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Mercer Type Variants of the Jensen−Steffensen Inequality-II

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Abstract. Jensen–Mercer's inequality for weights satisfying conditions as for the reversed Jensen-Steffensen inequality is proved. Several inequalities involving more than one monotonic function with reversed Jensen-Steffensen conditions are proved. Furthermore, a couple of general companion inequalities related to Mercer's inequality with reversed Jensen-Steffensen conditions are presented as well. Applications for the generalization of weighted Ky Fan's inequality, classical Power Mean, and classical Arithmetic–Geometric–Harmonic Mean inequalities are given as well.

Keywords: Monotonic function, Composite function, Reversed Jensen –Steffensen inequality.

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1. Introduction and preliminaries

One of the most important concepts in the theory of inequalities is the notion of convex functions and note that the definition of a convex function is, in fact, an inequality. The theory of convex functions has experienced a rapid

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development. This can be attributed to several causes: first, a great many areas in modern analysis directly or indirectly involve the application of convex functions; secondly, convex functions are closely related to the theory of inequalities, and many important inequalities are consequences of the applications of convex functions.

Let us start with convex functions definition extracted from [26, p. 1](see also [25, p. 1]). Throughout the article, we use T, J and $[\varepsilon,\theta]$ to denote intervals in $\mathbb{R}$. Furthermore all over the chapter we assume that $[\varepsilon,\theta]\subset \mathrm{T}$ and $\varepsilon<\theta$.

Definition 1.1. $\sqsupset : \mathrm{T}\to\mathbb{R}$ is named as convex–function if

$$\sqsupset(¥\lambda_1+(1-¥)\lambda_2)\leq ¥\sqsupset(\lambda_1)+(1-¥)\sqsupset(\lambda_2) \tag{1.1}$$

holds for each $\lambda_1,\lambda_2\in \mathrm{T}$ and $¥\in[0,1]$.

In order to eliminate confusion, here we define some notations. For the real n−tuple $\mathbf{w}=(\mathrm{w}_1,\ldots,\mathrm{w}_n)$, the following notations are valid.

$$\mathrm{W}_k=\sum_{i\in\Theta_k}\mathrm{w}_i,\quad k\in\{1,\ldots,n\},\quad \overline{\mathrm{W}}_k=\sum_{i\in\Theta^k}\mathrm{w}_i,\quad \text{and}\quad \Upsilon=\frac{1}{W_n}.$$

Of course, $W_n=\sum_{i\in\Theta_n}\mathrm{w}_i$.
As aforementioned that various powerful inequalities originated due to convex functions, and Jensen inequality is one of the highly important inequalities. Jensen inequality is established in the following proposition [26, p. 43] (see also [25, p. 6]).

Proposition 1.2. *If* $\mathbf{w}$ *is any nonnegative n−tuple such that* $\mathrm{W}_n>0$*, then for any n−tuple* $\mathbf{x}$ *in* T^n *and for any function* $\sqsupset$ *convex on* T *we have*

$$\sqsupset\left(\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i\right)\leq\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\sqsupset(\mathrm{x}_i). \tag{1.2}$$

There are many versions, variants and generalization of Jensen inequality, see for example [2], [5], [7], [8], and [10]–[23].

It is acceptable to question that in Jensen inequality the supposition "$\mathbf{w}$ is a nonnegative n−tuple" can be loosened at the surcharge of tightening $\mathbf{x}$ more strictly [29]. Steffensen (see [29]) was the pioneer to address this issue, but in the following proposition we stated a more generalized form (see also [26, p. 57]).

Proposition 1.3. *If* $\mathbf{w}$ *is any real n−tuple satisfying*

$$0\leq \mathrm{W}_j\leq \mathrm{W}_n,\quad (1\leq j\leq n),\quad \mathrm{W}_n>0, \tag{1.3}$$

then for any monotonic n−tuple $\mathbf{x}$ *in* T^n *such that* $\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i\in\mathrm{T}$*, and for any function* $\sqsupset$ *convex on* T*, (1.2) still holds.*

Under these supposition (1.2) is regarded as Jensen–Steffensen inequality.

Note that throughout the article, when we claim that n−tuple $\mathbf{x}$ is increasing (decreasing) it indicate that $\mathrm{x}_1 \leq \mathrm{x}_2 \leq \cdots \leq \mathrm{x}_n$ ($\mathrm{x}_1 \geq \mathrm{x}_2 \geq \cdots \geq \mathrm{x}_n$). Vasić and Pečarić in [30] state and prove the following result of reversed Jensen's inequality.

Proposition 1.4. *If* $\mathbf{w}$ *is any real* $n-$*tuple such that such that* $\mathrm{W}_n > 0$*, and*

$$\mathrm{w}_1 > 0, \quad \mathrm{w}_i \leq 0 \quad for \quad i \in \{2, \ldots, n\}, \tag{1.4}$$

then for any $n-$*tuple* $\mathbf{x}$ *in* T^n *such that* $\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i \in \mathrm{T}$*, and for any function* $\beth$ *convex on* T *the inequality*

$$\beth\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) \geq \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i), \tag{1.5}$$

holds which is actually reversed of (1.2)*.*

Remark 1.5. Popoviciu's and Aczél's inequalities can be obtained from reversed Jensen–inequality (see [25, p. 192]).

In [27] Pečarić applied the Fuchs generalization of majorization theorem [9] (see Section 2.2.6) to prove the necessary and sufficient condition for the validity of following Reversed Jensen−Steffensen inequality. (see also [26, p. 83] and [25, p. 6]).

Proposition 1.6. *If* $\mathbf{w}$ *is any real* $n-$*tuple with* $\mathrm{W}_n > 0$*, and* $\exists\, l \in \{1, \ldots, n\}$ *such that*

$$\mathrm{W}_j \leq 0 \;\; for \;\; j < l \qquad and \qquad \overline{W}_j \leq 0 \;\; for \;\; j > l, \tag{1.6}$$

then for any monotonic $n-$*tuple* $\mathbf{x}$ *in* T^n *such that* $\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i \in \mathrm{T}$ *and for any function* $\beth$ *convex on* T (1.5) *holds.*

Following Jensen inequality variant is given by Mercer in [23].

Proposition 1.7. *If* $\mathbf{w}$ *is any nonnegative* $n-$*tuple such that* $\mathrm{W}_n > 0$*, then for any* $n-$*tuple* $\mathbf{x}$ *in* $[\varepsilon, \theta]^n$ *and for any function* $\beth$ *convex on* T *we have*

$$\beth\left(\varepsilon + \theta - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) \leq \beth(\varepsilon) + \beth(\theta) - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i). \tag{1.7}$$

Inequality (1.7) is named as Jensen−Mercer inequality.

By applying some specific conditions on weights w_i for $i \in \{1, \ldots, n\}$, we obtain different variants of Proposition 1.7.

Particularly in [1] Abramovich et. al. gave the following generalization of Proposition 1.7.

Proposition 1.8. *If* $\mathbf{w}$ *is any real* $n-$*tuple satisfying* (1.3)*, then for any monotonic* $n-$*tuple* $\mathbf{x}$ *in* $[\varepsilon, \theta]^n$ *such that* $\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i \in [\varepsilon, \theta]$*, and for any function* $\beth$ *convex on* T*, the inequality* (1.7) *still holds.*

In [22] Matković and Pečarić gave the following impressive characteristic of Jensen–Mercer inequality a kind of powerful feature concerning weights.

Proposition 1.9. *If $n-$tuple* $\mathbf{w}$ *satisfy* (1.4)*, then for any $n-$tuple* $\mathbf{x}$ *in* $[\varepsilon,\theta]^n$ *such that* $\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i\in[\varepsilon,\theta]$ *and for any function* $\beth$ *convex on* T *the inequality* (1.7) *still holds.*

In current article our prime goal is to treat discrete reversed Jensen − Steffensen type inequality in a variety of ways. The article contains five sections. In Section 1, introduction and preliminaries are provided. In Section 2, we prove Jensen–Mercer inequality under reversed Jensen−Steffensen conditions . In Section 3, further generalization of Jensen–Mercer inequality involving two or more functions are given. In Section 4, we prove a couple of general companion inequalities. The last section covers our proven results applications.

For interested reader, here we mention that integral version results are published in our article [12].

2. Mercer's inequality with reversed jensen–steffensen conditions

Before we further proceed here we need an important property of convex function proved by Mercer in [23] (see also [4, Theorem 1]).

Lemma 2.1. *If* $\beth:[\varepsilon,\theta]\to\mathbb{R}$ *is convex function, then* $\forall\ \mathrm{x}\in[\varepsilon,\theta]$ *we have*

$$\beth(\varepsilon+\theta-\mathrm{x})\leq\beth(\varepsilon)+\beth(\theta)-\beth(\mathrm{x}). \tag{2.1}$$

Following the proof of Proposition 1.9 we state and prove Jensen–Mercer's inequality for real weights satisfying conditions as for the reversed Jensen–Steffensen inequality.

Theorem 2.2. *Let monotonic $n-tuple$* $\mathbf{x}$ *is such that* $\mathrm{x}_i\in[\varepsilon,\theta]$*, and real $n-tuple$* $\mathbf{w}$ *is such that* $\mathrm{W}_n>0$ *and* $\Upsilon\sum\limits_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i\in[\varepsilon,\theta]$*. We additionally choose* $\mathrm{l}\in\{1,\ldots,n\}$ *in such a way that condition* (1.6) *holds. Then for any convex function* $\beth:[\varepsilon,\theta]\to\mathbb{R}$*, the inequality* (1.7) *still holds.*

Proof. From Lemma 2.1, $\forall\ \mathrm{x}\in[\varepsilon,\theta]$ we have

$$\beth(\varepsilon+\theta-\mathrm{x})\leq\beth(\varepsilon)+\beth(\theta)-\beth(\mathrm{x}).$$

Further for $\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i \in [\varepsilon, \theta]$ we have

$$\begin{aligned}\beth\left(\varepsilon+\theta-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i\right)& \\ &\leq \beth(\varepsilon)+\beth(\theta)-\beth\left(\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i\right)\\ &\qquad\leq \beth(\varepsilon)+\beth(\theta)-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\beth(\mathrm{x}_i),\end{aligned}$$

last inequality is a consequence of Proposition 1.6. □

3. Reversed Jensen−Steffensen and Jensen−Mercer type inequalities involving two or more functions

In paper [1], Abramovich et. al. proved Jensen–Steffensen type inequality for a sequence of functions $\psi_0, \psi_1, \ldots, \psi_{\mathrm{q}}$, $\mathrm{q} \in \mathbb{N}$ where each ψ_i is either convex or concave for $i \in \{0, 1, \ldots, \mathrm{q}\}$ and ψ_i is monotonic for $i \in \mathbb{N}$. In current section, we would prove analogous results for reversed Jensen–Steffensen type inequality. Here, we need an important result for the convexity and concavity of composite functions which we would use repeatedly in our proofs, see [28, p. 16 and p. 20].

Lemma 3.1. *Let $\beth : \mathrm{T} \to \mathbb{R}$ and $\psi : J \to \mathbb{R}$ where range$(\beth) \subseteq J$.*

(a) *If either $\beth$ is convex on T and ψ is increasing convex on J , or $\beth$ is concave on T and ψ is decreasing convex, then the composite function $\psi \circ \beth$ is convex on T.*

(b) *If either $\beth$ is convex on T and ψ is decreasing concave on J, or $\beth$ is concave on T and ψ is increasing concave on J, then the composite function $\psi \circ \beth$ is concave on T.*

Here we present a theorem concerning composite functions.

Theorem 3.2. *Let $\beth : \mathrm{T} \to \mathrm{J}$ be a monotonic function and $\psi : \mathrm{J} \to \mathbb{R}$ be a function. Let monotonic $n-$tuple $\mathbf{x}$ is such that $\mathrm{x}_i \in \mathrm{T}$ and real $n-$tuple $\mathbf{w}$ is such that $\mathrm{W}_n > 0$, and $\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i \in \mathrm{T}$. We further choose $\mathrm{l} \in \{1, \ldots, n\}$ such that condition (1.6) holds:*
(i) If either $\beth$ is convex on T and ψ is increasing convex on J with $\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i) \in \mathrm{J}$, or $\beth$ is concave on T and ψ is decreasing convex on J with

$\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i) \in \mathrm{J}$, *then following inequalities hold.*

$$\psi \circ \beth\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) \geq \psi\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i)\right) \geq \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i(\psi\circ\beth)(\mathrm{x}_i). \quad (3.1)$$

(ii) If either $\beth$ *is convex on* T *and* ψ *is decreasing concave on* J *with* $\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i) \in \mathrm{J}$, *or* $\beth$ *is concave on* T *and* ψ *is increasing concave on* J *with* $\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i) \in \mathrm{J}$, *then inequalities in* (3.1) *hold in reversed direction.*

Proof. By Proposition 1.6, convexity of $\beth$ yields

$$\beth\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) \geq \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i).$$

Since $\beth\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) \in \mathrm{J}$ and $\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i) \in \mathrm{J}$. If ψ is increasing, then we have

$$\psi \circ \beth\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) \geq \psi \circ \left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i)\right).$$

Further $\beth$ is taken as monotonic function on T, $\beth(\mathbf{x}) = (\beth(\mathrm{x}_1), \dots, \beth(\mathrm{x}_n))$ be monotonic $n - tuple$ in J^n, *i.e.*, $\beth(\mathrm{x}_1) \leq \cdots \leq \beth(\mathrm{x}_n)$ or $\beth(\mathrm{x}_1) \geq \cdots \geq \beth(\mathrm{x}_n)$. Since $\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i) \in \mathrm{J}$, and $\mathrm{l} \in \{1, \dots, n\}$ which satisfies condition (1.6), thus, by applying Proposition 1.6 for convex function ψ we have

$$\psi \circ \left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i)\right) \geq \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i(\psi\circ\beth)(\mathrm{x}_i).$$

We will obtain the desired outcome by applying the transitive property of inequalities.

The left hand inequality of (3.1) still follows by the fact that ψ is decreasing when $\beth$ is concave and then the right hand inequality (3.1) follows by the fact that ψ is convex.

On the contrary, similar arguments provide the reversed of the inequality (3.1), i.e., either the reversed of left hand inequality of (3.1) follows by the fact that ψ is decreasing when $\beth$ is convex and the reversed of right hand inequality of (3.1) follows by the concavity of ψ, or the reversed of left hand inequality of (3.1) follows by the fact that ψ is increasing when $\beth$ is concave and the reversed of right hand inequality of (3.1) follows by the concavity of ψ. □

In order to give analogous result for Jensen–Mercer inequality we assume that $\varepsilon_1 = \min[\beth(\varepsilon), \beth(\theta)]$, $\theta_1 = \max[\beth(\varepsilon), \beth(\theta)]$ where $[\varepsilon_1, \theta_1] \subset \mathrm{J}$.

Theorem 3.3. *Let* $\beth : [\varepsilon, \theta] \to [\varepsilon_1, \theta_1]$ *be a monotonic function and* $\psi : [\varepsilon_1, \theta_1] \to \mathbb{R}$ *be a function. Let monotonic* $n-$*tuple* $\mathbf{x}$ *is such that* $\mathrm{x}_i \in [\varepsilon, \theta]$ *and real* $n-$*tuple* $\mathbf{w}$ *is such that* $\mathrm{W}_n > 0$ *and* $\Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \mathrm{x}_i \in [\varepsilon, \theta]$*. We further choose* $\mathrm{l} \in \{1, \ldots, n\}$ *such that condition* (1.6) *holds:*
(i)*If either* $\beth$ *is convex on* $[\varepsilon, \theta]$ *and* ψ *is increasing convex on* $[\varepsilon_1, \theta_1]$ *with* $\Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \beth(\mathrm{x}_i) \in [\varepsilon_1, \theta_1]$*, or* $\beth$ *is concave on* $[\varepsilon, \theta]$ *and* ψ *is decreasing convex on* $[\varepsilon_1, \theta_1]$ *with* $\Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \beth(\mathrm{x}_i) \in [\varepsilon_1, \theta_1]$*, then*

$$\begin{aligned}\psi \circ \beth &\left(\varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \mathrm{x}_i\right) \\ &\leq \psi\left(\beth(\varepsilon) + \beth(\theta) - \Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \beth(\mathrm{x}_i)\right) \\ &\leq (\psi \circ \beth)(\varepsilon) + (\psi \circ \beth)(\theta) - \Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i (\psi \circ \beth)(\mathrm{x}_i). \quad (3.2)\end{aligned}$$

(ii)*If either* $\beth$ *is convex on* $[\varepsilon, \theta]$ *and* ψ *is decreasing concave on* $[\varepsilon_1, \theta_1]$ *with* $\Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \beth(\mathrm{x}_i) \in [\varepsilon_1, \theta_1]$*, or* $\beth$ *is concave on* $[\varepsilon, \theta]$ *and* ψ *is increasing concave on* $[\varepsilon_1, \theta_1]$ *with* $\Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \beth(\mathrm{x}_i) \in [\varepsilon_1, \theta_1]$*, then reversed direction of inequalities in* (3.2) *hold.*

Proof. Follow "Theorem 3.2" proof but rather using "Proposition 1.6" use "Theorem 2.2". $\square$

It is stimulating that previous results related to two functions, which are either concave or convex, could be extended by induction to any set which consists of more than two functions fulfilling some conditions. Now we consider a set which consists of $\mathrm{q}+1$ functions where $\mathrm{q} \in \mathbb{N}$,

$$\beth : \mathrm{T} \longrightarrow \mathbb{R}, \quad \psi_1 : \mathrm{T}_1 \longrightarrow \mathbb{R}, \ \ldots, \ \psi_{\mathrm{q}} : \mathrm{T}_{\mathrm{q}} \longrightarrow \mathbb{R},$$

here $\mathrm{T}, \mathrm{T}_1, \ldots, \mathrm{T}_{\mathrm{q}}$ are intervals in real line such that

$$\beth(\mathrm{T}) \subseteq \mathrm{T}_1, \quad \psi_{\mathrm{k}}(\mathrm{T}_{\mathrm{k}}) \subseteq \mathrm{T}_{\mathrm{k}+1}, \quad \mathrm{k} \in \{1, \ldots, \mathrm{q}-1\}. \quad (3.3)$$

Now auxiliary functions $\mathrm{F}_{\mathrm{k}} : \mathrm{T} \longrightarrow \mathbb{R}$, $\mathrm{G}_{\mathrm{k}} : \mathrm{T}_{\mathrm{k}} \longrightarrow \mathbb{R}$, $\mathrm{k} \in \{1, \ldots, \mathrm{q}\}$, are well defined as

$$\mathrm{F}_{\mathrm{k}} : \mathrm{T} \longrightarrow \mathbb{R}, \qquad \mathrm{F}_{\mathrm{k}} = \psi_{\mathrm{k}} \circ \psi_{\mathrm{k}-1} \circ \cdots \circ \psi_1 \circ \beth, \quad (3.4)$$
$$\mathrm{G}_{\mathrm{k}} : \mathrm{T}_{\mathrm{k}} \longrightarrow \mathbb{R}, \qquad \mathrm{G}_{\mathrm{k}} = \psi_{\mathrm{q}} \circ \psi_{\mathrm{q}-1} \circ \cdots \circ \psi_{\mathrm{k}}, \quad (3.5)$$

and $\forall$ $\mathrm{k} \in \{1, \ldots, \mathrm{q}\}$, $[\varepsilon_{\mathrm{k}}, \theta_{\mathrm{k}}] \subseteq \mathrm{T}_{\mathrm{k}}$, where $\varepsilon_{\mathrm{k}} = \min[\psi_{\mathrm{k}-1}(\varepsilon), \psi_{\mathrm{k}-1}(\theta)]$, $\theta_{\mathrm{k}} = \max[\psi_{\mathrm{k}-1}(\varepsilon), \psi_{\mathrm{k}-1}(\theta)]$ with $\psi_0 = \daleth$, and

$$\daleth([\varepsilon, \theta]) \subseteq [\varepsilon_1, \theta_1], \text{ and } \psi_{\mathrm{k}}([\varepsilon_{\mathrm{k}}, \theta_{\mathrm{k}}]) \subseteq [\varepsilon_{\mathrm{k}+1}, \theta_{\mathrm{k}+1}], \ \forall \ \mathrm{k} \in \{1, \ldots, \mathrm{q}-1\} \ . \tag{3.6}$$

Now, we are assuming that each of the investigated functions of the set $\{\daleth, \psi_1, \ldots, \psi_{\mathrm{q}}\}$ is either concave or convex and also we assume that the following monotonicity condition is fulfilled. The following condition is taken from [1].

Monotonicity Condition(MC).

Indicate $\psi_0 = \daleth$. We consider that Monotonicity condition MC is satisfied by any set of functions $\psi_0, \psi_1, \ldots, \psi_{\mathrm{q}}$, $\mathrm{q} \in \mathbb{N}$, if following condition is satisfied by all pairs $(\psi_{\mathrm{k}}, \psi_{\mathrm{k}+1})$ where for all $\mathrm{k} \in \{0, 1, \ldots, \mathrm{q}-1\}$:

(1) when both functions ψ_{k} and $\psi_{\mathrm{k}+1}$ are either convex or concave, then $\psi_{\mathrm{k}+1}$ is increasing;
(2) when either ψ_{k} is convex and $\psi_{\mathrm{k}+1}$ is concave, or $\psi_{\mathrm{k}+1}$ is convex and ψ_{k} is concave, then $\psi_{\mathrm{k}+1}$ is decreasing.

Observe that in case MC is satisfied by the functions $\psi_0, \psi_1, \ldots, \psi_{\mathrm{q}}$, in that case all functions except could be ψ_0 are monotonic. Now, we state the following proposition, for proof see [12].

Proposition 3.4. *Let the functions* $\daleth, \psi_1, \ldots \psi_{\mathrm{q}}$ *be as above and satisfy MC. Let* F_{k}, $\mathrm{k} = 1, \ldots, \mathrm{q}$ *be defined by* (3.4)*. Then for all* $\mathrm{k} \in \{1, \ldots, \mathrm{q}\}$*, we have:*
(i) *If* ψ_{k} *is convex on* T_{k}*, then* F_{k} *is convex on* T*;*
(ii) *If* ψ_{k} *is concave on* T_{k}*, then* F_{k} *is concave on* T.

Now we extend Theorem 3.2 to the following general result.

Theorem 3.5. *Let* $\daleth : \mathrm{T} \to \mathbb{R}$ *and* $\psi_{\mathrm{k}} : \mathrm{T}_{\mathrm{k}} \to \mathbb{R}$, $\mathrm{k} \in \{1, \ldots, \mathrm{q}\}$, $(\mathrm{q} \in N)$ *be either concave or convex , satisfying* (3.3) *and MC. Additionally assume that* $\daleth$ *to be monotonic. Let monotonic* $n-tuple$ $\mathbf{x}$ *is such that for all* $i \in \Theta_n$ $\mathrm{x}_i \in \mathrm{T}$ *and real* $n-tuple$ $\mathbf{w}$ *is such that* $\mathrm{W}_n > 0$, $\Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \mathrm{x}_i \in \mathrm{T}$, *and* $\Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \daleth(\mathrm{x}_i) \in \mathrm{T}_1$*. We additionally choose* $\mathrm{l} \in \{1, \ldots, n\}$ *such that condition* (1.6) *holds. The auxiliary functions* F_{k} *be as defined in* (3.4) *and* $\Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \mathrm{F}_{\mathrm{k}}(\mathrm{x}_i) \in \mathrm{T}_{\mathrm{k}+1}$ *for* $\mathrm{k} \in \{1, \ldots, \mathrm{q}-1\}$*:*
(i) *If* ψ_{q} *is convex on* T_{q}*, then*

$$\begin{aligned} \mathrm{F}_{\mathrm{q}}\left(\Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \mathrm{x}_i\right) &\geq \mathrm{G}_1\left(\Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \daleth(\mathrm{x}_i)\right) \geq \mathrm{G}_2\left(\Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \mathrm{F}_1(\mathrm{x}_i)\right) \\ &\geq \cdots \geq \mathrm{G}_{\mathrm{q}}\left(\Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \mathrm{F}_{\mathrm{q}-1}(\mathrm{x}_i)\right) \geq \Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \mathrm{F}_{\mathrm{q}}(\mathrm{x}_i), \end{aligned} \tag{3.7}$$

where G_{k}*,* $\forall$ $\mathrm{k} \in \{1, \ldots, \mathrm{q}\}$ *be defined by* (3.5)*.*
(*ii*) *If* ψ_{q} *is concave on* T_{q}*, then inequalities in* (3.7) *hold in reversed direction.*

Proof. We prove this theorem by mathematical induction. For $\mathrm{q} = 1$ we set $\psi = \psi_1$ and it is seen with ease that the considered statement is actually Theorem 3.2 statement. By the induction hypothesis result is true for $\mathrm{q} - 1$, thus for the convexity of $\psi_{\mathrm{q}-1}$ we have

$$\begin{aligned}\mathrm{F}_{\mathrm{q}-1}\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{x}_i\right) &\geq \mathrm{G}_1\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \beth(\mathrm{x}_i)\right) \geq \mathrm{G}_2\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{F}_1(\mathrm{x}_i)\right)\\ &\geq \cdots \geq \mathrm{G}_{\mathrm{q}-1}\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{F}_{\mathrm{q}-2}(\mathrm{x}_i)\right) \geq \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{F}_{\mathrm{q}-1}(\mathrm{x}_i), \quad (3.8)\end{aligned}$$

and the reversed inequalities hold in (3.8) if $\psi_{\mathrm{q}-1}$ is concave. Now we claim that each term in (3.8) belongs to T_{q}.

Firstly, we claim

$$\mathrm{F}_{\mathrm{q}-1}\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{x}_i\right) \in \mathrm{T}_{\mathrm{q}}.$$

Given that

$$\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{x}_i \in \mathrm{T}.$$

Using (3.3) we have

$$\beth\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{x}_i\right) \in \mathrm{T}_1$$

and

$$\psi_1 \circ \beth\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{x}_i\right) \in \mathrm{T}_2,$$

by advancing in a similar way we get

$$\begin{aligned}&\psi_{\mathrm{q}-1} \circ \psi_{\mathrm{q}-2} \circ \cdots \circ \psi_1 \circ \beth\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{x}_i\right)\\ &= \mathrm{F}_{\mathrm{q}-1}\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{x}_i\right) \in \mathrm{T}_{\mathrm{q}}.\end{aligned}$$

Secondly, we claim

$$\mathrm{G}_1\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \beth(\mathrm{x}_i)\right) \in \mathrm{T}_{\mathrm{q}}.$$

Given that

$$\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\daleth(\mathrm{x}_i) \in \mathrm{T}_1.$$

Using (3.3) we have

$$\psi_1\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\daleth(\mathrm{x}_i)\right) \in \mathrm{T}_2$$

and

$$\psi_2\circ\psi_1\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\daleth(\mathrm{x}_i)\right) \in \mathrm{T}_3,$$

by advancing in a similar way we have

$$\begin{aligned}&\psi_{\mathrm{q}-1}\circ\psi_{\mathrm{q}-2}\circ\cdots\circ\psi_1\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\daleth(\mathrm{x}_i)\right)\\ &= \mathrm{G}_1\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\daleth(\mathrm{x}_i)\right) \in \mathrm{T}_{\mathrm{q}}.\end{aligned}$$

Finally, From the given condition we have

$$\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{F}_{\mathrm{q}-1}(\mathrm{x}_i) \in \mathrm{T}_{\mathrm{q}}.$$

Thus we conclude that each term in (3.8) belongs to T_{q}. Now let ψ_{q} is convex and by induction hypothesis $\psi_{\mathrm{q}-1}$ is also convex, so by MC $\psi_{\mathrm{q}} : \mathrm{T}_{\mathrm{q}} \longrightarrow \mathbb{R}$ is increasing, *i.e.*,

$$\begin{aligned}\psi_{\mathrm{q}}\circ\mathrm{F}_{\mathrm{q}-1}\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) &\geq \psi_{\mathrm{q}}\circ\mathrm{G}_1\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\daleth(\mathrm{x}_i)\right)\\ &\geq \psi_{\mathrm{q}}\circ\mathrm{G}_2\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{F}_1(\mathrm{x}_i)\right)\\ \geq\cdots\geq \psi_{\mathrm{q}}\circ\mathrm{G}_{\mathrm{q}-1}\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{F}_{\mathrm{q}-2}(\mathrm{x}_i)\right) &\geq \psi_{\mathrm{q}}\circ\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{F}_{\mathrm{q}-1}(\mathrm{x}_i)\right).\end{aligned} \tag{3.9}$$

By using MC, (3.3), (3.4) and (3.5) we have $(\mathrm{F}_{\mathrm{q}-1}(\mathrm{x}_1),\ldots\mathrm{F}_{\mathrm{q}-1}(\mathrm{x}_n))$ be monotonic $n-tuple$ in $\mathrm{T}_{\mathrm{q}}^n$. *i.e.*, $\mathrm{F}_{\mathrm{q}-1}(\mathrm{x}_1)\leq\cdots\leq\mathrm{F}_{\mathrm{q}-1}(\mathrm{x}_n)$ or $\mathrm{F}_{\mathrm{q}-1}(\mathrm{x}_1)\geq\cdots\geq\mathrm{F}_{\mathrm{q}-1}(\mathrm{x}_n)$.
Since $\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{F}_{\mathrm{q}-1}(\mathrm{x}_i) \in \mathrm{T}_{\mathrm{q}}$, and $\mathrm{l}\in\{1,\ldots,n\}$ which satisfies condition

(1.6), thus, by using Proposition 1.6 for convex function ψ_{q} we have

$$\begin{aligned}\mathrm{F_q}\left(\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i\right)&\geq \mathrm{G}_1\left(\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\beth(\mathrm{x}_i)\right)\geq \mathrm{G}_2\left(\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{F}_1(\mathrm{x}_i)\right)\\ &\geq\cdots\geq \mathrm{G_q}\left(\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{F}_{\mathrm{q}-1}(\mathrm{x}_i)\right)\geq\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{F_q}(\mathrm{x}_i).\end{aligned}\tag{3.10}$$

□

Also it is clear that Theorem 3.3 can be extended to the underlaying general result.

Theorem 3.6. *Let* $\beth:[\varepsilon,\theta]\to\mathbb{R}$, $\psi_{\mathrm{k}}:[\varepsilon_{\mathrm{k}},\theta_{\mathrm{k}}]\to\mathbb{R}$, $\mathrm{k}\in\{1,\ldots,\mathrm{q}\}$, $(\mathrm{q}\in N)$ *be either concave or convex satisfying* (3.6)*. Assume that* $\beth$ *and* ψ_{k}, $\mathrm{k}\in\{1,\ldots,\mathrm{q}\}$ *satisfies MC and also assume* $\beth$ *to be monotonic function. Let monotonic* $n-$ *tuple* $\mathbf{x}$ *is such that* $\mathrm{x}_i\in[\varepsilon,\theta]$ *and real* $n-$ *tuple* $\mathbf{w}$ *is such that* $\mathrm{W}_n>0$, $\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i\in[\varepsilon,\theta]$ *and* $\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\beth(\mathrm{x}_i)\in[\varepsilon_1,\theta_1]$*. We further choose* $\mathrm{l}\in\{1,\ldots,n\}$ *such that condition* (1.6) *holds. Define the auxiliary functions* $\mathrm{F_k}:[\varepsilon,\theta]\to\mathbb{R}$ *by* (3.4) *and* $\mathrm{G_k}:[\varepsilon_{\mathrm{k}},\theta_{\mathrm{k}}]\to\mathbb{R}$ *by* (3.5) *such that* $\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{F_k}(\mathrm{x}_i)\in[\varepsilon_{\mathrm{k}+1},\theta_{\mathrm{k}+1}]$ *for* $\mathrm{k}\in\{1,\ldots,\mathrm{q}-1\}$*.*

(i) If ψ_{q} *is convex on* $[\varepsilon_{\mathrm{q}},\theta_{\mathrm{q}}]$*, then*

$$\begin{aligned}\mathrm{F_q}\left(\varepsilon+\theta-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i\right)&\leq \mathrm{G}_1\left(\beth(\varepsilon)+\beth(\theta)-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\beth(\mathrm{x}_i)\right)\\ &\leq \mathrm{G}_2\left(\mathrm{F}_1(\varepsilon)+\mathrm{F}_1(\theta)-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{F}_1(\mathrm{x}_i)\right)\\ &\leq\cdots\leq \mathrm{G_q}\left(\mathrm{F}_{\mathrm{q}-1}(\varepsilon)+\mathrm{F}_{\mathrm{q}-1}(\theta)-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{F}_{\mathrm{q}-1}(\mathrm{x}_i)\right)\\ &\leq \mathrm{F_q}(\varepsilon)+\mathrm{F_q}(\theta)-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{F_q}(\mathrm{x}_i),\end{aligned}\tag{3.11}$$

where $\mathrm{G_k}$, $\forall\ \mathrm{k}\in\{1,\ldots,\mathrm{q}\}$ *be defined by* (3.5)*.*
(i) If ψ_{q} *is concave on* $[\varepsilon_{\mathrm{q}},\theta_{\mathrm{q}}]$*, then reversed of* (3.11) *inequalities hold.*

Proof. Simply follow proof of Theorem 3.5 but rather using “Proposition 1.6” and “Theorem 3.2” use “Theorem 2.2” and “Theorem 3.3” respectively. □

4. Companion inequalities related to the jensen–mercer inequality

A key feature of the function which is convex is the right and left derivatives existence on T^0 [28, pp. 4–12]. If $\beth$ is convex on T, then for any $\mathrm{x}\in\mathrm{T}^0$ the right derivative $\beth'_{+}(\mathrm{x})$ and the left derivative $\beth'_{-}(\mathrm{x})$ are increasing on T^0 and

$$\beth'_{-}(\mathrm{x}) \leq \beth'_{+}(\mathrm{x}) \text{ for all } \mathrm{x} \in \mathrm{T}^0.$$

It can also be shown that for any convex function $\beth$ the inequalities

$$\beth(\eth) + \phi^*(\eth)(\Bbbk - \eth) \leq \beth(\Bbbk), \quad \phi^*(\eth) \in [\beth'_{-}(\eth), \beth'_{+}(\eth)] \tag{4.1}$$

$$\beth(\Bbbk) \leq \beth(\eth) + \phi^*(\Bbbk)(\Bbbk - \eth), \quad \phi^*(\Bbbk) \in [\beth'_{-}(\eth), \beth'_{+}(\eth)] \tag{4.2}$$

hold for all $\Bbbk, \eth \in \mathrm{T}^0$.
One output of (4.1) and (4.2) is that $\beth : \mathrm{T} \to \mathbb{R}$ is convex function if and only if there is at least one line of support for $\beth$ at each $\mathrm{x}_o \in \mathrm{T}^0$. Additionally, $\beth$ is differentiable function if and only if the line of support at $\mathrm{x}_o \in \mathrm{T}^0$ is unique. In such a case, the support line is

$$\mathrm{A}(\mathrm{x}) = \beth(\mathrm{x}_o) + \beth^{'}(\mathrm{x}_o)(\mathrm{x} - \mathrm{x}_o).$$

In [3] a couple of companion inequalities to Mercer's inequality were proved under Jensen and Jensen–Steffensen condition. In the following theorem, we give companion inequalities to Mercer's inequality under reversed Jensen– Steffensen condition. For this purpose, we need following prerequisite constructions. Let the function $\beth : \mathrm{I} \to \mathbb{R}$ is convex. For $\beth'(\mathrm{x})$, where $\mathrm{x} \in \mathrm{T}$, we may take any element of $[\beth'_{-}(\mathrm{x}), \beth'_{+}(\mathrm{x})]$, however without loss of generality we can take $\beth'(\mathrm{x}) = \beth'_{+}(\mathrm{x})$(for the differentiability of $\beth$ we have of course $\beth'_{+}(\mathrm{x}) = \beth'_{-}(\mathrm{x}) = \beth'(\mathrm{x})$).

Theorem 4.1. *Let the monotonic $n-tuple$ $\mathbf{x}$ is such that $\mathrm{x}_i \in [\varepsilon, \theta]$ and real $n-tuple$ $\mathbf{w}$ is such that $\mathrm{W}_n > 0$ and $\Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \mathrm{x}_i \in [\varepsilon, \theta]$. We further choose $\mathrm{l} \in \{1, \ldots, n\}$ such that condition (1.6) holds. Then for any function $\beth : \mathrm{T} \to \mathbb{R}$ which is convex the following inequalities*

$$\begin{aligned}
&\beth(\mathrm{h}) + \beth'(\mathrm{h}) \left(\varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \mathrm{x}_i - \mathrm{h} \right) \\
&\leq \beth(\varepsilon) + \beth(\theta) - \Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \beth(\mathrm{x}_i) \\
&\leq \beth(\mathrm{d}) + \beth'(\varepsilon)(\varepsilon - \mathrm{d}) - \beth' \left(\Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \mathrm{x}_i \right) \left(\Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \mathrm{x}_i - \mathrm{d} \right) + \beth'(\theta)(\theta - \mathrm{d}) \\
&+ \beth \left(\Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \mathrm{x}_i \right) - \Upsilon \sum_{i \in \Theta_n} \mathrm{w}_i \beth(\mathrm{x}_i)
\end{aligned} \tag{4.3}$$

hold, for all $\mathrm{h}, \mathrm{d} \in [\varepsilon, \theta]$.

Proof. If in (4.1) we put $\eth = \mathrm{h}$ and $\Bbbk = \varepsilon + \theta - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i$, then we have

$$\begin{aligned}
&\daleth(\mathrm{h}) + \daleth'(\mathrm{h})\left(\varepsilon + \theta - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i - \mathrm{h}\right)\\
&\qquad\qquad \leq \daleth\left(\varepsilon + \theta - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right)\\
&\qquad\qquad\qquad\qquad \leq \daleth(\varepsilon) + \daleth(\theta) - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\daleth(\mathrm{x}_i). \qquad (4.4)
\end{aligned}$$

The second inequality in (4.4) is due to Theorem 2.2. Now we prove the right hand inequality of (4.3). Let $\Upsilon \sum\limits_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i$, $\mathrm{d} \in [\varepsilon, \theta]$.

We consider two cases.

Case1. $\Upsilon \sum\limits_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i \geq \mathrm{d}$. From (4.2) we have

$$\begin{aligned}
&\daleth(\varepsilon) - \daleth(\mathrm{d}) \leq \daleth'(\varepsilon)(\varepsilon - \mathrm{d})\\
&\daleth(\theta) - \daleth\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) \leq \daleth'(\theta)\left(\theta - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right).
\end{aligned}$$

Consider

$$\begin{aligned}
&\daleth(\varepsilon) + \daleth(\theta) - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\daleth(\mathrm{x}_i) \qquad (4.5)\\
&= \daleth(\mathrm{d}) + \daleth(\varepsilon) - \daleth(\mathrm{d}) + \daleth(\theta) - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\daleth(\mathrm{x}_i)\\
&\leq \daleth(\mathrm{d}) + \daleth'(\varepsilon)(\varepsilon - \mathrm{d}) + \daleth'(\theta)\left(\theta - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right)\\
&+ \daleth\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\daleth(\mathrm{x}_i)\\
&= \daleth(\mathrm{d}) + \daleth'(\varepsilon)(\varepsilon - \mathrm{d}) + \daleth'(\theta)(\theta - \mathrm{d}) - \daleth'(\theta)\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i - \mathrm{d}\right)\\
&+ \daleth\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\daleth(\mathrm{x}_i).
\end{aligned}$$

Since $\Upsilon \sum\limits_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i \leq \theta$, and derivative $\beth'$ is nondecreasing due to convexity of $\beth$, so we have $\beth'\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) \leq \beth'(\theta)$, thus (4.5) implies

$$\begin{aligned}
&\beth(\varepsilon) + \beth(\theta) - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i) \\
&\leq \beth(\mathrm{d}) + \beth'(\varepsilon)(\varepsilon - \mathrm{d}) + \beth'(\theta)(\theta - \mathrm{d}) - \beth'\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right)\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i - \mathrm{d}\right) \\
&\qquad + \beth\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i). \qquad (4.6)
\end{aligned}$$

Case2. $\Upsilon \sum\limits_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i \leq \mathrm{d}$.

$$\begin{aligned}
&\beth(\varepsilon) + \beth(\theta) - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i) \\
&= \beth(\mathrm{d}) + \beth(\varepsilon) + \beth(\theta) - \beth(\mathrm{d}) - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i) \\
&\leq \beth(\mathrm{d}) + \beth'(\varepsilon)\left(\varepsilon - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) + \beth'(\theta)(\theta - \mathrm{d})) \\
&\qquad + \beth\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i) \\
&= \beth(\mathrm{d}) + \beth'(\varepsilon)\left(\varepsilon - \mathrm{d}\right) + \beth'(\theta)(\theta - \mathrm{d}) + \beth'(\varepsilon)\left(\mathrm{d} - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) \\
&\qquad + \beth\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i). \qquad (4.7)
\end{aligned}$$

Since $\varepsilon \leq \Upsilon \sum\limits_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i$, and derivative $\beth'$ is nondecreasing due to convexity of $\beth$, so we have $\beth'(\varepsilon) \leq \beth'\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right)$, thus (4.7) implies

$$\begin{aligned}
&\beth(\varepsilon) + \beth(\theta) - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i) \\
&\leq \beth(\mathrm{d}) + \beth'(\varepsilon)\left(\varepsilon - \mathrm{d}\right) + \beth'(\theta)(\theta - \mathrm{d}) - \beth'\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right)\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i - \mathrm{d}\right) \\
&\qquad + \beth\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i), \qquad (4.8)
\end{aligned}$$

thus (4.6) is obtained again.
That is for any $\mathrm{d}, \Upsilon \sum\limits_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i \in [\varepsilon, \theta]$ the inequality in (4.8) holds. □

Corollary 4.2. *Following the assumptions of Theorem* 4.1 *we have*

$$\begin{aligned} 0 &\le \beth(\varepsilon) + \beth(\theta) - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i) - \beth(\hat{\mathrm{x}}) \\ &\le \beth'(\varepsilon)(\varepsilon-\theta) + \beth'(\theta)(\theta-\varepsilon) + \beth\left(\Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i\right) - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\beth(\mathrm{x}_i) \end{aligned} \tag{4.9}$$

where $\hat{\mathrm{x}} = \varepsilon + \theta - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i$.

Proof. Directly follows by putting for $\mathrm{h} = \mathrm{d} = \hat{\mathrm{x}}$ in (4.3). □

Remark 4.3. It is note worthy that all the results presented in Section 3 and Section 4 can also be obtained for Proposition 1.4 and Proposition 1.9 under specific assumptions.

5. Applications

Generalized Arithmetic Mean

$$\mathrm{A}_\mathrm{x} = \varepsilon + \theta - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i$$
$$\mathrm{A}'_\mathrm{x} = 1 - \mathrm{A}_\mathrm{x}.$$

Generalized Geometric Mean

$$\mathrm{G}_\mathrm{x} = \frac{\varepsilon\theta}{\left(\prod_{i=1}^n {\mathrm{x}_i}^{\mathrm{w}_i}\right)^{1/\mathrm{W}_n}}$$
$$\mathrm{G}'_\mathrm{x} = \frac{(1-\varepsilon)(1-\theta)}{\left(\prod_{i=1}^n (1-\mathrm{x}_i)^{\mathrm{w}_i}\right)^{1/\mathrm{W}_n}}.$$

Generalized Harmonic Mean

$$\mathrm{H}_\mathrm{x} = \left(\frac{1}{\varepsilon} + \frac{1}{\theta} - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \frac{1}{\mathrm{x}_i}\right)^{-1}$$
$$\mathrm{H}'_\mathrm{x} = 1 - \mathrm{H}_\mathrm{x}.$$

Generalized Power Mean

$$\mathrm{M}_\mathrm{x}^{[\mathrm{r}]} = \left(\varepsilon^\mathrm{r} + \theta^\mathrm{r} - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i\mathrm{x}_i^\mathrm{r}\right)^{\frac{1}{\mathrm{r}}}, \qquad \mathrm{r} \in \mathbb{R}\setminus\{0\}.$$

Theorem 5.1. *Following the suppositions of Theorem* 3.3*, additionally accompanying* $\beth : (0,1) \longrightarrow \mathbb{R}$, $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ *and* $\psi \circ \beth : (0,1) \longrightarrow \mathbb{R}$ *we obtain,* (*i*) *For* $0 < \mathrm{r} < 1$,

$$\left\{\left(\frac{1}{\varepsilon}-1\right)^{\mathrm{r}}+\left(\frac{1}{\theta}-1\right)^{\mathrm{r}}-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\left(\frac{1}{\mathrm{x}_i}-1\right)^{\mathrm{r}}\right\}^{\frac{-1}{\mathrm{r}}}$$
$$\leq\frac{\mathrm{G_x}}{\mathrm{G'_x}}\leq\frac{\mathrm{A_x}}{\mathrm{A'_x}}$$

(*ii*) *For* $\mathrm{r}=1$,

$$\frac{\mathrm{H_x}}{\mathrm{H'_x}}\leq\frac{\mathrm{G_x}}{\mathrm{G'_x}}\leq\frac{\mathrm{A_x}}{\mathrm{A'_x}}.$$

Proof. (*i*) Let $0<\mathrm{r}<1$, and we define

$$\daleth(\mathrm{x})=\ln\frac{\mathrm{x}}{1-\mathrm{x}},\quad\psi(\mathrm{x})=\exp^{-rx}$$
$$\psi\circ\daleth(\mathrm{x})=\left(\frac{1-\mathrm{x}}{\mathrm{x}}\right)^{\mathrm{r}}.$$

Then on $\mathrm{T}=(0,\frac{1}{2}]$ $\daleth$, is strictly increasing and strictly concave , while on $\mathrm{J}=\mathbb{R}$, ψ is strictly decreasing and strictly convex so (3.2) becomes

$$\left(\frac{1-\left(\varepsilon+\theta-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i\right)}{\varepsilon+\theta-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i}\right)^{\mathrm{r}}$$
$$\leq\exp-\mathrm{r}\left\{\ln\frac{\varepsilon}{1-\varepsilon}+\ln\frac{\theta}{1-\theta}-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\ln\frac{\mathrm{x}_i}{1-\mathrm{x}_i}\right\}$$
$$\leq\left(\frac{1-\varepsilon}{\varepsilon}\right)^{\mathrm{r}}+\left(\frac{1-\theta}{\theta}\right)^{\mathrm{r}}-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\left(\frac{1-\mathrm{x}_i}{\mathrm{x}_i}\right)^{\mathrm{r}}.$$

Applying ln and exp properties we get

$$\left(\frac{1-\left(\varepsilon+\theta-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i\right)}{\varepsilon+\theta-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i}\right)^{\mathrm{r}}$$
$$\leq\left(\frac{\varepsilon\theta}{(\prod_{i=1}^{n}\mathrm{x}_i{}^{\mathrm{w}_i})^{1/\mathrm{W}_n}}\Big\backslash\frac{(1-\varepsilon)(1-\theta)}{(\prod_{i=1}^{n}(1-\mathrm{x}_i)^{\mathrm{w}_i})^{1/\mathrm{W}_n}}\right)^{-\mathrm{r}}$$
$$\leq\left(\frac{1-\varepsilon}{\varepsilon}\right)^{\mathrm{r}}+\left(\frac{1-\theta}{\theta}\right)^{\mathrm{r}}-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\left(\frac{1-\mathrm{x}_i}{\mathrm{x}_i}\right)^{\mathrm{r}}.$$

Equivalently we have

$$\left(\frac{\mathrm{A'_x}}{\mathrm{A_x}}\right)^{\mathrm{r}}\leq\left(\frac{\mathrm{G'_x}}{\mathrm{G_x}}\right)^{\mathrm{r}}\leq\left(\frac{1-\varepsilon}{\varepsilon}\right)^{\mathrm{r}}+\left(\frac{1-\theta}{\theta}\right)^{\mathrm{r}}-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\left(\frac{1-\mathrm{x}_i}{\mathrm{x}_i}\right)^{\mathrm{r}}.$$

Since $\mathrm{f}(t)=t^{\frac{-1}{\mathrm{r}}}$, $t>0$ is strictly decreasing ,thus equivalently we have

$$\left\{\left(\frac{1-\varepsilon}{\varepsilon}\right)^{\mathrm{r}}+\left(\frac{1-\theta}{\theta}\right)^{\mathrm{r}}-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\left(\frac{1-\mathrm{x}_i}{\mathrm{x}_i}\right)^{\mathrm{r}}\right\}^{\frac{-1}{\mathrm{r}}}\leq\frac{\mathrm{G}_{\mathrm{x}}}{\mathrm{G}'_{\mathrm{x}}}\leq\frac{\mathrm{A}_{\mathrm{x}}}{\mathrm{A}'_{\mathrm{x}}}. \tag{5.1}$$

(ii) Taking $\mathrm{r}=1$ in (5.1) we obtain,

$$\frac{\mathrm{H}_{\mathrm{x}}}{\mathrm{H}'_{\mathrm{x}}}\leq\frac{\mathrm{G}_{\mathrm{x}}}{\mathrm{G}'_{\mathrm{x}}}\leq\frac{\mathrm{A}_{\mathrm{x}}}{\mathrm{A}'_{\mathrm{x}}}. \tag{5.2}$$

□

Remark 5.2. The right hand inequality in (5.2) is a generalized variant of weighted Ky Fan's inequality [25, pp. 25-28].

In [24] power means monotonicity properties proved. Here in Theorem 5.3 we obtained such properties as special case. Following theorem is a generalized form of monotonicity properties of Power Means established in [24].

Theorem 5.3. *Following the suppositions of Theorem* 3.3 *with* $[\varepsilon,\theta]\subseteq(0,\infty)$, $\beth:[\varepsilon,\theta]\longrightarrow[\varepsilon_1,\theta_1]$, $\psi:[\varepsilon_1,\theta_1]\longrightarrow\mathbb{R}$ *and* $\psi\circ\beth:[\varepsilon,\theta]\longrightarrow\mathbb{R}$ *we have,*

(i) *If* $\mathrm{r}<\mathrm{s}<0$

$$\mathrm{M}_{\mathrm{x}}^{[\mathrm{r}]}\leq\mathrm{M}_{\mathrm{x}}^{[\mathrm{s}]}\leq\mathrm{A}_{\mathrm{x}}.$$

(ii) *If* $\mathrm{r}<0<\mathrm{s}\leq1$

$$\mathrm{M}_{\mathrm{x}}^{[\mathrm{r}]}\leq\mathrm{G}_{\mathrm{x}}\leq\mathrm{A}_{\mathrm{x}}.$$

(iii) *If* $0<\mathrm{r}<\mathrm{s}\leq1$

$$\mathrm{M}_{\mathrm{x}}^{[\mathrm{r}]}\leq\mathrm{M}_{\mathrm{x}}^{[\mathrm{s}]}\leq\mathrm{A}_{\mathrm{x}}.$$

(iv) *If* $1\leq\mathrm{r}<\mathrm{s}$

$$\mathrm{A}_{\mathrm{x}}\leq\mathrm{M}_{\mathrm{x}}^{[\mathrm{r}]}\leq\mathrm{M}_{\mathrm{x}}^{[\mathrm{s}]}.$$

Proof. (i) Let $\mathrm{r}<\mathrm{s}<0$, $\beth:[\varepsilon,\theta]\longrightarrow[\varepsilon_1,\theta_1]$, $\psi:[\varepsilon_1,\theta_1]\longrightarrow\mathbb{R}$ and $\psi\circ\beth:[\varepsilon,\theta]\longrightarrow\mathbb{R}$ we have,

$$\beth(\mathrm{x})=\mathrm{x}^{\mathrm{s}},\qquad\psi(\mathrm{x})=\mathrm{x}^{\frac{\mathrm{r}}{\mathrm{s}}},\qquad\psi\circ\beth(\mathrm{x})=\mathrm{x}^{\mathrm{r}}.$$

Then $\beth$ and ψ are strictly convex while $\beth$ is strictly decreasing and ψ is increasing, thus (3.2) becomes

$$\left(\varepsilon+\theta-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i\right)^{\mathrm{r}}\leq\left(\varepsilon^{\mathrm{s}}+\theta^{\mathrm{s}}-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i^{\mathrm{s}}\right)^{\frac{\mathrm{r}}{\mathrm{s}}}$$
$$\leq\left(\varepsilon^{\mathrm{r}}+\theta^{\mathrm{r}}-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i^{\mathrm{r}}\right).$$

Since $\mathrm{f}(t) = t^{\frac{1}{\mathrm{r}}}$, $t > 0$ is strictly decreasing, thus equivalently we have

$$\left(\varepsilon^{\mathrm{r}} + \theta^{\mathrm{r}} - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{x}_i^{\mathrm{r}}\right)^{\frac{1}{\mathrm{r}}} \leq \left(\varepsilon^{\mathrm{s}} + \theta^{\mathrm{s}} - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{x}_i^{\mathrm{s}}\right)^{\frac{1}{\mathrm{s}}} \leq \left(\varepsilon + \theta - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{x}_i\right).$$

After substitution we get

$$\mathrm{M}_{\mathrm{x}}^{[\mathrm{r}]} \leq \mathrm{M}_{\mathrm{x}}^{[\mathrm{s}]} \leq \mathrm{A}_{\mathrm{x}}.$$

(ii) Let $\mathrm{r} < 0 < \mathrm{s} \leq 1$, $\beth : [\varepsilon, \theta] \longrightarrow [\varepsilon_1, \theta_1]$, $\psi : [\varepsilon_1, \theta_1] \longrightarrow \mathbb{R}$ and $\psi \circ \beth : [\varepsilon, \theta] \longrightarrow \mathbb{R}$ we have,

$$\beth(\mathrm{x}) = \ln \mathrm{x}^{\mathrm{s}}, \qquad \psi(\mathrm{x}) = \exp^{\frac{\mathrm{r}}{\mathrm{s}}\mathrm{x}}, \qquad \psi \circ \beth(\mathrm{x}) = \mathrm{x}^{\mathrm{r}}.$$

As $\beth$ is strictly increasing and strictly concave on the other hand ψ is decreasing and strictly convex, thus (3.2) becomes

$$\left(\varepsilon + \theta - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{x}_i\right)^{\mathrm{r}} \leq \exp^{\frac{\mathrm{r}}{\mathrm{s}}} \left(\ln \varepsilon^{\mathrm{s}} + \ln \theta^{\mathrm{s}} - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{x}_i^{\mathrm{s}}\right) \leq \left(\varepsilon^{\mathrm{r}} + \theta^{\mathrm{r}} - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{x}_i^{\mathrm{r}}\right).$$

Using basic property of exp and ln and we obtain

$$\left(\varepsilon + \theta - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{x}_i\right)^{\mathrm{r}} \leq \left(\frac{\varepsilon\theta}{(\prod_{i=1}^{n} {\mathrm{x}_i}^{\mathrm{w}_i})^{1/\mathrm{W}_n}}\right)^{\mathrm{r}} \leq \left(\varepsilon^{\mathrm{r}} + \theta^{\mathrm{r}} - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{x}_i^{\mathrm{r}}\right).$$

Since $\mathrm{f}(t) = t^{\frac{1}{\mathrm{r}}}$, $t > 0$ is strictly decreasing, thus equivalently we have

$$\left(\varepsilon^{\mathrm{r}} + \theta^{\mathrm{r}} - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{x}_i^{\mathrm{r}}\right)^{\frac{1}{\mathrm{r}}} \leq \frac{\varepsilon\theta}{(\prod_{i=1}^{n} {\mathrm{x}_i}^{\mathrm{w}_i})^{1/\mathrm{W}_n}} \leq \left(\varepsilon + \theta - \Upsilon \sum_{i\in\Theta_n} \mathrm{w}_i \mathrm{x}_i\right).$$

After substitution we get

$$\mathrm{M}_{\mathrm{x}}^{[\mathrm{r}]} \leq \mathrm{G}_{\mathrm{x}} \leq \mathrm{A}_{\mathrm{x}}.$$

(iii) Let $0 < \mathrm{r} < \mathrm{s} \leq 1$, $\beth : [\varepsilon, \theta] \longrightarrow [\varepsilon_1, \theta_1]$, $\psi : [\varepsilon_1, \theta_1] \longrightarrow \mathbb{R}$ and $\psi \circ \beth : [\varepsilon, \theta] \longrightarrow \mathbb{R}$ we have,

$$\beth(\mathrm{x}) = \mathrm{x}^{\mathrm{s}}, \qquad \psi(\mathrm{x}) = \mathrm{x}^{\frac{\mathrm{r}}{\mathrm{s}}}, \qquad \psi \circ \beth(\mathrm{x}) = \mathrm{x}^{\mathrm{r}}.$$

Then $\beth$ and ψ are strictly increasing and strictly concave so reversed of (3.2) takes the form of

$$\left(\varepsilon^{\mathrm{r}}+\theta^{\mathrm{r}}-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i^{\mathrm{r}}\right)\leq\left(\varepsilon^{\mathrm{s}}+\theta^{\mathrm{s}}-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i^{\mathrm{s}}\right)^{\frac{\mathrm{r}}{\mathrm{s}}}$$
$$\leq\left(\varepsilon+\theta-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i\right)^{\mathrm{r}}.$$

Since $\mathrm{f}(t)=t^{\frac{1}{\mathrm{r}}}$, $t>0$ is strictly increasing, thus equivalently we have

$$\left(\varepsilon^{\mathrm{r}}+\theta^{\mathrm{r}}-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i^{\mathrm{r}}\right)^{\frac{1}{\mathrm{r}}}\leq\left(\varepsilon^{\mathrm{s}}+\theta^{\mathrm{s}}-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i^{\mathrm{s}}\right)^{\frac{1}{\mathrm{s}}}$$
$$\leq\left(\varepsilon+\theta-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i\right).$$

After substitution we get

$$\mathrm{M}_{\mathrm{x}}^{[\mathrm{r}]}\leq\mathrm{M}_{\mathrm{x}}^{[\mathrm{s}]}\leq\mathrm{A}_{\mathrm{x}}.$$

(iv) Let $1\leq\mathrm{r}<\mathrm{s}$, $\beth:[\varepsilon,\theta]\longrightarrow[\varepsilon_1,\theta_1]$, $\psi:[\varepsilon_1,\theta_1]\longrightarrow\mathbb{R}$ and $\psi\circ\beth:[\varepsilon,\theta]\longrightarrow\mathbb{R}$ we have,

$$\beth(\mathrm{x})=\mathrm{x}^{\mathrm{r}},\qquad\psi(\mathrm{x})=\mathrm{x}^{\frac{\mathrm{s}}{\mathrm{r}}},\qquad\psi\circ\beth(\mathrm{x})=\mathrm{x}^{\mathrm{s}}.$$

Then $\beth$ and ψ are increasing and strictly convex so (3.2) takes the form

$$\left(\varepsilon+\theta-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i\right)^{\mathrm{s}}\leq\left(\varepsilon^{\mathrm{r}}+\theta^{\mathrm{r}}-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i^{\mathrm{r}}\right)^{\frac{\mathrm{s}}{\mathrm{r}}}$$
$$\leq\left(\varepsilon^{\mathrm{s}}+\theta^{\mathrm{s}}-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i^{\mathrm{s}}\right).$$

As $\mathrm{f}(t)=t^{\frac{1}{\mathrm{s}}}$, where $t>0$ is strictly increasing, thus equivalently we have

$$\left(\varepsilon+\theta-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i\right)\leq\left(\varepsilon^{\mathrm{r}}+\theta^{\mathrm{r}}-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i^{\mathrm{r}}\right)^{\frac{1}{\mathrm{r}}}$$
$$\leq\left(\varepsilon^{\mathrm{s}}+\theta^{\mathrm{s}}-\Upsilon\sum_{i\in\Theta_n}\mathrm{w}_i\mathrm{x}_i^{\mathrm{s}}\right)^{\frac{1}{\mathrm{s}}}.$$

equivalently we have

$$\mathrm{A}_{\mathrm{x}}\leq\mathrm{M}_{\mathrm{x}}^{[\mathrm{r}]}\leq\mathrm{M}_{\mathrm{x}}^{[\mathrm{s}]}.$$

□

Corollary 5.4. *Following the suppositions of Theorem* 3.3 *with* $[\varepsilon, \theta] \subseteq (0, \infty)$, $\beth : [\varepsilon, \theta] \longrightarrow [\varepsilon_1, \theta_1]$, $\psi : [\varepsilon_1, \theta_1] \longrightarrow \mathbb{R}$ *and* $\psi \circ \beth : [\varepsilon, \theta] \longrightarrow \mathbb{R}$ *we have,*

$$\mathrm{H_x} \leq \mathrm{G_x} \leq \mathrm{A_x}. \tag{5.3}$$

Proof. Simply follows by taking $\mathrm{r} = -1$ and $\mathrm{s} = 1$ in the assertion (ii) of Theorem 5.3. □

Remark 5.5. Inequalities (5.3) are the generalization of classical Arithmetic, Geometric and Harmonic Mean inequalities.

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