\alpha \in M$. Such a family $\mathfrak{F}$ is said to be **contractive** if there exists a function $\phi : (0,1) \to (0,1)$ such that for all $\alpha, \beta \in M$ and for all $t \in (0,1)$, we have

$$d(f_\alpha(t), f_\beta(t)) \leq \phi(t) d(\alpha, \beta).$$

Such a family $\mathfrak{F}$ is said to be **jointly continuous** if $t \to t_\circ$ in $[0,1]$ and $\alpha \to \alpha_\circ$ in M imply that $f_\alpha(t) \to f_{\alpha_\circ}(t_\circ)$ is in M.

These notions were discussed in normed linear spaces by Dotson [9] and it was observed that if M is a starshaped subset (of a normed linear space) with a star-center p then the family $\mathfrak{F} = \{f_\alpha : \alpha \in M\}$ defined by $f_\alpha(t) = (1-t)p+t\alpha$ is contractive if we take $\phi(t) = t$ for $0 < t < 1$, and is jointly continuous. It can be easily seen that the same is true for starshaped subsets of convex metric spaces with Property (I), by taking $f_\alpha(t) = W(\alpha, p, t)$ and therefore the class of subsets of X with the property of contractiveness and joint continuity contains the class of starshaped sets which in turn contains the class of convex sets. Also if S is an affine self mapping of M and $S(q) = q$, then $S(f_x(t)) = S((1-t)q + tx) = (1-t)S(q) + tS(x) = (1-t)q + tS(x) = f_{S(x)}(t)$ for all $x \in M$ and $t \in [0,1]$.

We extend the concept of C_q-commuting mappings to metric spaces having nonstarshaped domain in the following way:

If S,T are self mappings on a metric space (X, d), M a subset of X having a contractive jointly continuous family $\mathfrak{F} = \{f_\alpha : \alpha \in M\}$ with $q = f_{Tx}(0) \in F(S)$. Then T and S are s.t.b.

i) C_q-**commuting** if $STx = TSx$ for all $x \in C_q(S,T)$, where $C_q(S,T) = \cup\{C(S, T_k) : 0 \leq k \leq 1\}$ and $T_k x = \{f_{Tx}(k) : 0 < k < 1\}$;

ii) R-**subweakly commuting** on M if for all $x \in M$, there exists a real number $R > 0$ such that $d(TSx, STx) \leq R \ dist(Sx, Y_q^{Tx})$, where $Y_q^{Tx} = \{f_{Tx}(\lambda) : 0 < \lambda < 1\}$.

2. Common fixed points of C_q-commuting mappings

In this section we discuss the existence of common fixed points for a pair of C_q-commuting mappings in the framework of metric spaces.

The following result of Song [22] will be used in the sequel.

Lemma 2.1. *Let M be a subset of a metric space (X, d), and T, f and g are self mappings of M with cl $(T(M)) \subseteq f(M) \cap g(M)$. Suppose that cl $(T(M))$ is complete, and T, f and g satisfy:*

$$d(Tx, Ty) \leq h \max\{d(fx, gy), d(fx, Tx), d(gy, Ty), \frac{1}{2}[d(fx, Ty) + d(gy, Tx)]\}.$$

for all $x, y \in M$ and $h \in [0,1)$. If the pairs (T, f) and (T, g) are weakly compatible, then $F(T) \cap F(f) \cap F(g)$ is a singleton.

Theorem 2.2. *Let M be a complete subset of a metric space (X, d), and T, h and g be self mappings on M. Suppose that M has a jointly continuous contractive family $\mathfrak{F}$ with $hf_x(k) = f_{hx}(k)$ and $gf_x(k) = f_{gx}(k)$ for all $x \in M$ and $k \in (0,1)$, and $clT(M) \subset h(M) \cap g(M)$. If h, g are continuous on M, the pairs (T, h) and (T, g) are C_q-commuting and T satisfies,*

$$\begin{aligned} d(Tx, Ty) \quad \leq \quad & \max\{d(hx, gy), dist(hx, Y_{f_{Tx}(0)}^{Tx}), dist(gy, Y_{f_{Ty}(0)}^{Ty}), \\ & \tfrac{1}{2}[dist(hx, Y_{f_{Ty}(0)}^{Ty}) + dist(gy, Y_{f_{Tx}(0)}^{Tx})]\}, \end{aligned}$$

for all $x, y \in M$, then T, h and g have a common fixed point in M, provided one of the following conditions hold:

i) $clT(M)$ is compact and T is continuous;

ii) M is compact and T is continuous.

Proof. For each $n \geq 1$, define $T_n : M \to M$ by $T_n x = f_{Tx}(\lambda_n)$, $x \in M$ where $\langle \lambda_n \rangle$ is a sequence in $(0, 1)$ such that $\lambda_n \to 1$. Since $cl(T(M)) \subseteq h(M) \cap g(M)$, $cl\ (T_n(M)) \subseteq h(M) \cap g(M)$ for each n. As h and T are C_q-commuting and $hf_x(k) = f_{hx}(k)$, for each $x \in C_q(h, T)$ we have,

$$hT_n x = hf_{Tx}(\lambda_n) = f_{hTx}(\lambda_n) = f_{Thx}(\lambda_n) = T_n hx.$$

Thus $hT_n x = T_n hx$ for each $x \in C_q(h, T_n) \subset C_q(h, T)$. Hence h and T_n are weakly compatible for all n. Similarly, one can prove that g and T_n are weakly compatible for all n. Also

$$
\begin{aligned}
d(T_n x, T_n y) &= d(f_{Tx}(\lambda_n), f_{Ty}(\lambda_n)) \\
&\leq \phi(\lambda_n) d(Tx, Ty) \\
&\leq \phi(\lambda_n) \max\{d(hx, gy), dist(hx, Y^{Tx}_{f_{Tx}(0)}), dist(gy, Y^{Ty}_{f_{Ty}(0)}), \\
&\qquad \tfrac{1}{2}[dist(hx, Y^{Ty}_{f_{Ty}(0)}) + dist(gy, Y^{Tx}_{f_{Tx}(0)})]\}, \\
&\leq \phi(\lambda_n) \max\{d(hx, gy), d(hx, T_n x), d(gy, T_n y), \\
&\qquad \tfrac{1}{2}[d(hx, T_n y) + d(gy, T_n x)]\}
\end{aligned}
$$

for all $x, y \in M$.

(i) By Lemma 2.1, there exists some $x_n \in M$ such that x_n is a common fixed point of h, g and T_n for each $n \geq 1$. The compactness of $cl\ (T(M))$ implies that there exists a subsequence $\{Tx_{n_i}\}$ of $\{Tx_n\}$ such that $Tx_{n_i} \to y \in M$. Since $x_{n_i} = T_{n_i} x_{n_i} = f_{Tx_{n_i}}(\lambda_{n_i}) \to f_y(1) = y$, by the continuity of T, h and g, we have $y \in F(T) \cap F(h) \cap F(g)$.

(ii) The result follows from (i) as T is continuous.

$\square$

Corollary 2.3. *Let M be a complete q-starshaped subset of a convex metric space (X, d) with Property (I), and T,f and g are continuous self mappings on X. Suppose that $cl\ (T(M))$ is compact, f and g are affine on M, $q \in F(f) \cap F(g)$ and $T(M) \subset f(M) \cap g(M)$. If the pairs (T, f) and (T, g) are C_q-commuting and T satisfies,*

$$
\begin{aligned}
d(Tx, Ty) &\leq \max\{d(fx, gy), dist(fx, [q, Tx]), dist(gy, [q, Ty]), \\
&\qquad \tfrac{1}{2}[dist(fx, [q, Ty]) + dist(gy, [q, Tx])]\},
\end{aligned}
$$

for all $x, y \in M$, then T, f and g have a common fixed point in M.

Corollary 2.4. *([3]-Theorem 2.1) Let M be a complete q-starshaped subset of a p-normed space X, and T,f and g are continuous self mappings on X. Suppose that $cl\ (T(M))$ is compact, f and g are affine on M, $q \in F(f) \cap F(g)$ and $T(M) \subset f(M) \cap g(M)$. If the pairs (T, f) and (T, g) are C_q-commuting and T satisfies for all $x, y \in M$,*

$$
\begin{aligned}
\|Tx - Ty\|_p &\leq \max\{\|fx - gy\|_p, dist(fx, [Tx, q]), dist(gy, [Ty, q]), \\
&\qquad dist(fx, [Ty, q]), dist(gy, [Tx, q])\},
\end{aligned}
$$

then T, f and g have a common fixed point in M.

Corollary 2.5. *([22]-Theorem 2.4) Let M be a complete q-starshaped subset of a normed space X, and T,f and g be continuous self mappings on X. Suppose that cl $(T(M))$ is compact subset of $f(M) \cap g(M)$, f and g are are affine on M, $q \in F(f) \cap F(g)$. If the pairs (T, f) and (T, g) are C_q-commuting and T satisfies for all $x, y \in M$,*

$$\|Tx - Ty\| \quad \leq \quad \max\{\|fx - gy\|, dist(fx, [Tx, q]), dist(gy, [Ty, q]),$$
$$\tfrac{1}{2}[dist(fx, [Ty, q]), dist(gy, [Tx, q])]\},$$

then T, f and g have a common fixed point in M.

As R-subweakly commuting mappings are C_q-commuting, we have the following result.

Corollary 2.6. *Let M be a complete q-starshaped subset of a convex metric space (X, d) with Property (I), and T,f and g are continuous self mappings on X. Suppose that cl $(T(M))$ is compact, f and g are affine on M, $q \in F(f) \cap F(g)$ and $T(M) \subset f(M) \cap g(M)$. If the pairs (T, f) and (T, g) are R-subweakly commuting and T satisfies,*

$$d(Tx, Ty) \quad \leq \quad \max\{d(fx, gy), dist(fx, [q, Tx]), dist(gy, [q, Ty]),$$
$$\tfrac{1}{2}[dist(fx, [q, Ty]) + dist(gy, [q, Tx])]\},$$

for all $x, y \in M$, $\lambda \in [0, 1)$, then T, f and g have a common fixed point in M.

Corollary 2.7. *([3]-Corollary 2.2) Let M be a complete q-starshaped subset of a p-normed space X, and T,f and g are continuous self mappings on X. Suppose that cl $(T(M))$ is compact, f and g are affine on M, $q \in F(f) \cap F(g)$ and $T(M) \subset f(M) \cap g(M)$. If the pairs (T, f) and (T, g) are R-subweakly commuting and T satisfies,*

$$\|Tx - Ty\|_p \quad \leq \quad \max\{\|fx - gy\|_p, dist(fx, [Tx, q]), dist(gy, [Ty, q]),$$
$$dist(fx, [Ty, q]), dist(gy, [Tx, q])\},$$

for all $x, y \in M$, then T, f and g have a common fixed point in M.

Following result of Hussain and Rhoades ([12]-Theorem 2.1) will be used in the sequel.

Lemma 2.8. *Let M be a subset of a metric space (X, d), and S,T are weakly compatible self maps of M. Assume that cl $(T(M)) \subset S(M)$, cl $(T(M))$ is complete, and T and S satisfy for all $x, y \in M$ and $h \in [0, 1)$,*

$$d(Tx, Ty) \leq h \max\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\}.$$

Then $F(S) \cap F(T)$ is a singleton.

Using Lemma 2.8 and taking $h = g = S$, in Theorem 2.2, we prove the following result.

Theorem 2.9. *Let M be a complete subset of a metric space (X, d), and T, S are self maps of M such that cl $(T(M)) \subseteq S(M) = M$. Suppose that M has*

a jointly continuous contractive family $\mathfrak{F}$ *such that* $Sf_x(k) = f_{Sx}(k)$ *for each* $x \in M$ *and* $k \in (0,1)$, T *and* S *are* C_q-*commuting and* T *satisfies,*

$$d(Tx, Ty) \leq \max\{d(Sx, Sy), dist(Sx, Y_{f_{Tx}(0)}^{Tx}), dist(Sy, Y_{f_{Ty}(0)}^{Ty}),$$
$$dist(Sx, Y_{f_{Ty}(0)}^{Ty}), dist(Sy, Y_{f_{Tx}(0)}^{Tx})\},$$

for all $x, y \in M$. *If* T *is continuous then* $M \cap F(T) \cap F(S) \neq \emptyset$ *provided one of the following conditions holds:*

   i) $cl\ (T(M))$ *is compact and* S *is continuous;*
   ii) M *is compact and* S *is continuous;*
   iii) $F(S)$ *is bounded,* S *is continuous and* T *is a compact map;*
   iv) M *is bounded and* S *is demicompact and continuous.*

Proof. Define T_n as in Theorem 2.2 and proceeding, we shall get

$$d(T_n x, T_n y) \leq \lambda_n \max\{d(Sx, Sy), d(Sx, T_n x), d(Sy, T_n y), d(Sx, T_n y),$$
$$d(Sy, T_n x)\}$$

for all $x, y \in M$.

(i) Since $cl\ (T(M)$ is compact, $cl\ (T_n(M))$ is compact and so by Lemma 2.8, there exists some $x_n \in M$ such that $F(T_n) \cap F(S) = \{x_n\}$ for each n. The compactness of $cl\ (T(M))$ implies the existence of a subsequence $\{Tx_{n_i}\}$ of $\{Tx_n\}$ such that $Tx_{n_i} \to y \in M$. Since $x_{n_i} = T_{n_i} x_{n_i} = f_{Tx_{n_i}}(\lambda_{n_i}) \to f_y(1) = y$, by the continuity of T and S, we have $y \in F(T) \cap F(S)$. Hence $M \cap F(T) \cap F(S) \neq \emptyset$.

(ii) The result follows from (i).

(iii) As in (i), there is a unique $x_n \in M$ such that $x_n = T_n x_n = Sx_n$. As T is compact and $\{x_n\}$ being in $F(S)$ is bounded, $\{Tx_n\}$ has a subsequence $\{Tx_{n_i}\}$ such that $\{Tx_{n_i}\} \to y$ as $n_i \to \infty$. Since $x_{n_i} = T_{n_i} x_{n_i} = f_{Tx_{n_i}}(\lambda_{n_i}) \to f_y(1) = y$. Thus $x_{n_i} \to y$. So, by the continuity of T and S, we have $y \in F(T) \cap F(S)$.

(iv) By Lemma 2.8, for each $n \geq 1$, there is a unique $x_n \in M$ such that $x_n = T_n x_n = Sx_n$. Since M is bounded, $\{x_n\}$ is bounded and $d(x_n, Sx_n) \to 0$, therefore by the demicompactness of S, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ converging to $z \in M$. As T is continuous, $Tx_{n_i} \to Tz$. Also, $x_{n_i} = T_{n_i} x_{n_i} = f_{Tx_{n_i}}(\lambda_{n_i}) \to f_z(1) = z$ i.e. $x_{n_i} \to z$. Therefore, by the continuity of T and S, we have $z \in F(T) \cap F(S)$ and hence $M \cap F(T) \cap F(S) \neq \emptyset$. $\square$

Corollary 2.10. *Let* M *be a complete* q-*starshaped subset of a convex metric space* (X, d) *with Property (I), and* T, S *are self mappings of* M *such that* $cl\ (T(M)) \subseteq S(M)$, $q \in F(S)$ *and* S *is affine. Suppose that* T *and* S *are* C_q-*commuting and* T *satisfies,*

$$d(Tx, Ty) \leq \max\{d(Sx, Sy), dist(Sx, [q, Tx]), dist(Sy, [q, Ty]),$$
$$dist(Sx, [q, Ty]), dist(Sy, [q, Tx])\},$$

for all $x, y \in M$. *If* T *is continuous then* $F(T) \cap F(S) \neq \emptyset$ *provided one of the following conditions holds:*

   i) $cl\ (T(M))$ *is compact and* S *is continuous;*
   ii) M *is compact and* S *is continuous;*

iii) $F(S)$ is bounded, S is continuous and T is a compact map;
iv) M is bounded and S is demicompact and continuous;
v) T is hemicompact and S is continuous.

Proof. For the cases (i)-(iv), the result follows from Theorem 2.2.
(v) As in Theorem 2.2 (i), there is a unique $x_n \in M$ such that $x_n = T_n x_n = S x_n$. Consider

$$
\begin{aligned}
d(x_n, Tx_n) &= d(T_n x_n, Tx_n) \\
&= d(W(Tx_n, q, \lambda_n), Tx_n) \\
&\leq \lambda_n d(Tx_n, Tx_n) + (1 - \lambda_n) d(q, Tx_n) \\
&= (1 - \lambda_n) \, d(q, Tx_n) \to 0,
\end{aligned}
$$

the hemicompactness of T implies that $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ converging to some $z \in M$. Therefore, by the continuity of T and S, we have $y \in F(T) \cap F(S)$. $\qquad\square$

Remark 2.1. Theorem 2.9 generalizes and extends Theorem 2.2 of Hussain and Rhoades [12] and extends the corresponding results of Narang and Chandok [17] from R-subweakly to C_q-commuting mappings.

Corollary 2.11. *Let M be a closed subset of a convex metric space (X, d) with Property (I), and T, S are continuous self mappings of M such that $T(M) \subseteq S(M)$. Suppose S is affine, $p \in F(S)$, M is p-starshaped, and cl $(T(M))$ is compact. If T and S are R-subweakly commuting and satisfy*

$$
\begin{aligned}
d(Tx, Ty) \leq \max\{&d(Sx, Sy), dist(Sx, W(Tx, p, \lambda)), dist(Sy, W(Ty, p, \lambda)), \\
&dist(Sx, W(Ty, p, \lambda)), dist(Sy, W(Tx, p, \lambda))\},
\end{aligned}
$$

for all $x, y \in M$, $\lambda \in [0, 1)$, then $M \cap F(T) \cap F(S) \neq \emptyset$.

Corollary 2.12. *([21]-Theorem 2.2) Let M be a closed subset of a normed space X, and T, S are continuous self mappings of M such that $T(M) \subseteq S(M)$. Suppose S is linear, $p \in F(S)$, M is p-starshaped, and cl $(T(M))$ is compact. If T and S are R-subweakly commuting and satisfy*

$$
\begin{aligned}
\|Tx - Ty\| \leq \max\{&\|Sx - Sy\|, dist(Sx, [Tx, p]), dist(Sy, [Ty, p]), \\
&\tfrac{1}{2}[dist(Sx, [Ty, p]) + dist(Sy, [Tx, p])]\},
\end{aligned}
$$

for all $x, y \in M$, then $M \cap F(T) \cap F(S) \neq \emptyset$.

Corollary 2.13. *Let M be a closed q-starshaped subset of a convex metric space (X, d) with Property (I), and T, f are R-subweakly commuting mappings on M such that $T(M) \subset f(M)$, $q \in F(f)$ and cl $(T(M))$ is compact. If T is continuous f-nonexpansive and f is affine on M, then T, f have a common fixed point in M.*

Corollary 2.14. *([3]-Corollary 2.3) Let M be a closed q-starshaped subset of a p-normed space X, and T, f are R-subweakly commuting mappings on M such that $T(M) \subset f(M)$, $q \in F(f)$ and cl $(T(M))$ is compact. If T is continuous f-nonexpansive and f is affine on M, then T and f have a common fixed point in M.*

Remark 2.2. If $f = g = S =$ identity mapping, Theorems 2.2 and 2.9 extend and generalize the corresponding results of Beg, Shahzad and Iqbal [4], Dotson [8] [9], Guay et al. [10], Habiniak [11]and of Narang and Chandok [16].

3. Invariant approximation and common fixed points of C_q-commuting mappings

In this section, we obtain results on best approximations as fixed points of C_q-commuting mappings in the setting of metric spaces and convex metric spaces.

Theorem 3.1. *Let M be a subset of a metric space (X, d), T,h and g be self mappings of M. Suppose that $D = P_M(u)$ is nonempty, and has a jointly continuous contractive family $\mathfrak{F}$ with $hf_x(k) = f_{hx}(k)$ and $gf_x(k) = f_{gx}(k)$ for all $x \in M$ and $k \in (0, 1)$. If h and g are continuous, $clT(D) \subseteq D = h(D) \cap g(D)$, pairs (T, h) and (T, g) are C_q-commuting and T satisfies,*

$$d(Tx, Ty) \leq \begin{cases} d(hx, gu) & , \ if \ y = u \\ Q(x, y) & , \ if \ y \in D \end{cases}$$

where

$$Q(x, y) = \max\{d(hx, gy), dist(hx, Y^{Tx}_{f_{Tx}(0)}), dist(gy, Y^{Ty}_{f_{Ty}(0)}),$$
$$\tfrac{1}{2}[dist(hx, Y^{Ty}_{f_{Ty}(0)}) + dist(gy, Y^{Tx}_{f_{Tx}(0)})]\};$$

for all $x \in D \cup \{u\}$, then $P_M(u) \cap F(T) \cap F(h) \cap F(g) \neq \emptyset$, provided one of the following condition is satisfied:

i) $clT(D)$ is compact and T is continuous;

ii) D is compact and T is continuous.

Proof. Since $cl\,(T(D)) \subset D = h(D) \cap g(D)$ is compact, the result follows from Theorem 2.2. $\qquad\square$

Corollary 3.2. *([22]-Theorem 3.1) Let M be a subset of a normed linear space X, T,f and g be continuous self mappings of M. Suppose that $D = P_M(u)$ is nonempty, q-starshaped, $cl\,(T(D)) \subset D$ is compact, $D = f(D) \cap g(D)$, f and g are q-affine on D. If the pair (T, f) and (T, g) are C_q-commuting and T satisfies,*

$$\|Tx - Ty\| \leq \begin{cases} \|fx - gu\| & , \ if \ y = u \\ Q(x, y) & , \ if \ y \in D \end{cases}$$

where

$$Q(x, y) = \max\{\|fx - gy\|, dist(fx, [Tx, q]), dist(gy, [Ty, q]),$$
$$\tfrac{1}{2}[dist(fx, [Ty, q]) + dist(gy, [Tx, q])]\},$$

for all $x \in D \cup \{u\}$, then $P_M(u) \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Theorem 3.3. *Let M be a subset of a convex metric space (X, d), T,h and g be self mappings of X such that $u \in F(T, h, g)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Suppose that $D = P_M(u)$ is complete, has a jointly continuous contractive family $\mathfrak{F}$ with $hf_x(k) = f_{hx}(k)$ and $gf_x(k) = f_{gx}(k)$ for all $x \in M$*

and $k \in (0, 1)$, h and g are continuous on D, and $D = hD = gD$. If cl (D) is compact and the pair (T, h) and (T, g) are C_q-commuting and T satisfies,

$$d(Tx, Ty) \leq \begin{cases} d(hx, gu) & , \quad if \ y = u \\ Q(x, y) & , \quad if \ y \in D \end{cases}$$

where

$$\begin{aligned} Q(x, y) = & \ \max\{d(hx, gy), dist(hx, Y^{Tx}_{f_{Tx}(0)}), dist(gy, Y^{Ty}_{f_{Ty}(0)}), \\ & \tfrac{1}{2}[dist(hx, Y^{Ty}_{f_{Ty}(0)}) + dist(gy, Y^{Tx}_{f_{Tx}(0)})]\}; \end{aligned}$$

for all $x \in D \cup \{u\}$, then $P_M(u) \cap F(T) \cap F(h) \cap F(g) \neq \emptyset$, provided one of the following conditions holds:
   i) $clT(D)$ is compact and T is continuous;
   ii) D is compact and T is continuous.

Proof. Let $x \in D$. For any $k \in (0, 1)$, we have

$$d(W(u, x, k), u) \leq kd(u, u) + (1 - k)d(x, u) = (1 - k)d(x, u) \leq dist(u, M).$$

By Lemma 3.2 of [1] it follows that the line segment $\{W(u, x, k) : 0 < k < 1\}$ and the set M are disjoint i.e. x is not in the interior of M and so $x \in \partial M \cap M$. Since $T(\partial M \cap M) \subset M$, Tx must be in M. Since $hx \in D$, $u \in F(T, h, g)$, by the given contractive condition we have, $d(Tx, u) = d(Tx, Tu) \leq d(hx, gu) = d(hx, u) \leq dist(u, M)$. Therefore $Tx \in D$, $T(D) \subset D = hD = gD$. The result now follows from Theorem 2.2.

$\square$

Corollary 3.4. *([3]-Theorem 3.2) Let M be a subset of a p-normed space X, T,f and g be self mappings of X such that $u \in F(T, f, g)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Suppose that $D = P_M(u)$ is complete, q-starshaped with $q \in F(f) \cap F(g)$, f and g are affine and continuous on D, and $D = fD = gD$. If cl (D) is compact and the pair (T, f) and (T, g) are C_q-commuting and T satisfies,*

$$\|Tx - Ty\|_p \leq \begin{cases} \|fx - gu\|_p & , \quad if \ y = u \\ Q(x, y) & , \quad if \ y \in D \end{cases}$$

where

$$\begin{aligned} Q(x, y) = & \ \max\{\|fx - gy\|_p, dist(fx, [Tx, q]), dist(gy, [Ty, q]), \\ & \tfrac{1}{2}[dist(fx, [Ty, q]) + dist(gy, [Tx, q])]\}, \end{aligned}$$

for all $x \in D \cup \{u\}$, and $\lambda \in [0, 1)$, then $P_M(u) \cap F(T) \cap F(f) \cap F(g)$ is nonempty.

Corollary 3.5. *Let M be a subset of a convex metric space (X, d) with Property (I), T,f and g be self mappings of X such that $u \in F(T, f, g)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Suppose that $D = P_M(u)$ is complete, q-starshaped with $q \in F(f) \cap F(g)$, f and g are affine and continuous on D, and $D = fD = gD$.*

If cl (D) is compact and the pair (T, f) and (T, g) are R-subweakly commuting and T satisfies,

$$d(Tx, Ty) \leq \begin{cases} d(fx, gu) & , \quad if \ \ y = u \\ Q(x, y) & , \quad if \ \ y \in D \end{cases}$$

where

$$Q(x, y) \quad = \quad \max\{d(fx, gy), dist(fx, [q, Tx]), dist(gy, [q, Ty]),$$
$$\tfrac{1}{2}[dist(fx, [q, Ty]) + dist(gy, [q, Tx])]\},$$

for all $x \in D \cup \{u\}$, then $P_M(u) \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Corollary 3.6. *([3]-Corollary 3.3) Let M be a subset of a p-normed space X, T, f and g be self mappings of X such that $u \in F(T, f, g)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Suppose that $D = P_M(u)$ is complete, q-starshaped with $q \in F(f) \cap F(g)$, f and g are affine and continuous on D, and $D = fD = gD$. If cl (D) is compact and the pair (T, f) and (T, g) are R-subweakly commuting and T satisfies,*

$$\|Tx - Ty\|_p \leq \begin{cases} \|fx - gu\|_p & , \quad if \ \ y = u \\ Q(x, y) & , \quad if \ \ y \in D \end{cases}$$

where

$$Q(x, y) \quad = \quad \max\{\|fx - gy\|_p, dist(fx, [Tx, q]), dist(gy, [Ty, q]),$$
$$\tfrac{1}{2}[dist(fx, [Ty, q]) + dist(gy, [Tx, q])]\},$$

for all $x \in D \cup \{u\}$, then $P_M(u) \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Theorem 3.7. *Let M be a subset of a convex metric space (X, d), T and S be self mappings of X such that $u \in F(T, S)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Suppose that $D = P_M(u)$ is nonempty, closed, and has a contractive jointly continuous family $\mathfrak{F}$ such that $Sf_x(k) = f_{Sx}(k)$ for each $x \in M$ and $k \in (0, 1)$. If $D = SD$ and the pair (S, T) is C_q-commuting, continuous on D and T satisfies,*

(3.1) $$d(Tx, Ty) \leq \begin{cases} d(Sx, Su) & , \quad if \ \ y = u \\ Q(x, y) & , \quad if \ \ y \in D \end{cases}$$

where

$$Q(x, y) \quad = \quad \max\{d(Sx, Sy), dist(Sx, Y^{Tx}_{f_{Tx}(0)}), dist(Sy, Y^{Ty}_{f_{Ty}(0)}),$$
$$dist(Sx, Y^{Ty}_{f_{Ty}(0)}), dist(Sy, Y^{Tx}_{f_{Tx}(0)})\},$$

for all $x \in D \cup \{u\}$, then $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$, provided one of the following conditions holds:

i) cl $(T(D))$ is compact and S is continuous;

ii) D is compact and S is continuous;

iii) D is complete, $F(S)$ is bounded, S is continuous and T is a compact map;

iv) S is demicompact and continuous.

Proof. Let $x \in D$. For any $k \in (0, 1)$, we have

$$d(W(u, x, k), u) \leq kd(u, u) + (1 - k)d(x, u) = (1 - k)d(x, u) \leq dist(u, M).$$

From Lemma 3.2 [1] it follows that the line segment $\{W(u, x, k) : 0 < k < 1\}$ and the set M are disjoint i.e. x is not in the interior of M and so $x \in \partial M \cap M$. Since $T(\partial M \cap M) \subset M$, Tx must be in M. Since $Sx \in D$, $u \in F(T, S)$ and T and S satisfy (3.1), we have $d(Tx, u) = d(Tx, Tu) \leq d(Sx, Su) = d(Sx, u) = dist(u, M)$ i.e. $Tx \in D$. Theorem 2.9 (i)-(iv) then imply that $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$. $\qquad\square$

Corollary 3.8. *([12]-Theorem 2.6) Let M be a subset of a p-normed space X, T and S be self mappings of X such that $u \in F(T, S)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Suppose that $D = P_M(u)$ is nonempty, closed, q-starshaped with $q \in F(S)$, S is affine, $D = SD$. If $cl\ (T(D))$ is compact and the pair (S, T) is C_q-commuting, continuous on D and T satisfies,*

$$\|Tx - Ty\|_p \leq \begin{cases} \|Sx - Su\|_p & , & if\ y = u \\ Q(x, y) & , & if\ y \in D \end{cases}$$

where

$$\begin{aligned} Q(x, y) \quad = \quad & \max\{\|Sx - Sy\|_p, dist(Sx, [Tx, q]), dist(Sy, [Ty, q]), \\ & dist(Sx, [Ty, q]), dist(Sy, [Tx, q])\}, \end{aligned}$$

for all $x \in D \cup \{u\}$, then $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$.

Theorem 3.9. *Let M be a subset of a convex metric space (X, d), T and S be self mappings of X such that $u \in F(T, S)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Suppose that $D = P_M(u) \cap C_M^S(u)$, where $C_M^S(u) = \{x \in M : Sx \in P_M(u)\}$, is closed, and has a contractive jointly continuous family $\mathfrak{F}$ such that $Sf_x(k) = f_{Sx}(k)$ for each $x \in M$ and $k \in (0, 1)$. If $D = SD$, $cl\ (T(D))$ is compact, S is nonexpansive on $P_M(u) \cup \{u\}$ and the pair (S, T) is C_q-commuting, continuous on D and T satisfies,*

$$(3.2) \qquad d(Tx, Ty) \leq \begin{cases} d(Sx, Su) & , & if\ y = u \\ Q(x, y) & , & if\ y \in D \end{cases}$$

where

$$\begin{aligned} Q(x, y) \quad = \quad & \max\{d(Sx, Sy), dist(Sx, Y_{f_{Tx}(0)}^{Tx}), dist(Sy, Y_{f_{Ty}(0)}^{Ty}), \\ & dist(Sx, Y_{f_{Ty}(0)}^{Ty}), dist(Sy, Y_{f_{Tx}(0)}^{Tx})\}, \end{aligned}$$

for all $x \in D \cup \{u\}$, then $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$.

Proof. Proceeding as in Theorem 3.7, we obtain $Tx \in P_M(u)$. Since S is nonexpansive on $P_M(u) \cup \{u\}$ and T satisfies (3.2), we obtain $d(STx, u) = d(STx, STu) \leq d(Tx, Tu) \leq d(Sx, Su) = dist(u, M)$. Therefore $STx \in P_M(u)$ and so $Tx \in C_M^S(u)$. Hence $Tx \in D$ and consequently, $cl\ (T(D)) \subset D = S(D)$. Theorem 2.9(i) then implies that $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$. $\qquad\square$

Corollary 3.10. *([12]-Theorem 2.7) Let M be a subset of a p-normed space, T and S be self mappings of X such that $u \in F(T, S)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Suppose that $D = P_M(u) \cap C_M^S(u)$ closed, q-starshaped with $q \in F(S)$, S is affine, $D = SD$. If $cl\,(T(D))$ is compact, S is nonexpansive on $P_M(u) \cup \{u\}$ and the pair (S, T) is C_q-commuting, continuous on D and T satisfies,*

$$\|Tx - Ty\|_p \leq \left\{ \begin{array}{ll} \|Sx - Su\|_p & , \quad if \;\; y = u \\ Q(x, y) & , \quad if \;\; y \in D \end{array} \right.$$

where

$$\begin{aligned} Q(x, y) \quad = \quad & \max\{\|Sx - Sy\|_p, dist(Sx, [Tx, q]), dist(Sy, [Ty, q]), \\ & dist(Sx, [Ty, q]), dist(Sy, [Tx, q])\}, \end{aligned}$$

for all $x \in D \cup \{u\}$, then $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$.

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