

## Mercer Type Variants of the Jensen–Steffensen Inequality-II

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**ABSTRACT.** Jensen–Mercer’s inequality for weights satisfying conditions as for the reversed Jensen–Steffensen inequality is proved. Several inequalities involving more than one monotonic function with reversed Jensen–Steffensen conditions are proved. Furthermore, a couple of general companion inequalities related to Mercer’s inequality with reversed Jensen–Steffensen conditions are presented as well. Applications for the generalization of weighted Ky Fan’s inequality, classical Power Mean, and classical Arithmetic–Geometric–Harmonic Mean inequalities are given as well.

**Keywords:** Monotonic function, Composite function, Reversed Jensen–Steffensen inequality.

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### 1. INTRODUCTION AND PRELIMINARIES

One of the most important concepts in the theory of inequalities is the notion of convex functions and note that the definition of a convex function is, in fact, an inequality. The theory of convex functions has experienced a rapid

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development. This can be attributed to several causes: first, a great many areas in modern analysis directly or indirectly involve the application of convex functions; secondly, convex functions are closely related to the theory of inequalities, and many important inequalities are consequences of the applications of convex functions.

Let us start with convex functions definition extracted from [26, p. 1](see also [25, p. 1]). Throughout the article, we use  $T$ ,  $J$  and  $[\varepsilon, \theta]$  to denote intervals in  $\mathbb{R}$ . Furthermore all over the chapter we assume that  $[\varepsilon, \theta] \subset T$  and  $\varepsilon < \theta$ .

**Definition 1.1.**  $\mathfrak{J} : T \rightarrow \mathbb{R}$  is named as convex-function if

$$\mathfrak{J}(\Upsilon\lambda_1 + (1 - \Upsilon)\lambda_2) \leq \Upsilon\mathfrak{J}(\lambda_1) + (1 - \Upsilon)\mathfrak{J}(\lambda_2) \quad (1.1)$$

holds for each  $\lambda_1, \lambda_2 \in T$  and  $\Upsilon \in [0, 1]$ .

In order to eliminate confusion, here we define some notations. For the real  $n$ -tuple  $\mathbf{w} = (w_1, \dots, w_n)$ , the following notations are valid.

$$W_k = \sum_{i \in \Theta_k} w_i, \quad k \in \{1, \dots, n\}, \quad \bar{W}_k = \sum_{i \in \Theta^k} w_i, \quad \text{and} \quad \Upsilon = \frac{1}{W_n}.$$

Of course,  $W_n = \sum_{i \in \Theta_n} w_i$ .

As aforementioned that various powerful inequalities originated due to convex functions, and Jensen inequality is one of the highly important inequalities. Jensen inequality is established in the following proposition [26, p. 43] (see also [25, p. 6]).

**Proposition 1.2.** *If  $\mathbf{w}$  is any nonnegative  $n$ -tuple such that  $W_n > 0$ , then for any  $n$ -tuple  $\mathbf{x}$  in  $T^n$  and for any function  $\mathfrak{J}$  convex on  $T$  we have*

$$\mathfrak{J}\left(\Upsilon \sum_{i \in \Theta_n} w_i x_i\right) \leq \Upsilon \sum_{i \in \Theta_n} w_i \mathfrak{J}(x_i). \quad (1.2)$$

There are many versions, variants and generalization of Jensen inequality, see for example [2], [5], [7], [8], and [10]–[23].

It is acceptable to question that in Jensen inequality the supposition “ $\mathbf{w}$  is a nonnegative  $n$ -tuple” can be loosened at the surcharge of tightening  $\mathbf{x}$  more strictly [29]. Steffensen (see [29]) was the pioneer to address this issue, but in the following proposition we stated a more generalized form (see also [26, p. 57]).

**Proposition 1.3.** *If  $\mathbf{w}$  is any real  $n$ -tuple satisfying*

$$0 \leq W_j \leq W_n, \quad (1 \leq j \leq n), \quad W_n > 0, \quad (1.3)$$

*then for any monotonic  $n$ -tuple  $\mathbf{x}$  in  $T^n$  such that  $\Upsilon \sum_{i \in \Theta_n} w_i x_i \in T$ , and for any function  $\mathfrak{J}$  convex on  $T$ , (1.2) still holds.*

Under these supposition (1.2) is regarded as Jensen–Steffensen inequality.

Note that throughout the article, when we claim that  $n$ -tuple  $\mathbf{x}$  is increasing (decreasing) it indicate that  $x_1 \leq x_2 \leq \dots \leq x_n$  ( $x_1 \geq x_2 \geq \dots \geq x_n$ ). Vasić and Pečarić in [30] state and prove the following result of reversed Jensen's inequality.

**Proposition 1.4.** *If  $\mathbf{w}$  is any real  $n$ -tuple such that  $W_n > 0$ , and*

$$w_1 > 0, \quad w_i \leq 0 \quad \text{for } i \in \{2, \dots, n\}, \quad (1.4)$$

*then for any  $n$ -tuple  $\mathbf{x}$  in  $T^n$  such that  $\Upsilon \sum_{i \in \Theta_n} w_i x_i \in T$ , and for any function  $\sqsupset$  convex on  $T$  the inequality*

$$\sqsupset \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \geq \Upsilon \sum_{i \in \Theta_n} w_i \sqsupset(x_i), \quad (1.5)$$

*holds which is actually reversed of (1.2).*

*Remark 1.5.* Popoviciu's and Aczél's inequalities can be obtained from reversed Jensen–inequality (see [25, p. 192]).

In [27] Pečarić applied the Fuchs generalization of majorization theorem [9] (see Section 2.2.6) to prove the necessary and sufficient condition for the validity of following Reversed Jensen–Steffensen inequality. (see also [26, p. 83] and [25, p. 6]).

**Proposition 1.6.** *If  $\mathbf{w}$  is any real  $n$ -tuple with  $W_n > 0$ , and  $\exists l \in \{1, \dots, n\}$  such that*

$$W_j \leq 0 \quad \text{for } j < l \quad \text{and} \quad \overline{W}_j \leq 0 \quad \text{for } j > l, \quad (1.6)$$

*then for any monotonic  $n$ -tuple  $\mathbf{x}$  in  $T^n$  such that  $\Upsilon \sum_{i \in \Theta_n} w_i x_i \in T$  and for any function  $\sqsupset$  convex on  $T$  (1.5) holds.*

Following Jensen inequality variant is given by Mercer in [23].

**Proposition 1.7.** *If  $\mathbf{w}$  is any nonnegative  $n$ -tuple such that  $W_n > 0$ , then for any  $n$ -tuple  $\mathbf{x}$  in  $[\varepsilon, \theta]^n$  and for any function  $\sqsupset$  convex on  $T$  we have*

$$\sqsupset \left( \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \leq \sqsupset(\varepsilon) + \sqsupset(\theta) - \Upsilon \sum_{i \in \Theta_n} w_i \sqsupset(x_i). \quad (1.7)$$

Inequality (1.7) is named as Jensen–Mercer inequality.

By applying some specific conditions on weights  $w_i$  for  $i \in \{1, \dots, n\}$ , we obtain different variants of Proposition 1.7.

Particularly in [1] Abramovich et. al. gave the following generalization of Proposition 1.7.

**Proposition 1.8.** *If  $\mathbf{w}$  is any real  $n$ -tuple satisfying (1.3), then for any monotonic  $n$ -tuple  $\mathbf{x}$  in  $[\varepsilon, \theta]^n$  such that  $\Upsilon \sum_{i \in \Theta_n} w_i x_i \in [\varepsilon, \theta]$ , and for any function  $\sqsupset$  convex on  $T$ , the inequality (1.7) still holds.*

In [22] Matković and Pečarić gave the following impressive characteristic of Jensen–Mercer inequality a kind of powerful feature concerning weights.

**Proposition 1.9.** *If  $n$ -tuple  $\mathbf{w}$  satisfy (1.4), then for any  $n$ -tuple  $\mathbf{x}$  in  $[\varepsilon, \theta]^n$  such that  $\Upsilon \sum_{i \in \Theta_n} w_i x_i \in [\varepsilon, \theta]$  and for any function  $\sqsupset$  convex on  $T$  the inequality (1.7) still holds.*

In current article our prime goal is to treat discrete reversed Jensen – Steffensen type inequality in a variety of ways. The article contains five sections. In Section 1, introduction and preliminaries are provided. In Section 2, we prove Jensen–Mercer inequality under reversed Jensen–Steffensen conditions. In Section 3, further generalization of Jensen–Mercer inequality involving two or more functions are given. In Section 4, we prove a couple of general companion inequalities. The last section covers our proven results applications.

For interested reader, here we mention that integral version results are published in our article [12].

## 2. MERCER'S INEQUALITY WITH REVERSED JENSEN–STEFFENSEN CONDITIONS

Before we further proceed here we need an important property of convex function proved by Mercer in [23] (see also [4, Theorem 1]).

**Lemma 2.1.** *If  $\sqsupset : [\varepsilon, \theta] \rightarrow \mathbb{R}$  is convex function, then  $\forall x \in [\varepsilon, \theta]$  we have*

$$\sqsupset(\varepsilon + \theta - x) \leq \sqsupset(\varepsilon) + \sqsupset(\theta) - \sqsupset(x). \quad (2.1)$$

Following the proof of Proposition 1.9 we state and prove Jensen–Mercer's inequality for real weights satisfying conditions as for the reversed Jensen–Steffensen inequality.

**Theorem 2.2.** *Let monotonic  $n$ -tuple  $\mathbf{x}$  is such that  $x_i \in [\varepsilon, \theta]$ , and real  $n$ -tuple  $\mathbf{w}$  is such that  $W_n > 0$  and  $\Upsilon \sum_{i \in \Theta_n} w_i x_i \in [\varepsilon, \theta]$ . We additionally choose  $l \in \{1, \dots, n\}$  in such a way that condition (1.6) holds. Then for any convex function  $\sqsupset : [\varepsilon, \theta] \rightarrow \mathbb{R}$ , the inequality (1.7) still holds.*

*Proof.* From Lemma 2.1,  $\forall x \in [\varepsilon, \theta]$  we have

$$\sqsupset(\varepsilon + \theta - x) \leq \sqsupset(\varepsilon) + \sqsupset(\theta) - \sqsupset(x).$$

Further for  $\Upsilon \sum_{i \in \Theta_n} w_i x_i \in [\varepsilon, \theta]$  we have

$$\begin{aligned} & \beth \left( \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \\ & \leq \beth(\varepsilon) + \beth(\theta) - \beth \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \\ & \leq \beth(\varepsilon) + \beth(\theta) - \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i), \end{aligned}$$

last inequality is a consequence of Proposition 1.6.  $\square$

### 3. REVERSED JENSEN–STEFFENSEN AND JENSEN–MERCER TYPE INEQUALITIES INVOLVING TWO OR MORE FUNCTIONS

In paper [1], Abramovich et. al. proved Jensen–Steffensen type inequality for a sequence of functions  $\psi_0, \psi_1, \dots, \psi_q$ ,  $q \in \mathbb{N}$  where each  $\psi_i$  is either convex or concave for  $i \in \{0, 1, \dots, q\}$  and  $\psi_i$  is monotonic for  $i \in \mathbb{N}$ . In current section, we would prove analogous results for reversed Jensen–Steffensen type inequality. Here, we need an important result for the convexity and concavity of composite functions which we would use repeatedly in our proofs, see [28, p. 16 and p. 20].

**Lemma 3.1.** *Let  $\beth : T \rightarrow \mathbb{R}$  and  $\psi : J \rightarrow \mathbb{R}$  where  $\text{range}(\beth) \subseteq J$ .*

- (a) *If either  $\beth$  is convex on  $T$  and  $\psi$  is increasing convex on  $J$ , or  $\beth$  is concave on  $T$  and  $\psi$  is decreasing convex, then the composite function  $\psi \circ \beth$  is convex on  $T$ .*
- (b) *If either  $\beth$  is convex on  $T$  and  $\psi$  is decreasing concave on  $J$ , or  $\beth$  is concave on  $T$  and  $\psi$  is increasing concave on  $J$ , then the composite function  $\psi \circ \beth$  is concave on  $T$ .*

Here we present a theorem concerning composite functions.

**Theorem 3.2.** *Let  $\beth : T \rightarrow J$  be a monotonic function and  $\psi : J \rightarrow \mathbb{R}$  be a function. Let monotonic  $n$  – tuple  $\mathbf{x}$  is such that  $x_i \in T$  and real  $n$  – tuple  $\mathbf{w}$  is such that  $W_n > 0$ , and  $\Upsilon \sum_{i \in \Theta_n} w_i x_i \in T$ . We further choose  $l \in \{1, \dots, n\}$*

*such that condition (1.6) holds:*

- (i) *If either  $\beth$  is convex on  $T$  and  $\psi$  is increasing convex on  $J$  with  $\Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \in J$ , or  $\beth$  is concave on  $T$  and  $\psi$  is decreasing convex on  $J$  with*

$\Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \in J$ , then following inequalities hold.

$$\psi \circ \beth \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \geq \psi \left( \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \right) \geq \Upsilon \sum_{i \in \Theta_n} w_i (\psi \circ \beth)(x_i). \quad (3.1)$$

(ii) If either  $\beth$  is convex on  $T$  and  $\psi$  is decreasing concave on  $J$  with  $\Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \in J$ , or  $\beth$  is concave on  $T$  and  $\psi$  is increasing concave on  $J$  with  $\Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \in J$ , then inequalities in (3.1) hold in reversed direction.

*Proof.* By Proposition 1.6, convexity of  $\beth$  yields

$$\beth \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \geq \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i).$$

Since  $\beth \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \in J$  and  $\Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \in J$ . If  $\psi$  is increasing, then we have

$$\psi \circ \beth \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \geq \psi \circ \left( \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \right).$$

Further  $\beth$  is taken as monotonic function on  $T$ ,  $\beth(\mathbf{x}) = (\beth(x_1), \dots, \beth(x_n))$  be monotonic  $n$ -tuple in  $J^n$ , i.e.,  $\beth(x_1) \leq \dots \leq \beth(x_n)$  or  $\beth(x_1) \geq \dots \geq \beth(x_n)$ . Since  $\Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \in J$ , and  $1 \in \{1, \dots, n\}$  which satisfies condition (1.6),

thus, by applying Proposition 1.6 for convex function  $\psi$  we have

$$\psi \circ \left( \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \right) \geq \Upsilon \sum_{i \in \Theta_n} w_i (\psi \circ \beth)(x_i).$$

We will obtain the desired outcome by applying the transitive property of inequalities.

The left hand inequality of (3.1) still follows by the fact that  $\psi$  is decreasing when  $\beth$  is concave and then the right hand inequality (3.1) follows by the fact that  $\psi$  is convex.

On the contrary, similar arguments provide the reversed of the inequality (3.1), i.e., either the reversed of left hand inequality of (3.1) follows by the fact that  $\psi$  is decreasing when  $\beth$  is convex and the reversed of right hand inequality of (3.1) follows by the concavity of  $\psi$ , or the reversed of left hand inequality of (3.1) follows by the fact that  $\psi$  is increasing when  $\beth$  is concave and the reversed of right hand inequality of (3.1) follows by the concavity of  $\psi$ .  $\square$

In order to give analogous result for Jensen–Mercer inequality we assume that  $\varepsilon_1 = \min[\beth(\varepsilon), \beth(\theta)]$ ,  $\theta_1 = \max[\beth(\varepsilon), \beth(\theta)]$  where  $[\varepsilon_1, \theta_1] \subset J$ .

**Theorem 3.3.** Let  $\beth : [\varepsilon, \theta] \rightarrow [\varepsilon_1, \theta_1]$  be a monotonic function and  $\psi : [\varepsilon_1, \theta_1] \rightarrow \mathbb{R}$  be a function. Let monotonic  $n$ -tuple  $\mathbf{x}$  is such that  $x_i \in [\varepsilon, \theta]$  and real  $n$ -tuple  $\mathbf{w}$  is such that  $W_n > 0$  and  $\Upsilon \sum_{i \in \Theta_n} w_i x_i \in [\varepsilon, \theta]$ . We further

choose  $l \in \{1, \dots, n\}$  such that condition (1.6) holds:

(i) If either  $\beth$  is convex on  $[\varepsilon, \theta]$  and  $\psi$  is increasing convex on  $[\varepsilon_1, \theta_1]$  with  $\Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \in [\varepsilon_1, \theta_1]$ , or  $\beth$  is concave on  $[\varepsilon, \theta]$  and  $\psi$  is decreasing convex on  $[\varepsilon_1, \theta_1]$  with  $\Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \in [\varepsilon_1, \theta_1]$ , then

$$\begin{aligned} \psi \circ \beth \left( \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \\ \leq \psi \left( \beth(\varepsilon) + \beth(\theta) - \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \right) \\ \leq (\psi \circ \beth)(\varepsilon) + (\psi \circ \beth)(\theta) - \Upsilon \sum_{i \in \Theta_n} w_i (\psi \circ \beth)(x_i). \quad (3.2) \end{aligned}$$

(ii) If either  $\beth$  is convex on  $[\varepsilon, \theta]$  and  $\psi$  is decreasing concave on  $[\varepsilon_1, \theta_1]$  with  $\Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \in [\varepsilon_1, \theta_1]$ , or  $\beth$  is concave on  $[\varepsilon, \theta]$  and  $\psi$  is increasing concave on  $[\varepsilon_1, \theta_1]$  with  $\Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \in [\varepsilon_1, \theta_1]$ , then reversed direction of inequalities in (3.2) hold.

*Proof.* Follow “Theorem 3.2” proof but rather using “Proposition 1.6” use “Theorem 2.2”.  $\square$

It is stimulating that previous results related to two functions, which are either concave or convex, could be extended by induction to any set which consists of more than two functions fulfilling some conditions. Now we consider a set which consists of  $q + 1$  functions where  $q \in \mathbb{N}$ ,

$$\beth : T \longrightarrow \mathbb{R}, \quad \psi_1 : T_1 \longrightarrow \mathbb{R}, \quad \dots, \quad \psi_q : T_q \longrightarrow \mathbb{R},$$

here  $T, T_1, \dots, T_q$  are intervals in real line such that

$$\beth(T) \subseteq T_1, \quad \psi_k(T_k) \subseteq T_{k+1}, \quad k \in \{1, \dots, q-1\}. \quad (3.3)$$

Now auxiliary functions  $F_k : T \longrightarrow \mathbb{R}$ ,  $G_k : T_k \longrightarrow \mathbb{R}$ ,  $k \in \{1, \dots, q\}$ , are well defined as

$$F_k : T \longrightarrow \mathbb{R}, \quad F_k = \psi_k \circ \psi_{k-1} \circ \dots \circ \psi_1 \circ \beth, \quad (3.4)$$

$$G_k : T_k \longrightarrow \mathbb{R}, \quad G_k = \psi_q \circ \psi_{q-1} \circ \dots \circ \psi_k, \quad (3.5)$$

and  $\forall k \in \{1, \dots, q\}$ ,  $[\varepsilon_k, \theta_k] \subseteq T_k$ , where  $\varepsilon_k = \min [\psi_{k-1}(\varepsilon), \psi_{k-1}(\theta)]$ ,  $\theta_k = \max [\psi_{k-1}(\varepsilon), \psi_{k-1}(\theta)]$  with  $\psi_0 = \sqsupset$ , and

$$\sqsupset([\varepsilon, \theta]) \subseteq [\varepsilon_1, \theta_1], \text{ and } \psi_k([\varepsilon_k, \theta_k]) \subseteq [\varepsilon_{k+1}, \theta_{k+1}], \forall k \in \{1, \dots, q-1\}. \quad (3.6)$$

Now, we are assuming that each of the investigated functions of the set  $\{\sqsupset, \psi_1, \dots, \psi_q\}$  is either concave or convex and also we assume that the following monotonicity condition is fulfilled. The following condition is taken from [1].

**Monotonicity Condition(MC).**

Indicate  $\psi_0 = \sqsupset$ . We consider that Monotonicity condition MC is satisfied by any set of functions  $\psi_0, \psi_1, \dots, \psi_q$ ,  $q \in \mathbb{N}$ , if following condition is satisfied by all pairs  $(\psi_k, \psi_{k+1})$  where for all  $k \in \{0, 1, \dots, q-1\}$ :

- (1) when both functions  $\psi_k$  and  $\psi_{k+1}$  are either convex or concave, then  $\psi_{k+1}$  is increasing;
- (2) when either  $\psi_k$  is convex and  $\psi_{k+1}$  is concave, or  $\psi_{k+1}$  is convex and  $\psi_k$  is concave, then  $\psi_{k+1}$  is decreasing.

Observe that in case MC is satisfied by the functions  $\psi_0, \psi_1, \dots, \psi_q$ , in that case all functions except could be  $\psi_0$  are monotonic. Now, we state the following proposition, for proof see [12].

**Proposition 3.4.** *Let the functions  $\sqsupset, \psi_1, \dots, \psi_q$  be as above and satisfy MC. Let  $F_k$ ,  $k = 1, \dots, q$  be defined by (3.4). Then for all  $k \in \{1, \dots, q\}$ , we have:*

- (i) *If  $\psi_k$  is convex on  $T_k$ , then  $F_k$  is convex on  $T$ ;*
- (ii) *If  $\psi_k$  is concave on  $T_k$ , then  $F_k$  is concave on  $T$ .*

Now we extend Theorem 3.2 to the following general result.

**Theorem 3.5.** *Let  $\sqsupset : T \rightarrow \mathbb{R}$  and  $\psi_k : T_k \rightarrow \mathbb{R}$ ,  $k \in \{1, \dots, q\}$ , ( $q \in \mathbb{N}$ ) be either concave or convex, satisfying (3.3) and MC. Additionally assume that  $\sqsupset$  to be monotonic. Let monotonic  $n$ -tuple  $\mathbf{x}$  is such that for all  $i \in \Theta_n$   $x_i \in T$  and real  $n$ -tuple  $\mathbf{w}$  is such that  $W_n > 0$ ,  $\Upsilon \sum_{i \in \Theta_n} w_i x_i \in T$ ,*

*and  $\Upsilon \sum_{i \in \Theta_n} w_i \sqsupset(x_i) \in T_1$ . We additionally choose  $l \in \{1, \dots, n\}$  such that condition (1.6) holds. The auxiliary functions  $F_k$  be as defined in (3.4) and  $\Upsilon \sum_{i \in \Theta_n} w_i F_k(x_i) \in T_{k+1}$  for  $k \in \{1, \dots, q-1\}$ :*

- (i) *If  $\psi_q$  is convex on  $T_q$ , then*

$$\begin{aligned} F_q \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) &\geq G_1 \left( \Upsilon \sum_{i \in \Theta_n} w_i \sqsupset(x_i) \right) \geq G_2 \left( \Upsilon \sum_{i \in \Theta_n} w_i F_1(x_i) \right) \\ &\geq \dots \geq G_q \left( \Upsilon \sum_{i \in \Theta_n} w_i F_{q-1}(x_i) \right) \geq \Upsilon \sum_{i \in \Theta_n} w_i F_q(x_i), \quad (3.7) \end{aligned}$$

where  $G_k, \forall k \in \{1, \dots, q\}$  be defined by (3.5).

(ii) If  $\psi_q$  is concave on  $T_q$ , then inequalities in (3.7) hold in reversed direction.

*Proof.* We prove this theorem by mathematical induction. For  $q = 1$  we set  $\psi = \psi_1$  and it is seen with ease that the considered statement is actually Theorem 3.2 statement. By the induction hypothesis result is true for  $q - 1$ , thus for the convexity of  $\psi_{q-1}$  we have

$$\begin{aligned} F_{q-1} \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) &\geq G_1 \left( \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \right) \geq G_2 \left( \Upsilon \sum_{i \in \Theta_n} w_i F_1(x_i) \right) \\ &\geq \dots \geq G_{q-1} \left( \Upsilon \sum_{i \in \Theta_n} w_i F_{q-2}(x_i) \right) \geq \Upsilon \sum_{i \in \Theta_n} w_i F_{q-1}(x_i), \quad (3.8) \end{aligned}$$

and the reversed inequalities hold in (3.8) if  $\psi_{q-1}$  is concave. Now we claim that each term in (3.8) belongs to  $T_q$ .

**Firstly**, we claim

$$F_{q-1} \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \in T_q.$$

Given that

$$\Upsilon \sum_{i \in \Theta_n} w_i x_i \in T.$$

Using (3.3) we have

$$\beth \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \in T_1$$

and

$$\psi_1 \circ \beth \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \in T_2,$$

by advancing in a similar way we get

$$\begin{aligned} &\psi_{q-1} \circ \psi_{q-2} \circ \dots \circ \psi_1 \circ \beth \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \\ &= F_{q-1} \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \in T_q. \end{aligned}$$

**Secondly**, we claim

$$G_1 \left( \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \right) \in T_q.$$

Given that

$$\Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \in T_1.$$

Using (3.3) we have

$$\psi_1 \left( \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \right) \in T_2$$

and

$$\psi_2 \circ \psi_1 \left( \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \right) \in T_3,$$

by advancing in a similar way we have

$$\begin{aligned} & \psi_{q-1} \circ \psi_{q-2} \circ \cdots \circ \psi_1 \left( \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \right) \\ &= G_1 \left( \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \right) \in T_q. \end{aligned}$$

**Finally,** From the given condition we have

$$\Upsilon \sum_{i \in \Theta_n} w_i F_{q-1}(x_i) \in T_q.$$

Thus we conclude that each term in (3.8) belongs to  $T_q$ . Now let  $\psi_q$  is convex and by induction hypothesis  $\psi_{q-1}$  is also convex, so by MC  $\psi_q : T_q \rightarrow \mathbb{R}$  is increasing, *i.e.*,

$$\begin{aligned} \psi_q \circ F_{q-1} \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) &\geq \psi_q \circ G_1 \left( \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \right) \\ &\geq \psi_q \circ G_2 \left( \Upsilon \sum_{i \in \Theta_n} w_i F_1(x_i) \right) \\ &\geq \cdots \geq \psi_q \circ G_{q-1} \left( \Upsilon \sum_{i \in \Theta_n} w_i F_{q-2}(x_i) \right) \geq \psi_q \circ \left( \Upsilon \sum_{i \in \Theta_n} w_i F_{q-1}(x_i) \right). \end{aligned} \tag{3.9}$$

By using MC, (3.3), (3.4) and (3.5) we have  $(F_{q-1}(x_1), \dots, F_{q-1}(x_n))$  be monotonic  $n$ -tuple in  $T_q^n$ . *i.e.*,  $F_{q-1}(x_1) \leq \cdots \leq F_{q-1}(x_n)$  or  $F_{q-1}(x_1) \geq \cdots \geq F_{q-1}(x_n)$ .

Since  $\Upsilon \sum_{i \in \Theta_n} w_i F_{q-1}(x_i) \in T_q$ , and  $l \in \{1, \dots, n\}$  which satisfies condition

(1.6), thus, by using Proposition 1.6 for convex function  $\psi_q$  we have

$$\begin{aligned} F_q \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) &\geq G_1 \left( \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \right) \geq G_2 \left( \Upsilon \sum_{i \in \Theta_n} w_i F_1(x_i) \right) \\ &\geq \cdots \geq G_q \left( \Upsilon \sum_{i \in \Theta_n} w_i F_{q-1}(x_i) \right) \geq \Upsilon \sum_{i \in \Theta_n} w_i F_q(x_i). \end{aligned} \quad (3.10)$$

□

Also it is clear that Theorem 3.3 can be extended to the underlying general result.

**Theorem 3.6.** Let  $\beth : [\varepsilon, \theta] \rightarrow \mathbb{R}$ ,  $\psi_k : [\varepsilon_k, \theta_k] \rightarrow \mathbb{R}$ ,  $k \in \{1, \dots, q\}$ , ( $q \in \mathbb{N}$ ) be either concave or convex satisfying (3.6). Assume that  $\beth$  and  $\psi_k$ ,  $k \in \{1, \dots, q\}$  satisfies MC and also assume  $\beth$  to be monotonic function. Let monotonic  $n$ -tuple  $\mathbf{x}$  is such that  $x_i \in [\varepsilon, \theta]$  and real  $n$ -tuple  $\mathbf{w}$  is such that  $W_n > 0$ ,  $\Upsilon \sum_{i \in \Theta_n} w_i x_i \in [\varepsilon, \theta]$  and  $\Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \in [\varepsilon_1, \theta_1]$ . We further choose  $l \in \{1, \dots, n\}$  such that condition (1.6) holds. Define the auxiliary functions  $F_k : [\varepsilon, \theta] \rightarrow \mathbb{R}$  by (3.4) and  $G_k : [\varepsilon_k, \theta_k] \rightarrow \mathbb{R}$  by (3.5) such that  $\Upsilon \sum_{i \in \Theta_n} w_i F_k(x_i) \in [\varepsilon_{k+1}, \theta_{k+1}]$  for  $k \in \{1, \dots, q-1\}$ .

(i) If  $\psi_q$  is convex on  $[\varepsilon_q, \theta_q]$ , then

$$\begin{aligned} F_q \left( \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) &\leq G_1 \left( \beth(\varepsilon) + \beth(\theta) - \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \right) \\ &\leq G_2 \left( F_1(\varepsilon) + F_1(\theta) - \Upsilon \sum_{i \in \Theta_n} w_i F_1(x_i) \right) \\ &\leq \cdots \leq G_q \left( F_{q-1}(\varepsilon) + F_{q-1}(\theta) - \Upsilon \sum_{i \in \Theta_n} w_i F_{q-1}(x_i) \right) \\ &\leq F_q(\varepsilon) + F_q(\theta) - \Upsilon \sum_{i \in \Theta_n} w_i F_q(x_i), \end{aligned} \quad (3.11)$$

where  $G_k$ ,  $\forall k \in \{1, \dots, q\}$  be defined by (3.5).

(i) If  $\psi_q$  is concave on  $[\varepsilon_q, \theta_q]$ , then reversed of (3.11) inequalities hold.

*Proof.* Simply follow proof of Theorem 3.5 but rather using “Proposition 1.6” and “Theorem 3.2” use “Theorem 2.2” and “Theorem 3.3” respectively. □

#### 4. COMPANION INEQUALITIES RELATED TO THE JENSEN–MERCER INEQUALITY

A key feature of the function which is convex is the right and left derivatives existence on  $T^0$  [28, pp. 4–12]. If  $\beth$  is convex on  $T$ , then for any  $x \in T^0$  the

right derivative  $\varpi'_+(x)$  and the left derivative  $\varpi'_-(x)$  are increasing on  $T^0$  and

$$\varpi'_-(x) \leq \varpi'_+(x) \text{ for all } x \in T^0.$$

It can also be shown that for any convex function  $\varpi$  the inequalities

$$\varpi(\vartheta) + \phi^*(\vartheta)(\mathbb{k} - \vartheta) \leq \varpi(\mathbb{k}), \quad \phi^*(\vartheta) \in [\varpi'_-(\vartheta), \varpi'_+(\vartheta)] \quad (4.1)$$

$$\varpi(\mathbb{k}) \leq \varpi(\vartheta) + \phi^*(\mathbb{k})(\mathbb{k} - \vartheta), \quad \phi^*(\mathbb{k}) \in [\varpi'_-(\vartheta), \varpi'_+(\vartheta)] \quad (4.2)$$

hold for all  $\mathbb{k}, \vartheta \in T^0$ .

One output of (4.1) and (4.2) is that  $\varpi : T \rightarrow \mathbb{R}$  is convex function if and only if there is at least one line of support for  $\varpi$  at each  $x_o \in T^0$ . Additionally,  $\varpi$  is differentiable function if and only if the line of support at  $x_o \in T^0$  is unique. In such a case, the support line is

$$A(x) = \varpi(x_o) + \varpi'(x_o)(x - x_o).$$

In [3] a couple of companion inequalities to Mercer's inequality were proved under Jensen and Jensen–Steffensen condition. In the following theorem, we give companion inequalities to Mercer's inequality under reversed Jensen–Steffensen condition. For this purpose, we need following prerequisite constructions. Let the function  $\varpi : I \rightarrow \mathbb{R}$  is convex. For  $\varpi'(x)$ , where  $x \in T$ , we may take any element of  $[\varpi'_-(x), \varpi'_+(x)]$ , however without loss of generality we can take  $\varpi'(x) = \varpi'_+(x)$  (for the differentiability of  $\varpi$  we have of course  $\varpi'_+(x) = \varpi'_-(x) = \varpi'(x)$ ).

**Theorem 4.1.** *Let the monotonic  $n$ -tuple  $\mathbf{x}$  is such that  $x_i \in [\varepsilon, \theta]$  and real  $n$ -tuple  $\mathbf{w}$  is such that  $W_n > 0$  and  $\Upsilon \sum_{i \in \Theta_n} w_i x_i \in [\varepsilon, \theta]$ . We further choose  $l \in \{1, \dots, n\}$  such that condition (1.6) holds. Then for any function  $\varpi : T \rightarrow \mathbb{R}$  which is convex the following inequalities*

$$\begin{aligned} & \varpi(h) + \varpi'(h) \left( \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i - h \right) \\ & \leq \varpi(\varepsilon) + \varpi(\theta) - \Upsilon \sum_{i \in \Theta_n} w_i \varpi(x_i) \\ & \leq \varpi(d) + \varpi'(\varepsilon)(\varepsilon - d) - \varpi' \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i - d \right) + \varpi'(\theta)(\theta - d) \\ & \quad + \varpi \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) - \Upsilon \sum_{i \in \Theta_n} w_i \varpi(x_i) \end{aligned} \quad (4.3)$$

hold, for all  $h, d \in [\varepsilon, \theta]$ .

*Proof.* If in (4.1) we put  $\vartheta = h$  and  $k = \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i$ , then we have

$$\begin{aligned} \mathfrak{J}(h) + \mathfrak{J}'(h) \left( \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i - h \right) \\ \leq \mathfrak{J} \left( \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \\ \leq \mathfrak{J}(\varepsilon) + \mathfrak{J}(\theta) - \Upsilon \sum_{i \in \Theta_n} w_i \mathfrak{J}(x_i). \quad (4.4) \end{aligned}$$

The second inequality in (4.4) is due to Theorem 2.2. Now we prove the right hand inequality of (4.3). Let  $\Upsilon \sum_{i \in \Theta_n} w_i x_i, d \in [\varepsilon, \theta]$ .

We consider two cases.

**Case1.**  $\Upsilon \sum_{i \in \Theta_n} w_i x_i \geq d$ . From (4.2) we have

$$\begin{aligned} \mathfrak{J}(\varepsilon) - \mathfrak{J}(d) &\leq \mathfrak{J}'(\varepsilon)(\varepsilon - d) \\ \mathfrak{J}(\theta) - \mathfrak{J} \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) &\leq \mathfrak{J}'(\theta) \left( \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right). \end{aligned}$$

Consider

$$\begin{aligned} &\mathfrak{J}(\varepsilon) + \mathfrak{J}(\theta) - \Upsilon \sum_{i \in \Theta_n} w_i \mathfrak{J}(x_i) \quad (4.5) \\ &= \mathfrak{J}(d) + \mathfrak{J}(\varepsilon) - \mathfrak{J}(d) + \mathfrak{J}(\theta) - \Upsilon \sum_{i \in \Theta_n} w_i \mathfrak{J}(x_i) \\ &\leq \mathfrak{J}(d) + \mathfrak{J}'(\varepsilon)(\varepsilon - d) + \mathfrak{J}'(\theta) \left( \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \\ &\quad + \mathfrak{J} \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) - \Upsilon \sum_{i \in \Theta_n} w_i \mathfrak{J}(x_i) \\ &= \mathfrak{J}(d) + \mathfrak{J}'(\varepsilon)(\varepsilon - d) + \mathfrak{J}'(\theta)(\theta - d) - \mathfrak{J}'(\theta) \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i - d \right) \\ &\quad + \mathfrak{J} \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) - \Upsilon \sum_{i \in \Theta_n} w_i \mathfrak{J}(x_i). \end{aligned}$$

Since  $\Upsilon \sum_{i \in \Theta_n} w_i x_i \leq \theta$ , and derivative  $\beth'$  is nondecreasing due to convexity of  $\beth$ , so we have  $\beth'(\Upsilon \sum_{i \in \Theta_n} w_i x_i) \leq \beth'(\theta)$ , thus (4.5) implies

$$\begin{aligned} & \beth(\varepsilon) + \beth(\theta) - \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \\ & \leq \beth(d) + \beth'(\varepsilon)(\varepsilon - d) + \beth'(\theta)(\theta - d) - \beth' \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i - d \right) \\ & \quad + \beth \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) - \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i). \quad (4.6) \end{aligned}$$

**Case2.**  $\Upsilon \sum_{i \in \Theta_n} w_i x_i \leq d$ .

$$\begin{aligned} & \beth(\varepsilon) + \beth(\theta) - \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \\ & = \beth(d) + \beth(\varepsilon) + \beth(\theta) - \beth(d) - \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \\ & \leq \beth(d) + \beth'(\varepsilon) \left( \varepsilon - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) + \beth'(\theta)(\theta - d) \\ & \quad + \beth \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) - \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \\ & = \beth(d) + \beth'(\varepsilon)(\varepsilon - d) + \beth'(\theta)(\theta - d) + \beth'(\varepsilon) \left( d - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \\ & \quad + \beth \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) - \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i). \quad (4.7) \end{aligned}$$

Since  $\varepsilon \leq \Upsilon \sum_{i \in \Theta_n} w_i x_i$ , and derivative  $\beth'$  is nondecreasing due to convexity of  $\beth$ , so we have  $\beth'(\varepsilon) \leq \beth'(\Upsilon \sum_{i \in \Theta_n} w_i x_i)$ , thus (4.7) implies

$$\begin{aligned} & \beth(\varepsilon) + \beth(\theta) - \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \\ & \leq \beth(d) + \beth'(\varepsilon)(\varepsilon - d) + \beth'(\theta)(\theta - d) - \beth' \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i - d \right) \\ & \quad + \beth \left( \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) - \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i), \quad (4.8) \end{aligned}$$

thus (4.6) is obtained again.

That is for any  $d, \Upsilon \sum_{i \in \Theta_n} w_i x_i \in [\varepsilon, \theta]$  the inequality in (4.8) holds.  $\square$

**Corollary 4.2.** *Following the assumptions of Theorem 4.1 we have*

$$\begin{aligned} 0 &\leq \beth(\varepsilon) + \beth(\theta) - \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) - \beth(\hat{x}) \\ &\leq \beth'(\varepsilon)(\varepsilon - \theta) + \beth'(\theta)(\theta - \varepsilon) + \beth\left(\Upsilon \sum_{i \in \Theta_n} w_i x_i\right) - \Upsilon \sum_{i \in \Theta_n} w_i \beth(x_i) \end{aligned} \quad (4.9)$$

where  $\hat{x} = \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i$ .

*Proof.* Directly follows by putting for  $h = d = \hat{x}$  in (4.3).  $\square$

*Remark 4.3.* It is note worthy that all the results presented in Section 3 and Section 4 can also be obtained for Proposition 1.4 and Proposition 1.9 under specific assumptions.

## 5. APPLICATIONS

### Generalized Arithmetic Mean

$$\begin{aligned} A_x &= \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i \\ A'_x &= 1 - A_x. \end{aligned}$$

### Generalized Geometric Mean

$$\begin{aligned} G_x &= \frac{\varepsilon\theta}{(\prod_{i=1}^n x_i^{w_i})^{1/W_n}} \\ G'_x &= \frac{(1-\varepsilon)(1-\theta)}{(\prod_{i=1}^n (1-x_i)^{w_i})^{1/W_n}}. \end{aligned}$$

### Generalized Harmonic Mean

$$\begin{aligned} H_x &= \left( \frac{1}{\varepsilon} + \frac{1}{\theta} - \Upsilon \sum_{i \in \Theta_n} w_i \frac{1}{x_i} \right)^{-1} \\ H'_x &= 1 - H_x. \end{aligned}$$

### Generalized Power Mean

$$M_x^{[r]} = \left( \varepsilon^r + \theta^r - \Upsilon \sum_{i \in \Theta_n} w_i x_i^r \right)^{\frac{1}{r}}, \quad r \in \mathbb{R} \setminus \{0\}.$$

**Theorem 5.1.** *Following the suppositions of Theorem 3.3, additionally accompanying  $\beth : (0, 1) \rightarrow \mathbb{R}$ ,  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  and  $\psi \circ \beth : (0, 1) \rightarrow \mathbb{R}$  we obtain,*

(i) For  $0 < r < 1$ ,

$$\left\{ \left( \frac{1}{\varepsilon} - 1 \right)^r + \left( \frac{1}{\theta} - 1 \right)^r - \Upsilon \sum_{i \in \Theta_n} w_i \left( \frac{1}{x_i} - 1 \right)^r \right\}^{\frac{-1}{r}} \\ \leq \frac{G_x}{G'_x} \leq \frac{A_x}{A'_x}$$

(ii) For  $r = 1$ ,

$$\frac{H_x}{H'_x} \leq \frac{G_x}{G'_x} \leq \frac{A_x}{A'_x}.$$

*Proof.* (i) Let  $0 < r < 1$ , and we define

$$\mathfrak{J}(x) = \ln \frac{x}{1-x}, \quad \psi(x) = \exp^{-rx} \\ \psi \circ \mathfrak{J}(x) = \left( \frac{1-x}{x} \right)^r.$$

Then on  $T = (0, \frac{1}{2}]$   $\mathfrak{J}$ , is strictly increasing and strictly concave, while on  $J = \mathbb{R}$ ,  $\psi$  is strictly decreasing and strictly convex so (3.2) becomes

$$\left( \frac{1 - (\varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i)}{\varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i} \right)^r \\ \leq \exp -r \left\{ \ln \frac{\varepsilon}{1-\varepsilon} + \ln \frac{\theta}{1-\theta} - \Upsilon \sum_{i \in \Theta_n} w_i \ln \frac{x_i}{1-x_i} \right\} \\ \leq \left( \frac{1-\varepsilon}{\varepsilon} \right)^r + \left( \frac{1-\theta}{\theta} \right)^r - \Upsilon \sum_{i \in \Theta_n} w_i \left( \frac{1-x_i}{x_i} \right)^r.$$

Applying  $\ln$  and  $\exp$  properties we get

$$\left( \frac{1 - (\varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i)}{\varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i} \right)^r \\ \leq \left( \frac{\varepsilon \theta}{(\prod_{i=1}^n x_i^{w_i})^{1/W_n}} \setminus \frac{(1-\varepsilon)(1-\theta)}{(\prod_{i=1}^n (1-x_i)^{w_i})^{1/W_n}} \right)^{-r} \\ \leq \left( \frac{1-\varepsilon}{\varepsilon} \right)^r + \left( \frac{1-\theta}{\theta} \right)^r - \Upsilon \sum_{i \in \Theta_n} w_i \left( \frac{1-x_i}{x_i} \right)^r.$$

Equivalently we have

$$\left( \frac{A'_x}{A_x} \right)^r \leq \left( \frac{G'_x}{G_x} \right)^r \leq \left( \frac{1-\varepsilon}{\varepsilon} \right)^r + \left( \frac{1-\theta}{\theta} \right)^r - \Upsilon \sum_{i \in \Theta_n} w_i \left( \frac{1-x_i}{x_i} \right)^r.$$

Since  $f(t) = t^{-\frac{1}{r}}$ ,  $t > 0$  is strictly decreasing, thus equivalently we have

$$\left\{ \left( \frac{1-\varepsilon}{\varepsilon} \right)^r + \left( \frac{1-\theta}{\theta} \right)^r - \Upsilon \sum_{i \in \Theta_n} w_i \left( \frac{1-x_i}{x_i} \right)^r \right\}^{-\frac{1}{r}} \leq \frac{G_x}{G'_x} \leq \frac{A_x}{A'_x}. \quad (5.1)$$

(ii) Taking  $r = 1$  in (5.1) we obtain,

$$\frac{H_x}{H'_x} \leq \frac{G_x}{G'_x} \leq \frac{A_x}{A'_x}. \quad (5.2)$$

□

*Remark 5.2.* The right hand inequality in (5.2) is a generalized variant of weighted Ky Fan's inequality [25, pp. 25-28].

In [24] power means monotonicity properties proved. Here in Theorem 5.3 we obtained such properties as special case. Following theorem is a generalized form of monotonicity properties of Power Means established in [24].

**Theorem 5.3.** *Following the suppositions of Theorem 3.3 with  $[\varepsilon, \theta] \subseteq (0, \infty)$ ,  $\beth : [\varepsilon, \theta] \longrightarrow [\varepsilon_1, \theta_1]$ ,  $\psi : [\varepsilon_1, \theta_1] \longrightarrow \mathbb{R}$  and  $\psi \circ \beth : [\varepsilon, \theta] \longrightarrow \mathbb{R}$  we have,*

(i) *If  $r < s < 0$*

$$M_x^{[r]} \leq M_x^{[s]} \leq A_x.$$

(ii) *If  $r < 0 < s \leq 1$*

$$M_x^{[r]} \leq G_x \leq A_x.$$

(iii) *If  $0 < r < s \leq 1$*

$$M_x^{[r]} \leq M_x^{[s]} \leq A_x.$$

(iv) *If  $1 \leq r < s$*

$$A_x \leq M_x^{[r]} \leq M_x^{[s]}.$$

*Proof.* (i) Let  $r < s < 0$ ,  $\beth : [\varepsilon, \theta] \longrightarrow [\varepsilon_1, \theta_1]$ ,  $\psi : [\varepsilon_1, \theta_1] \longrightarrow \mathbb{R}$  and  $\psi \circ \beth : [\varepsilon, \theta] \longrightarrow \mathbb{R}$  we have,

$$\beth(x) = x^s, \quad \psi(x) = x^{\frac{r}{s}}, \quad \psi \circ \beth(x) = x^r.$$

Then  $\beth$  and  $\psi$  are strictly convex while  $\beth$  is strictly decreasing and  $\psi$  is increasing, thus (3.2) becomes

$$\begin{aligned} \left( \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right)^r &\leq \left( \varepsilon^s + \theta^s - \Upsilon \sum_{i \in \Theta_n} w_i x_i^s \right)^{\frac{r}{s}} \\ &\leq \left( \varepsilon^r + \theta^r - \Upsilon \sum_{i \in \Theta_n} w_i x_i^r \right). \end{aligned}$$

Since  $f(t) = t^{\frac{1}{r}}$ ,  $t > 0$  is strictly decreasing, thus equivalently we have

$$\begin{aligned} \left( \varepsilon^r + \theta^r - \Upsilon \sum_{i \in \Theta_n} w_i x_i^r \right)^{\frac{1}{r}} &\leq \left( \varepsilon^s + \theta^s - \Upsilon \sum_{i \in \Theta_n} w_i x_i^s \right)^{\frac{1}{s}} \\ &\leq \left( \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right). \end{aligned}$$

After substitution we get

$$M_x^{[r]} \leq M_x^{[s]} \leq A_x.$$

(ii) Let  $r < 0 < s \leq 1$ ,  $\beth : [\varepsilon, \theta] \longrightarrow [\varepsilon_1, \theta_1]$ ,  $\psi : [\varepsilon_1, \theta_1] \longrightarrow \mathbb{R}$  and  $\psi \circ \beth : [\varepsilon, \theta] \longrightarrow \mathbb{R}$  we have,

$$\beth(x) = \ln x^s, \quad \psi(x) = \exp^{\frac{r}{s}} x, \quad \psi \circ \beth(x) = x^r.$$

As  $\beth$  is strictly increasing and strictly concave on the other hand  $\psi$  is decreasing and strictly convex, thus (3.2) becomes

$$\begin{aligned} \left( \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right)^r &\leq \exp^{\frac{r}{s}} \left( \ln \varepsilon^s + \ln \theta^s - \Upsilon \sum_{i \in \Theta_n} w_i x_i^s \right) \\ &\leq \left( \varepsilon^r + \theta^r - \Upsilon \sum_{i \in \Theta_n} w_i x_i^r \right). \end{aligned}$$

Using basic property of exp and ln and we obtain

$$\left( \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right)^r \leq \left( \frac{\varepsilon \theta}{(\prod_{i=1}^n x_i^{w_i})^{1/W_n}} \right)^r \leq \left( \varepsilon^r + \theta^r - \Upsilon \sum_{i \in \Theta_n} w_i x_i^r \right).$$

Since  $f(t) = t^{\frac{1}{r}}$ ,  $t > 0$  is strictly decreasing, thus equivalently we have

$$\left( \varepsilon^r + \theta^r - \Upsilon \sum_{i \in \Theta_n} w_i x_i^r \right)^{\frac{1}{r}} \leq \frac{\varepsilon \theta}{(\prod_{i=1}^n x_i^{w_i})^{1/W_n}} \leq \left( \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right).$$

After substitution we get

$$M_x^{[r]} \leq G_x \leq A_x.$$

(iii) Let  $0 < r < s \leq 1$ ,  $\beth : [\varepsilon, \theta] \longrightarrow [\varepsilon_1, \theta_1]$ ,  $\psi : [\varepsilon_1, \theta_1] \longrightarrow \mathbb{R}$  and  $\psi \circ \beth : [\varepsilon, \theta] \longrightarrow \mathbb{R}$  we have,

$$\beth(x) = x^s, \quad \psi(x) = x^{\frac{r}{s}}, \quad \psi \circ \beth(x) = x^r.$$

Then  $\mathfrak{J}$  and  $\psi$  are strictly increasing and strictly concave so reversed of (3.2) takes the form of

$$\begin{aligned} \left( \varepsilon^r + \theta^r - \Upsilon \sum_{i \in \Theta_n} w_i x_i^r \right) &\leq \left( \varepsilon^s + \theta^s - \Upsilon \sum_{i \in \Theta_n} w_i x_i^s \right)^{\frac{r}{s}} \\ &\leq \left( \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right)^r. \end{aligned}$$

Since  $f(t) = t^{\frac{1}{r}}$ ,  $t > 0$  is strictly increasing, thus equivalently we have

$$\begin{aligned} \left( \varepsilon^r + \theta^r - \Upsilon \sum_{i \in \Theta_n} w_i x_i^r \right)^{\frac{1}{r}} &\leq \left( \varepsilon^s + \theta^s - \Upsilon \sum_{i \in \Theta_n} w_i x_i^s \right)^{\frac{1}{s}} \\ &\leq \left( \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right). \end{aligned}$$

After substitution we get

$$M_x^{[r]} \leq M_x^{[s]} \leq A_x.$$

(iv) Let  $1 \leq r < s$ ,  $\mathfrak{J} : [\varepsilon, \theta] \rightarrow [\varepsilon_1, \theta_1]$ ,  $\psi : [\varepsilon_1, \theta_1] \rightarrow \mathbb{R}$  and  $\psi \circ \mathfrak{J} : [\varepsilon, \theta] \rightarrow \mathbb{R}$  we have,

$$\mathfrak{J}(x) = x^r, \quad \psi(x) = x^{\frac{s}{r}}, \quad \psi \circ \mathfrak{J}(x) = x^s.$$

Then  $\mathfrak{J}$  and  $\psi$  are increasing and strictly convex so (3.2) takes the form

$$\begin{aligned} \left( \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right)^s &\leq \left( \varepsilon^r + \theta^r - \Upsilon \sum_{i \in \Theta_n} w_i x_i^r \right)^{\frac{s}{r}} \\ &\leq \left( \varepsilon^s + \theta^s - \Upsilon \sum_{i \in \Theta_n} w_i x_i^s \right). \end{aligned}$$

As  $f(t) = t^{\frac{1}{s}}$ , where  $t > 0$  is strictly increasing, thus equivalently we have

$$\begin{aligned} \left( \varepsilon + \theta - \Upsilon \sum_{i \in \Theta_n} w_i x_i \right) &\leq \left( \varepsilon^r + \theta^r - \Upsilon \sum_{i \in \Theta_n} w_i x_i^r \right)^{\frac{1}{r}} \\ &\leq \left( \varepsilon^s + \theta^s - \Upsilon \sum_{i \in \Theta_n} w_i x_i^s \right)^{\frac{1}{s}}. \end{aligned}$$

equivalently we have

$$A_x \leq M_x^{[r]} \leq M_x^{[s]}.$$

□

**Corollary 5.4.** *Following the suppositions of Theorem 3.3 with  $[\varepsilon, \theta] \subseteq (0, \infty)$ ,  $\sqsupset : [\varepsilon, \theta] \longrightarrow [\varepsilon_1, \theta_1]$ ,  $\psi : [\varepsilon_1, \theta_1] \longrightarrow \mathbb{R}$  and  $\psi \circ \sqsupset : [\varepsilon, \theta] \longrightarrow \mathbb{R}$  we have,*

$$H_x \leq G_x \leq A_x. \quad (5.3)$$

*Proof.* Simply follows by taking  $r = -1$  and  $s = 1$  in the assertion (ii) of Theorem 5.3.  $\square$

**Remark 5.5.** Inequalities (5.3) are the generalization of classical Arithmetic, Geometric and Harmonic Mean inequalities.

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