Iranian Journal of Mathematical Sciences and Informatics
Vol. 21, No. 2 (2026), pp 17-32
DOI: 10.66224/ijmsi.21.2.17

Characterization of Radical of A-Ideals in MV-Modules

S. Saidi Goraghani[a*], R. A. Borzooei[b]

[a]Department of Mathematics, Farhangian University, Tehran, Iran
[b]Department of Mathematics, Faculty of Mathematical Sciences,
Shahid Beheshti University, Tehran, Iran

E-mail: siminsaidi@yahoo.com
E-mail: borzooei@sbu.ac.ir

Abstract. In [16], we characterized the elements of radical of an ideal in MV-algebras. In this paper, we try to give a characterization for elements of radical of an A-ideal in MV-modules. For this purpose, we first present the definition of the envelope of an A-ideal in MV-modules and then verify the relation between the envelope of A-ideals and the radical of A-ideals in MV-modules. Finally, in a variety of circumstances, we offer a formula for identifying the elements of radical of an A-ideal in MV-modules.

Keywords: MV-Algebra, MV-Module, Radical of A-ideal, Envelope of A-ideal.

2000 Mathematics subject classification: 06D35, 06F25.

1. Introduction

MV-algebras are algebraic structures corresponding to the Łukasiewicz infinite valued propositional calculus. Their definition was introduced by C.C. Chang [2, 3] and continued by many researcher (see [1, 4, 5, 6, 7, 8, 9, 10, 16, 14, 18, 17, 19]). These algebras have other equivalents such as CN-algebras,

*Corresponding Author

Received 08 June 2022; Accepted 24 February 2023

Wajsberg algebras, bounded commutative BCK-algebras and so on. PMV-algebras were introduced in 2001. These algebraic structures are equivalent to a certain class of l-rings with a strong unit, and the introduction of modules on such algebras seems natural. This equivalence created the idea of defining MV-modules. So Di Nola and his colleagues introduced MV-modules as an action of a PMV-algebra over an MV-algebra [6].

In all algebraic structures, the study of ideals is of particular importance. One of the most important topics related to the ideals of an algebra is the study of the decomposition of ideals. In [16], by considering definition of ideals in MV-algebras, we studied decomposition of ideals in MV-algebras. Then in an implicative MV-algebra M, we introduced the following formula to identify the elements of $rad(B)$ (the intersection of all prime ideals of M, including ideal B):

$$rad(B) = \{x \in M : \forall P \in \mathcal{PI}_I(M), \exists c \in M \setminus P \text{ such that } c \wedge x \in B\},$$

where $\mathcal{I}(M)$ is the set of all ideals of M, $\mathcal{PI}(M)$ is the set of all prime ideals of M and $\mathcal{PI}_J(M)$ is the set of all prime ideals of M that contain $J \in \mathcal{I}(M)$. Also, we presented definitions of radical of A-ideals in MV-modules by prime A-ideals (it was defined by maximal A-ideals, too [10]) and primary decomposition of them.

In this paper, to complete previous studies, we try to provide a formula for identifying the radical elements of an A-ideal. For this purpose, we study A-ideals, the envelopes of A-ideals, and the relation between the envelope of an A-ideal and the radical of an A-ideal in MV-modules. Also, in specific cases, we study the radical of quotient MV-modules. Finally, we introduce a method for identifying the elements of radical of an A-ideal in an MV-module.

2. Preliminaries

Algebraic structure $M = (M, \oplus, \diamond, 0)$ of type $(2, 1, 0)$ is called an *MV-algebra*, if it satisfies the following equations:
$(MV1)$ $(M, \oplus, 0)$ is a commutative monoid,
$(MV2)$ $(\mu^\diamond)^\diamond = \mu$,
$(MV3)$ $0^\diamond \oplus \mu = 0^\diamond$,
$(MV4)$ $(\mu^\diamond \oplus \delta)^\diamond \oplus h = (\delta^\diamond \oplus \mu)^\diamond \oplus \mu$, for every $\mu, \delta \in M$.

If we define the constant $1 = 0^\diamond$ and operations $\circledcirc$ and $\ominus$ by $\mu \circledcirc \delta = (\mu^\diamond \oplus \delta^\diamond)^\diamond$ and $\mu \ominus \delta = \mu \circledcirc \delta^\diamond$, then:
$(MV5)$ $(\mu \oplus \delta) = (\mu^\diamond \circledcirc \delta^\diamond)^\diamond$,
$(MV6)$ $\mu \oplus 1 = 1$,
$(MV7)$ $(\mu \ominus \delta) \oplus \delta = (\delta \ominus \mu) \oplus \mu$,
$(MV8)$ $\mu \oplus \mu^\diamond = 1$, for every $\mu, \delta \in M$ (see [4]).

We know that $(M, \circledcirc, 1)$ is a commutative monoid. If we define auxiliary operations $\vee$ and $\wedge$ on M by $\mu \vee \delta = (\mu \circledcirc \delta^{\diamond}) \oplus \delta$ and $\mu \wedge \delta = \mu \circledcirc (\mu^{\diamond} \oplus \delta)$, for every $\mu, \delta \in M$, then $(M, \vee, \wedge, 0)$ is a *bounded distributive lattice*.

In an MV-algebra M, the following conditions are equivalent:
(i) $\mu^{\diamond} \oplus \delta = 1$,
(ii) $\mu \circledcirc \delta^{\diamond} = 0$,
(iii) $\delta = \mu \oplus (\delta \ominus \mu)$,
(iv) $\exists \lambda \in M$ such that $\mu \oplus \lambda = \delta$, for every $\mu, \delta, \lambda \in M$.
For any two elements μ, δ of the MV-algebra M, $\mu \leq \delta$ if and only if μ, δ satisfy the above equivalent conditions $(i) - (iv)$.

An ideal of MV-algebra M is a subset I of M, satisfying the following conditions:
$(I1)$: $0 \in I$,
$(I2)$: $\mu \leq \delta$ and $\delta \in I$ imply $\mu \in I$,
$(I3)$: $\mu \oplus \delta \in I$, for every $\mu, \delta \in I$.

The intersection of any family of ideals of M is still an ideal of M. For every subset $B \subseteq M$, the intersection of all ideals of M containing B is denoted by $\prec B \succ$. A proper ideal I of M is a prime ideal of μ if and only if $\mu \ominus \delta \in I$ or $\delta \ominus \mu \in I$ (or $\mu \wedge \delta \in I$ implies that $\mu \in I$ or $\delta \in I$), for every $\mu, \delta \in M$.

In an MV-algebra M, the *distance function* $d : M \times M \to M$ is defined by $d(\mu, \delta) = (\mu \ominus \delta) \oplus (\delta \ominus \mu)$ which satisfies
(i): $d(\mu, \delta) = 0$ if and only if $\mu = \delta$,
(ii): $d(\mu, \delta) = d(\delta, \mu)$,
(iii): $d(\mu, \nu) \leq d(\mu, \delta) \oplus d(\delta, \nu)$,
(iv): $d(\mu, \delta) = d(\mu^{\diamond}, \delta^{\diamond})$,
(v): $d(\mu \oplus \nu, \delta \oplus \gamma) \leq d(\mu, \delta) \oplus d(\nu, \gamma)$, for every $\mu, \delta, \nu, \gamma \in M$.

Let B be an ideal of MV-algebra M. We denote $\mu \equiv_B \delta$ if and only if $d(\mu, \delta) \in B$, for every $\mu, \delta \in M$. So $\equiv_B$ is a congruence relation on M. The equivalence class containing μ is denoted by $\frac{\mu}{B}$ and $\frac{M}{B} = \{\frac{\mu}{B} : \mu \in M\}$. Then $(\frac{M}{B}, \oplus, \diamond, \frac{0}{B})$ is an MV-algebra, where $(\frac{\mu}{B})^{\diamond} = \frac{\mu^{\diamond}}{B}$ and $\frac{\mu}{B} \oplus \frac{\delta}{B} = \frac{\mu \oplus \delta}{B}$, for all $\mu, \delta \in M$. Let K and K' be two MV-algebras.

A mapping $\psi : K \to K'$ is called an *MV-homomorphism* if:
$(H1)$: $\psi(0) = 0$,
$(H2)$: $\psi(\mu \oplus \delta) = \psi(\mu) \oplus \psi(\delta)$,
$(H3)$: $\psi(\mu^{\diamond}) = (\psi(\mu))^{\diamond}$, for every $\mu, \delta \in K$.
If ψ is one to one (onto), then ψ is called an MV-monomorphism (epimorphism), and if ψ is onto and one to one, then ψ is called an MV-isomorphism (see [4]).

Lemma 2.1. [4] *In every MV-algebra M, the natural order " $\leq$ " has the following properties:*

(i) $\mu \leq \delta$ *if and only if* $\delta^\diamond \leq \mu^\diamond$,
(ii) if $\mu \leq \delta$, *then* $\mu \oplus \lambda \leq \delta \oplus \lambda$, *for every* $\mu, \delta, \lambda \in M$.

Lemma 2.2. [4] *Let* M *be an* MV*-algebra and* $B \subseteq M$. *If* $B = \emptyset$, *then* $\prec B \succ = \{0\}$. *If* $B \neq \emptyset$, *then*

$$\prec B \succ = \{x \in M : x \leq \underbrace{b_1 \oplus \cdots \oplus b_n}_{n\ times},\ for\ some\ b_1, \cdots b_n \in B\}.$$

Definition 2.3. [6, 7] (i) In MV-algebra M, a *partial addition* is defined as follows:
$\mu + \delta$ is defined if and only if $\mu \leq \delta^\diamond$, and in this case, $\mu + \delta = \mu \oplus \delta$, for every $\mu, \delta \in M$. Moreover, if $\lambda + \mu \leq \lambda + \delta$, then $\mu \leq \delta$, for any $\lambda \in M$.
(ii) A *product* MV*-algebra* (or PMV-algebra, for short) is an algebraic structure $A = (A, \oplus, ., ^\diamond, 0)$, where $(A, \oplus, ^\diamond, 0)$ is an MV-algebra and "." is a binary associative operation on A such that the following property is satisfied: if $\mu + \delta$ is defined, then $\mu.\lambda + \delta.\lambda$ and $\lambda.\mu + \lambda.\delta$ are defined and $(\mu + \delta).\lambda = \mu.\lambda + \delta.\lambda$, $\lambda.(\mu + \delta) = \lambda.\mu + \lambda.\delta$, for every $\mu, \delta, \lambda \in A$, where "$+$" is the partial addition on A.

A unity for the product is an element $e \in A$ such that $e.\mu = \mu.e = \mu$, for every $\mu \in A$. If A has a unity for the product, then $e = 1$. A PMV-homomorphism is an MV-homomorphism which also commutes with the product operation.

Definition 2.4. [6] Let $A = (A, \oplus, ., ^\diamond, 0)$ be a PMV-algebra, $M = (M, \oplus, ^\diamond, 0)$ be an MV-algebra and the operation $\Phi : A \times M \longrightarrow M$ be defined by $\Phi(\alpha, \mu) = \alpha\mu$, which satisfies the following axioms:
$(AM1)$ If $\mu + \delta$ is defined in M, then $\alpha\mu + \alpha\delta$ is defined in M and $\alpha(\mu + \delta) = \alpha\mu + \alpha\delta$,
$(AM2)$ If $\alpha + \gamma$ is defined in A, then $\alpha\mu + \gamma\mu$ is defined in M and $(\alpha + \gamma)\mu = \alpha\mu + \gamma\mu$,
$(AM3)$ $(\alpha.\gamma)\mu = \alpha(\gamma\mu)$, for every $\alpha, \gamma \in A$ and $\mu, \delta \in M$, where "$+$" is the partial addition on M.
Then M is called a (left) MV*-module* over A or briefly an A-module. We say M is a *unitary* MV-module if A has a unity for the product, that is
$(AM4)$ $1_A\mu = \mu$, for every $\mu \in M$.

Lemma 2.5. [6] *Let* A *be a* PMV*-algebra and* M *be an* A*-module. Then*
(a) $0\mu = 0$,
(b) $\alpha 0 = 0$,
(c) $\alpha\mu^\diamond \leq (\alpha\mu)^\diamond$,
(d) $\alpha^\diamond\mu \leq (\alpha\mu)^\diamond$,
(e) $(\alpha\mu)^\diamond = \alpha^\diamond\mu + (1\mu)^\diamond$,
(f) $\mu \leq \delta$ *implies* $\alpha\mu \leq \alpha\delta$,
(g) $a \leq \gamma$ *implies* $\alpha\mu \leq \gamma\mu$,

(h) *$\alpha(\mu \oplus y) \leq \alpha\mu \oplus \alpha\delta$,*
(i) *$d(\alpha\mu, \alpha\delta) \leq \alpha d(\mu, \delta)$,*
(j) *if $\mu \equiv_I \delta$, then $\alpha\mu \equiv_I \alpha\delta$, where I is an ideal of A,*
(k) *if M is a unitary MV-module, then $(\alpha\mu)^\diamond = \alpha^\diamond\mu + \mu^\diamond$, for every $\alpha, \gamma \in A$ and $\mu, \delta \in M$.*

Lemma 2.6. [15] *Let A be a PMV-algebra and M be an A-module. Then $d(\alpha\mu, \gamma\mu) \leq d(\alpha, \gamma)\mu$, for every $\alpha, \gamma \in A$ and $\mu \in M$.*

Definition 2.7. [6] Let A be a PMV-algebra and M be an A-module. Then an ideal $N \subseteq M$ is called an A-*ideal* of M if $(I4)$ $\alpha\mu \in N$, for every $\alpha \in A$ and $\mu \in N$. Moreover, N is called a prime A-ideal of M, if $\alpha\mu \in N$ implies $\mu \in N$ or $\alpha \in (N : M) = \{\alpha \in A : \alpha M \subseteq B\}$, for any $\alpha \in A$ and $\mu \in M$.

Definition 2.8. [15] Let A be a PMV-algebra, M be an A-module and B be an A-ideal of M. Then the radical of B is the intersection of all prime A-ideals of M including B, and is denoted by $Rad_M(B)$ or $Rad(B)$. We have $Rad(B) = B$, where B is a prime A-ideal of M. Also, if there is no prime A-ideal of M including B, then we set $Rad(B) = M$.

Definition 2.9. [1] Let A be a PMV-algebra, and M be an A-module.
(i) If $\emptyset \neq T \subseteq M$ and

$$M = \{\sum_{i=1}^{n} x_i t_i : x_i \in A, t_i \in T, \text{ for } 1 \leq i \leq n \text{ and } x_1t_1 + \cdots + x_nt_n \text{ is defined in } M\},$$

then we say that T has *generated* M, where " $+$ " is the partial addition. In this case, we set $M =< T >$ (Similarly, $< T >$ can be defined, if T is an infinite set). If T is a finite set, then M is called a *finitely generated* A-module. Also, if T has only one element, then $M =< T >$ is called a *cyclic* A-module.
(ii) Let $\emptyset \neq T \subseteq M$. We say that T is a *basis* for M if $M =< T >$ and if $\sum_{i=1}^{n} x_i t_i = 0$, where $x_i \in A$ and $t_i \in T$, then $x_i = 0$, for every $1 \leq i \leq n$ (that is, T is a linearly independent set).
(iii) If M has a nonempty basis, then M is called a *free A-module*.

Definition 2.10. [18] Let M be a unitary A-module and $\sum_{i=1}^{n} a_i^\diamond m_i \leq (\sum_{i=1}^{n} a_i m_i)^\diamond$, for every $n \in \mathbb{N}$, then M is called an $A_\mathbb{N}$-module, where $a_i \in A$ and $m_i \in M$ and $1 \leq i \leq n$.

Note: From now on, in this paper, we let A be a PMV-algebra, and M be an MV-algebra. Also, let $\mathcal{I}(A)$ ($\mathcal{I}(M)$) be the set of all ideals of A (A-ideals of M), $\mathcal{PI}(A)$ ($\mathcal{PI}(M)$) is the set of all prime ideals of A (prime A-ideals of M), and $\mathcal{PI}_J(A)$ ($\mathcal{PI}_J(M)$) is the set of all prime ideals of A that contain $J \in \mathcal{I}(A)$ (prime A-ideals of M that contain $J \in \mathcal{I}(M)$).

3. Characterization of elements of radical of A-ideals

In this section, we define the notion of the envelope of an A-ideal in MV-modules. Then by using of notion, as a fundamental result, we formulate the elements of the radical of an A-ideal in MV-modules.

Definition 3.1. Let M be an A-module and B be an A-ideal of M. Then the *envelope* of B is defined by

$$\begin{aligned} E_M(B) &= \{x \in M : \exists m \in M, a \in A; x = am \ and\ \forall P \in \mathcal{PI}(A), \\ &\quad \exists c \in A \setminus P \ such\ that\ (c \wedge a)m \in B\}. \end{aligned}$$

Remark 3.2. Let M be a unitary A-module, and suppose that A has not any prime ideal. Then by vacuity, it is proved that $E_M(B) = M$, for every A-ideal B of M.

Example 3.3. (i) Let $A = \{0, a_1, a_2, a_3\}$ and the operations "$\oplus$" and "." on A are defined as follows:

$\oplus$	0	a_1	a_2	a_3
0	0	a_1	a_2	a_3
a_1	a_1	a_1	a_3	a_3
a_2	a_2	a_3	a_2	a_3
a_3	a_3	a_3	a_3	a_3

.	0	a_1	a_2	a_3
0	0	0	0	0
a_1	0	a_1	0	a_1
a_2	0	0	a_2	a_2
a_3	0	a_1	a_2	a_3

Consider $0^\diamond = a_3$, $a_1^\diamond = a_2$, $a_2^\diamond = a_1$ and $a_3^\diamond = 0$. Then we can show that $A = (A, \oplus, ^\diamond, ., 0)$ is a PMV-algebra and $M = (A, \oplus, ^\diamond, 0)$ is an MV-algebra. Now, let the operation $\Phi : A \times M \longrightarrow M$ be defined by $\Phi(a, b) = a.b$, for every $a \in A$ and $b \in M$. It is easy to show that M is an MV-module on A and $B = \{0, a_1\}$ is an A-ideal of M. Then $E_M(B) = \{0, a_1\}$. Because A has two prime ideals $P_1 = \{0, a_1\}$ and $P_2 = \{0, a_2\}$. We have

$$\begin{aligned} E_M(B) &= \{x \in M : \exists m \in M, a \in A; x = am \ and\ for\ P = P_1, P_2 \\ &\quad \exists\ c \in A \setminus P \ such\ that\ (c \wedge a)m = 0 \ or\ a_1\}. \end{aligned}$$

Consider $x = a_1$. Then we have $a_1 = a_1.a_1$, that $a = m = a_1$. If $P = P_1$, then we have $c = a_3 \in A \setminus P_1$ such that $(a_3 \wedge a_1)a_1 \in B$. Also, if $P = P_2$, then we have $c = a_3 \in A \setminus P_2$ such that $(a_3 \wedge a_1)a_1 \in B$. Hence $a_1 \in E_M(B)$. Similarly, we see that $0 \in E_M(B)$. Note that $a_2 \notin E_M(B)$. Because we have $a_2 = a_2.a_3 = a_2.a_2 = a_3.a_2$. For example, if we consider $a_2 = a_2.a_3$ and $P = P_1$, then there is no $c \in A \setminus P_1$ such that $(c \wedge a_2)a_3 \in B$. Similarly, we can show that for other cases.

(ii) Let A be all the subsets of $\{a_1, a_2, a_3, a_4\}$, and M be all the subsets of $\{a_1, a_2, a_3\}$. If we consider $\oplus := \cup$ and $\odot := \cap$, then A is a PMV-algebra, and M is an MV-algebra. Also, M is a unitary A-module (with the external operation defined by $TX = T \cap X$, for any $T \in A$ and $X \in M$). It is easy to see that $B = \{\emptyset, \{a_2\}\}$ is an A-ideal of M, and $E_M(B) = \{\emptyset, \{a_2\}\}$.

(iii) Let A be a PMV-algebra and M be an MV-algebra. If we define $\Phi : A \times M \longrightarrow M$ by $\Phi(a,m) = am = 0$, for every $a \in A$ and $m \in M$, then M becomes an A-module. Then for every A-ideal B of M, we have $E_M(B) = \{0\}$. Because for every $0 \neq x \in M$, there is no any $a \in A$ and $m \in M$ such that $x = am$.
(iv) Let $A = \{0, a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ and the operations "$\oplus$" and "." on A be defined as follows:

$\oplus$	0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
0	0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_1	a_1	a_1	a_4	a_5	a_4	a_5	a_7	a_7
a_2	a_2	a_4	a_2	a_6	a_4	a_7	a_6	a_7
a_3	a_3	a_5	a_6	a_3	a_7	a_5	a_6	a_7
a_4	a_4	a_4	a_4	a_7	a_4	a_7	a_7	a_7
a_5	a_5	a_5	a_7	a_5	a_7	a_5	a_7	a_7
a_6	a_6	a_7	a_6	a_6	a_7	a_7	a_6	a_7
a_7	a_7	a_7	a_7	a_7	a_7	a_7	a_7	a_7

$\cdot$	0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
0	0	0	0	0	0	0	0	0
a_1	0	a_1	0	0	a_1	a_1	0	a_1
a_2	0	0	a_2	0	a_2	0	a_2	a_2
a_3	0	0	0	a_3	0	a_3	a_3	a_3
a_4	0	a_1	a_2	0	a_4	a_1	0	a_4
a_5	0	a_1	0	a_3	a_1	a_5	a_3	a_5
a_6	0	0	a_2	a_3	a_2	a_3	a_6	a_6
a_7	0	a_1	a_2	a_3	a_4	a_5	a_6	a_7

Consider $0^\diamond = a_7,\ a_1^\diamond = a_6,\ a_2^\diamond = a_5,\ a_3^\diamond = a_4,\ a_4^\diamond = a_3,\ a_5^\diamond = a_2,\ a_6^\diamond = a_1$ and $a_7^\diamond = 0$. Then it is easy to show that $A = (A, \oplus, ^\diamond, ., 0)$ is a PMV-algebra. Let $M = \{0, a_1, a_2, a_3\}$ be the MV-algebra that is defined in (i), and the operation $\Phi : A \times M \longrightarrow M$ be defined by $\Phi(a,b) = a.b$, for every $a \in A$ and $b \in M$. Then it is easy to show that M is an A-module. We have $\mathcal{I}(\mathcal{M}) = \{\{0\},\ \{0,a_1\},\ \{0,a_2\}\}$. Then $E_M(B) = B$, for every $B \in \mathcal{I}(\mathcal{M})$.

The following example shows that, $E_M(B)$ is not an A-ideal of M in general.

Example 3.4. In Example 3.3(iv). If we define $\Phi : A \times M \longrightarrow M$ by $\Phi(a,m) = a_3$, for every $a \in A$ and $m \in M$, then M is an A-module. Now, for A-ideal $B = \{0\}$ of M, we have $E_M(B) = \emptyset$ that is not an A-ideal of M.

Proposition 3.5. *Let M be a unitary A-module and B be an A-ideal of M. Then*

(i) $B \subseteq \prec E_M(B) \succ \subseteq Rad(B)$;
(ii) *If* B *is a prime* A*-ideal of* M*, then* $Rad(B) = \prec E_M(B) \succ$.

Proof. (i) Let $b \in B$. Consider $b = 1b = (1 \wedge 1)b$. Now, since $c = 1 \in A \setminus P$, for every $P \in \mathcal{PI}(A)$, we have $b \in E_M(B)$, and so $B \subseteq \prec E_M(B) \succ$.
By definitions of $\prec E_M(B) \succ$ and $Rad(B)$, it is clear that $\prec E_M(B) \succ \subseteq Rad(B)$.
(ii) Assume B is a prime A-ideal of M. Then $Rad(B) = \bigcap_{B \subseteq P \in \mathcal{PI}(M)} P = B$ and so by (i), $Rad(B) = \prec E_M(B) \succ$. □

The following example shows that in general $\prec E_M(B) \succ \nsubseteq B$ and $Rad(B) \nsubseteq \prec E_M(B) \succ$.

Example 3.6. (i) The standard PMV-algebra $[0,1]$ is a $[0,1]$-module. We set $A = M = [0,1]$. Then $B = \{0\}$ is an A-ideal of $[0,1]$. Also, $\{0\}$ is the only ideal of A. Note that $\frac{1}{3} \ominus \frac{2}{3} \notin \{0\}$. Also, $\frac{2}{3} \ominus \frac{1}{3} \notin \{0\}$. It means that $\{0\}$ is not a prime ideal of $[0,1]$. Therefore, by Remark 3.2, $E_M(B) = [0,1]$.
(ii) Let $A = \{0, e\}$ be the Boolean algebra with two elements. Consider the standard MV-algebra $M = [0,1]$. If $\Phi : A \times M \longrightarrow M$ is defined by $\Phi(0,m) = 0$ and $\Phi(e,m) = m$, then M is an A-module. Consider the A-ideal $B = \{0\}$ of M. Then we have $Rad(B) = [0,1]$ and $\prec E_M(B) \succ = \{0\}$.

Lemma 3.7. *Let* M *and* M' *be two* A*-modules, and* $\wp : M \longrightarrow M'$ *be an* A*-epimorphism. Then*
(i) $Ker\wp \subseteq E_M(\wp^{-1}(B'))$;
(ii) $E_M(\wp^{-1}(B')) \subseteq \wp^{-1}(E_{M'}(B'))$;
(iii) $\wp(E_M(B)) = E_{M'}(\wp(B))$;
(iv) $\prec \wp^{-1}(E_{M'}(B')) \succ = \prec E_M(\wp^{-1}(B')) \succ$*, where* B *is an* A*-ideal of* M *containing* $ker\wp$ *and* B' *is an* A*-ideal of* M'.

Proof. (i) We have

$$\begin{aligned} E_M(\wp^{-1}(B')) \quad = \quad & \{x \in M : \exists m \in M, a \in A; x = am \text{ and } \forall P \in \mathcal{PI}(A), \\ & \exists\, c \in A \setminus P \text{ such that } (c \wedge a)m \in \wp^{-1}(B')\} = \{x \in M : \\ & \exists\, m \in M, a \in A; x = am \text{ and } \forall P \in \mathcal{PI}(A), \exists c \in A \setminus P \\ & \text{such that } (c \wedge a)\wp(m) = \wp((c \wedge a)m) \in \wp^{-1}(B')\}. \end{aligned}$$

Now, for every $x \in Ker\wp$, we can consider $a = c = 1$ and $m = x$. Since $(1 \wedge 1)\wp(x) = 0 \in B'$, we have $x \in E_M(\wp^{-1}(B'))$. Therefore, $Ker\wp \subseteq E_M(\wp^{-1}(B'))$.

(ii) Let $x \in E_M(\wp^{-1}(B'))$. Then there are $a \in A$ and $m \in M$ such that $x = am$. Also, for every $P \in \mathcal{PI}(A)$, there is $c \in A \setminus P$ such that $(c \wedge a)m \in \wp^{-1}(B')$. It means that $(c \wedge a)\wp(m) = \wp((c \wedge a)m) \in B'$ and so $\wp(am) = a\wp(m) \in E_{M'}(B')$. It results that $x = am \in \wp^{-1}(E_{M'}(B'))$. Therefore, $E_M(\wp^{-1}(B')) \subseteq \wp^{-1}(E_{M'}(B'))$.

(iii) Let $y \in \wp(E_M(B))$. Then there is $x \in E_M(B)$ such that $\wp(x) = y$. So there is $a \in A$, $m \in M$; $x = am$ and for every $P \in \mathcal{PI}(A)$ there is $c \in A \backslash P$ such that $(c \wedge a)m \in B$. Hence $y = a\wp(m)$ and $(c \wedge a)\wp(m) \in \wp(B)$. It means that $y \in E_{M'}(\wp(B))$ and so $\wp(E_M(B)) \subseteq E_{M'}(\wp(B))$. Now, let $y \in E_{M'}(\wp(B))$. Then there is $a \in A$, $m' \in M'$; $y = am'$ and so for every $P \in \mathcal{PI}(A)$ there is $c \in A \setminus P$ such that $(c \wedge a)m' \in \wp(B)$. Since $\wp$ is onto, there is $m \in M$ such that $\wp(m) = m'$. Then $(c \wedge a)\wp(m) = \wp((c \wedge a)m) \in \wp(B)$ and so there is $x \in B$ such that $\wp((c \wedge a)m) = \wp(x)$. It results that

$$(\wp(x))^\diamond \oplus \wp((c \wedge a)m) = 1 \text{ and so } \wp(x^\diamond \oplus (c \wedge a)m) = (\wp(0))^\diamond.$$

Hence

$$\wp((x^\diamond \oplus (c \wedge a)m))^\diamond = \wp((x^\diamond \oplus (c \wedge a)m)^\diamond) = \wp(0) = 0$$

and so $(x^\diamond \oplus (c \wedge a)m)^\diamond \in ker\wp \subseteq B$. It means that $(c \wedge a)m \ominus x \in B$. Now, since $x \in B$, we have $(c \wedge a)m \in B$ and so $y \in \wp(E_M(B))$. Hence, $E_{M'}(\wp(B)) \subseteq \wp(E_M(B))$ and therefore, $\wp(E_M(B)) = E_{M'}(\wp(B))$.

(iv) Let B' be an A-ideal of M'. By (i), $E_M(\wp^{-1}(B')) \subseteq \wp^{-1}(E_{M'}(B'))$ and so

$$\prec E_M(\wp^{-1}(B')) \succ \subseteq \prec \wp^{-1}(E_{M'}(B')) \succ .$$

Now, let $x \in \wp^{-1}(E_{M'}(B'))$. Then $\wp(x) \in E_{M'}(B')$ and so there are $a \in A$, $m' \in M'$ and $c \in A \setminus P$, for all $P \in \mathcal{PI}(A)$ such that $\wp(x) = am'$ and $(c \wedge a)m' \in B'$. Since $\wp$ is onto, there is $y \in M$ such that $\wp(y) = m'$. Hence

$$\wp((c \wedge a)y) = (c \wedge a)\wp(y) = (c \wedge a)m' \in B'.$$

So $(c \wedge a)y \in \wp^{-1}(B')$ and so $ay \in E_M(\wp^{-1}(B'))$. On the other hand,

$$\begin{aligned} \wp(x \ominus ay) &= \wp(x \odot (ay)^\diamond) = \wp((x^\diamond \oplus ay)^\diamond) = (\wp(x^\diamond \oplus ay))^\diamond = (\wp(x^\diamond) \oplus \wp(ay))^\diamond \\ &= ((am')^\diamond \oplus a\wp(y))^\diamond = ((am')^\diamond \oplus am')^\diamond = 1^\diamond = 0. \end{aligned}$$

Then $x \ominus ay \in ker\wp$. By (ii), $ker\wp \subseteq \prec E_M(\wp^{-1}(B')) \succ$. Since $ay \in \prec E_M(\wp^{-1}(B')) \succ$, we have $x \in \prec E_M(\wp^{-1}(B')) \succ$. Hence, $\prec \wp^{-1}(E_{M'}(B')) \succ \subseteq \prec E_M(\wp^{-1}(B')) \succ$ and therefore,

$$\prec \wp^{-1}(E_{M'}(B')) \succ = \prec E_M(\wp^{-1}(B')) \succ .$$

□

Theorem 3.8. *Let M and M' be two A-modules, $\wp : M \longrightarrow M'$ be an A-epimorphism and B be an A-ideal of M containing $ker\wp$.*
(i) If $Rad(B) = \prec E_M(B) \succ$, then

$$Rad(\wp(B)) = \prec E_{M'}(\wp(B)) \succ$$

(ii) If B' is an A-ideal of M' and $Rad(B') = \prec (E_{M'}(B') \succ$, then

$$Rad(\wp^{-1}(B')) = \prec E_M(\wp^{-1}(B')) \succ$$

Proof. (i) It is routine to show that there is an isomorphism between all proper A-ideals of M containing B and all proper A-ideals of M' containing $\wp(B)$. Then $\wp(Rad(B)) = Rad(\wp(B))$ and by Lemma 3.7(iii),

$$Rad(\wp(B)) = \wp(Rad(B)) = \wp(\prec (E_M(B) \succ) = \prec \wp(E_M(B)) \succ = \prec E_{M'}(\wp(B)) \succ .$$

(ii) Similarly to (i), we have $\wp^{-1}(Rad(B')) = Rad(\wp^{-1}(B'))$ and by Lemma 3.7(iv),

$$\begin{aligned} Rad(\wp^{-1}(B')) &= \wp^{-1}(Rad(B')) = \wp^{-1}(\prec (E_{M'}(B') \succ) = \prec \wp^{-1}(E_{M'}(B')) \succ \\ &= \prec E_M(\wp^{-1}(B')) \succ . \end{aligned}$$

□

Proposition 3.9. (i) *Let $f : A \longrightarrow S$ be a PMV-homomorphism. Then every S-module can be considered as an A-module.*
(ii) *Let $f : A \longrightarrow S$ be a PMV-epimorphism. If for every A-module M and for every A-ideal B of M, $Rad(B) = \prec E_M(B) \succ$, then $Rad(B') = \prec E_{M'}(B') \succ$, for every S-module M' and for every A-ideal B' of M'.*

Proof. (i) Let Ω be an S-module. If we define

$$\Phi : A \times \Omega \longrightarrow \Omega \text{ by } \Phi(a, x) = f(a)x,$$

then Ω is an A-module. Because for every $a, b \in A$ and $x, y \in \Omega$, we have :
$(A\Omega1)$ Let $x + y$ is defined in Ω. Then by $(S\Omega1)$, we have

$$a(x + y) = \Phi(a, x + y) = f(a)(x + y) = f(a)x + f(a)y = ax + ay.$$

$(A\Omega2)$ Let $a + b$ is defined in A. Then $a \leq b^\diamond$ and so $a^\diamond \oplus b^\diamond = 1$. Hence

$$(f(a))^\diamond \oplus (f(b))^\diamond = f(a^\diamond) \oplus f(b^\diamond) = f(a^\diamond \oplus b^\diamond) = f(1) = 1.$$

and so $f(a) \leq (f(b))^\diamond$. It means that $f(a) + f(b)$ is defined in S. Then by $(S\Omega2)$, we have

$$(a+b)x = \Phi(a+b, x) = f(a+b)x = (f(a)+f(b))x = f(a)x+f(b)x = ax+bx.$$

$(A\Omega3)$ By $(S\Omega3)$,

$$(a.b)x = f(a.b)x = (f(a).f(b))x = f(a)(f(b)x) = a(bx).$$

(ii) By (i), every S-module can be considered as an A-module. Since f is onto, every A-module can also be considered an S-module. In other words, B (B_A) is a prime A-ideal of M as an A-module if and only if B (B_S) is a prime A-ideal of M as an S-module. Then

$$RadB'_S = RadB'_A = \prec E_M(B'_A) \succ = \prec E_M(B'_S) \succ .$$

□

Note. $IM = \{\sum_{i=1}^{n} a_i m_i : a_i \in I, m_i \in M \text{ and } n \in \mathbb{N}\}$, where I is an ideal of A.

The following example shows that in general, IM is not an A-ideal of M.

Example 3.10. In Example 3.3(iv), $I = \{0, a_2, a_3, a_6\}$ is an ideal of A. It is easy to see that $IM = \{0, a_2, a_3\}$. Note that IM is not an A-ideal of M.

Lemma 3.11. *Let B be an A-ideal of M. If $B \subseteq (B : M)M$, then*
(i) $(B : M)M$ is an A-ideal of M.
(ii) $\dfrac{M}{(B : M)M}$ is an $\dfrac{A}{(B : M)}$-module.

Proof. (i) It is easy to see that $(B : M)$ is an ideal of A. At first, we show that $(B : M)M$ is an ideal of M.
(I_1) It is clear that $0 \in (B : M)M$.
(I_2) Let $x \in M$, $y \in (B : M)M$ and $x \le y$. Since $y \in (B : M)M$, for $1 \le i \le n$, there are $a_i \in (B : M)$ and $m_i \in M$ such that $y = \sum_{i=1}^{n} a_i m_i$. Now, since $a_i \in (B : M)$, we have $a_i M \subseteq B$, for $1 \le i \le n$. It means that $y \in B$ and so $x \in B$. Since $B \subseteq (B : M)M$, we have $x \in (B : M)M$.
(I_3) Let $x, y \in (B : M)M$. Then $x = \sum_{i=1}^{n} a_i m_i$ and $y = \sum_{i=1}^{n} b_i t_i$, for some $a_i, b_i \in (B : M)$ and $m_i, t_i \in M$. By renumbering, we have $x \oplus y = \sum_{i=1}^{n} c_i k_i$, where $c_i \in (B : M)$ and $k_i \in M$. Then $x \oplus y \in (B : M)M$. Hence $(B : M)M$ is an ideal of M.
(I_4) Let $a \in A$ and $y = \sum_{i=1}^{n} a_i m_i \in (B : M)M$. By Lemma 2.5($h$), we have

$$\begin{aligned} ay &= a\sum_{i=1}^{n} a_i m_i = a(a_1 m_1 \oplus a_2 m_2 \oplus \cdots \oplus a_n m_n) \le a(a_1 m_1) \oplus a(a_2 m_2) \oplus \\ &\quad \cdots \oplus a(a_n m_n) = (a.a_1) m_1 \oplus (a.a_2) m_2 \oplus \cdots \oplus (a.a_n) m_n. \end{aligned}$$

Now, by (I_2) and (I_3), we get $ay \in (B : M)M$. Therefore, $(B : M)M$ is an A-ideal of M.
(ii) By (i), we can have quotient MV-algebra $\dfrac{M}{(B : M)M}$. Consider the map

$$\Psi : \frac{A}{(B : M)} \times \frac{M}{(B : M)M} \longrightarrow \frac{M}{(B : M)M} \text{ by } \Psi\left(\frac{a}{(B : M)}, \frac{m}{(B : M)M}\right) = \frac{am}{(B : M)M},$$

for every $\dfrac{a}{(B : M)} \in \dfrac{A}{(B : M)}$ and $\dfrac{m}{(B : M)M} \in \dfrac{M}{(B : M)M}$. At first, we show that Ψ is well defined. Let $\dfrac{a_1}{(B : M)} = \dfrac{a_2}{(B : M)}$ and $\dfrac{m_1}{(B : M)} = \dfrac{m_2}{(B : M)}$, for every $\dfrac{a_1}{(B : M)}, \dfrac{a_2}{(B : M)} \in \dfrac{A}{(B : M)}$ and $\dfrac{m_1}{(B : M)}, \dfrac{m_2}{(B : M)} \in \dfrac{M}{(B : M)M}$. Then

$$d(a_1, a_2) \in (B : M) \text{ and } d(m_1, m_2) \in (B : M)M.$$

We must prove that $d(a_1 m_1, a_2 m_2) \in (B : M)M$. We have

$$d(a_1 m_1, a_2 m_2) \le d(a_1 m_1, a_1 m_2) \oplus d(a_1 m_2, a_2 m_2).$$

By Lemmas 2.5(i) and 2.6,

$$d(a_1m_1, a_1m_2) \leq a_1d(m_1, m_2) \text{ and } d(a_1m_2, a_2m_2) \leq d(a_1, a_2)m_2.$$

Note that

$$a_1d(m_1, m_2), d(a_1, a_2)m_2 \in (B : M)M.$$

Hence $d(a_1m_1, a_2m_2) \in (B : M)M$ and so $\frac{a_1m_1}{(B:M)} = \frac{a_2m_2}{(B:M)}$. It results that Ψ is well defined. Now, we show that $\frac{M}{(B:M)M}$ is an $\frac{A}{(B:M)}$-module:

($\frac{A}{(B:M)}\frac{M}{(B:M)M}1$): Let $\frac{m_1}{(B:M)M}, \frac{m_2}{(B:M)M} \in \frac{M}{(B:M)M}$ and $\frac{a}{(B:M)} \in \frac{A}{(B:M)}$. If $\frac{m_1}{(B:M)M} + \frac{m_2}{(B:M)M}$ is defined, then

$$\frac{m_1}{(B:M)M} \leq (\frac{m_2}{(B:M)M})^{\diamond} = \frac{m_2^{\diamond}}{(B:M)M}$$

and so by Lemma 2.5(f),

$$\frac{am_1}{(B:M)M} \leq \frac{am_2^{\diamond}}{(B:M)M} \leq \frac{(am_2)^{\diamond}}{(B:M)M}.$$

It results that $\frac{am_1}{(B:M)M} + \frac{am_2}{(B:M)M}$ is defined. Now, by $(AM1)$, it is routine to see that

$$\frac{a}{(B:M)}(\frac{m_1}{(B:M)M} + \frac{m_2}{(B:M)M}) = \frac{am_1}{(B:M)M} + \frac{am_2}{(B:M)M}.$$

($\frac{A}{(B:M)}\frac{M}{(B:M)M}2$): Let $\frac{a_1}{(B:M)}, \frac{a_2}{(B:M)} \in \frac{A}{(B:M)}$ and $\frac{m}{(B:M)M} \in \frac{M}{(B:M)M}$. Also, let $\frac{a_1}{(B:M)} + \frac{a_2}{(B:M)}$ is defined. Then $\frac{a_1}{(B:M)} \leq (\frac{a_2}{(B:M)})^{\diamond}$ and so by Lemma 2.5(f, c), $\frac{a_1m}{(B:M)M} + \frac{a_2m}{(B:M)M}$ is defined. Now, by $(AM2)$, we have

$$(\frac{a_1}{(B:M)} + \frac{a_2}{(B:M)})\frac{m}{(B:M)M} = \frac{a_1m}{(B:M)M} + \frac{a_2m}{(B:M)M}.$$

($\frac{A}{(B:M)}\frac{M}{(B:M)M}3$): By $(AM3)$, the proof is easy. $\square$

In general, not necessarily $B \subseteq (B : M)M$.

Example 3.12. (i) In Example 3.3(i), let $B = \{0, a_1\}$. Then $(B : M) = B$ and $(B : M)M = \{0, a_1\}$. Note that $B \subseteq (B : M)M$.
(ii) In Example 3.3(iii), let $\{0\} \neq B$ be an A-ideal of M. Then $(B : M) = A$ and so $(B : M)M = \{0\}$. Note that $B \nsubseteq (B : M)M$.

Theorem 3.13. *Let B be an A-ideal of M and $Rad_W(C) = \prec E_W(C) \succ$, for every $\frac{A}{(B:M)}$-module W and A-ideal C of W. Then $Rad(B) = \prec E_M(B) \succ$.*

Proof. By Lemma 3.11(ii), $\dfrac{M}{(B:M)M}$ is a $\dfrac{A}{(B:M)}$-module. By the same reason, we can consider $\dfrac{M}{(B:M)M}$ as an A-module, too. We have that $f: A \longrightarrow \dfrac{A}{(B:M)}$ is a PMV-epimorphism. Since for every $\dfrac{A}{(B:M)}$-module W and A-ideal C of W, $Rad_W(C)$ is generated by $E_W(C)$, we can use Proposition 3.9(ii). So for every A-module W and A-ideal C of W, $Rad_W(C)$ is generated by $E_W(C)$, too. Now, consider the A-epimorphism $\varphi: M \longrightarrow \dfrac{M}{(B:M)M}$. Note that $\dfrac{B}{(B:M)M}$ is an A-ideal of $\dfrac{M}{(B:M)M}$. Since $B = \varphi^{-1}(\dfrac{M}{(B:M)M})$, by Theorem 3.8$(ii)$, we conclude that $Rad(B) = \prec (E_M(B) \succ$. □

In the following theorem, we present some conditions to identify the radical elements of an A-ideal in MV-modules.

Lemma 3.14. *Let A be unital, and X be a nonempty set. Then*
(i) $A^{(X)}$ (the copies of A that are indexed with X) have a nonempty basis (we denote it by $K = \{\varepsilon_x\}_{x\in X}$).
(ii) Denote $i: X \longrightarrow A^{(X)}$ the function sending x to ε_x. Then for any $f: \{\varepsilon_x\}_{x\in X} \longrightarrow G$ there exists a unique A-homomorphism $h: A^{(X)} \longrightarrow G$ such that $hoi = f$, where G is an arbitrary $A_{\mathbb{N}}$-module.
(iii) Every $A_{\mathbb{N}}$-module is a homomorphic image of a free A-module.

Proof. (i) We construct a basis for $A^{(X)}$. Let

$$\varepsilon_j(x) = \begin{cases} 1 & \text{if } j = x \\ 0 & \text{otherwise} \end{cases}$$

We show that $K = \{\varepsilon_x\}_{x\in X}$ is a basis for $A^{(X)}$. Let $\{\gamma_j\}_{j\in X} \in A^{(X)}$. Since $1.\gamma = \gamma.1 = \gamma$, for every $\gamma \in A$, we have

$$\begin{aligned} \{\gamma_j\}_{j\in X} &= \{\gamma_1, 0, \cdots\} \oplus \{0, \gamma_2, 0, \cdots\} \oplus \cdots \oplus \{0, \cdots, \gamma_n \cdots\} \\ &= \{\gamma_1.1, 0, \cdots\} \oplus \{0, \gamma_2.1, 0, \cdots\} \oplus \cdots \oplus \{0, \cdots, \gamma_n.1 \cdots\} \\ &= \gamma_1.\{1, 0, \cdots\} \oplus \gamma_2.\{0, 1, \cdots\} \oplus \cdots \oplus \gamma_n.\{0, \cdots, 1 \cdots\} \\ &= \gamma_1.\varepsilon_1 \oplus \gamma_2.\varepsilon_2 \oplus \cdots \oplus \gamma_n.\varepsilon_n \oplus \cdots. \end{aligned}$$

Note that $\gamma_1\varepsilon_1 + \cdots + \gamma_n\varepsilon_n + \cdots$ is defined, and the elements of X have been renumbered by $\mathbb{N}$. Hence, $A^{(X)} =< K >$. Now, let $\sum_{j\in X} \gamma_j.\varepsilon_j = 0$. Then we can see that $\gamma_j = 0$, for every $j \in X$.

(ii) By (i), K is a basis for $A^{(X)}$. We define

$$h: A^{(X)} \longrightarrow G \text{ by } h(\sum_{t\in X} a_t.\varepsilon_t) = \sum_{t\in X} a_t.f(\varepsilon_t),$$

where $a_t \in A$, for every $t \in X$. Let $\sum_{t\in X} a_t.\varepsilon_t = \sum_{t\in X} b_t.\varepsilon_t$, for every $a_t, b_t \in A$. Then $\{a_t\}_{t\in X} = \{b_t\}_{t\in X}$ and so $a_t = b_t$, for every $t \in X$. Hence,

$\sum_{t\in X} a_t.f(\varepsilon_t) = \sum_{t\in X} b_t.f(\varepsilon_t)$ and so h is well defined. It is clear that $h(0) = 0$. Let $\sum_{t\in X} a_t.\varepsilon_t + \sum_{t\in X} b_t.\varepsilon_t$ be defined in $A^{(X)}$. We have

$$\sum_{t\in X} a_t.\varepsilon_t + \sum_{t\in X} b_t.\varepsilon_t = \{a_t + b_t\}_{t\in X}.$$

Then $a_t + b_t$ is defined in A and so by $(AG1)$, $a_t f(\varepsilon_t) + b_t f(\varepsilon_t)$ is defined in G and

$$(a_t + b_t)f(\varepsilon_t) = a_t f(\theta_t) + b_t f(\theta_t)$$

for every $t \in X$. Hence,

$$\begin{aligned} h(\sum_{t\in X} a_t.\varepsilon_t + \sum_{t\in X} b_t.\varepsilon_t) &= h(\{a_t + b_t\}_{t\in X}) = \sum_{t\in X}(a_t + b_t)f(\varepsilon_t) \\ &= \sum_{t\in X}(a_t f(\varepsilon_t) + b_t f(\varepsilon_t)) = \sum_{t\in X}(a_t f(\varepsilon_t) \oplus b_t f(\varepsilon_t)) \\ &= \sum_{t\in X} a_t f(\varepsilon_t) \oplus \sum_{t\in X} b_t f(\varepsilon_t) = h(\sum_{t\in X} a_t.\varepsilon_t) \oplus h(\sum_{t\in X} b_t.\varepsilon_t). \end{aligned}$$

Now, let $a \in A$ and $\sum_{t\in X} a_t.\varepsilon_t \in A^{(X)}$. Then

$$\begin{aligned} h(a.\sum_{t\in X} a_t.\varepsilon_t) &= h(\{a.a_t\}_{t\in X}) = \sum_{t\in X}(a.a_t)f(\varepsilon_t) = \sum_{t\in X} a(a_t f(\varepsilon_t)) \\ &= a\sum_{t\in X} a_t f(\varepsilon_t) = ah\{a_t\}_{t\in X}. \end{aligned}$$

Finally, since

$$1 = (h(0))^\diamond = h(0^\diamond) = h(1) = h(\sum_{t=1}^{n} \varepsilon_t) = \sum_{t\in X} f(\varepsilon_t)$$

we have

$$\begin{aligned} h((\sum_{t\in X} a_t\varepsilon_t)^\diamond) &= h((\{a_t\}_{t\in X})^\diamond) = h(\{a_t^\diamond\}_{t\in X}) = h(\sum_{t\in X} a_t^\diamond \varepsilon_t) = \sum_{t\in X} a_t^\diamond f(\varepsilon_t) \\ &= \sum_{t\in X} a_t^\diamond f(\varepsilon_t) \oplus (\sum_{t\in X} f(\varepsilon_t))^\diamond = (\sum_{t\in X} a_t f(\varepsilon_t))^\diamond \\ &= (h(\sum_{t\in X} a_t\varepsilon_t))^\diamond, \end{aligned}$$

for every $\sum_{t\in X} a_t.\varepsilon_t \in A^{(X)}$. Then h is an A-homomorphism. On the other hand, $hoi(\varepsilon_t) = h(\varepsilon_t) = f(\varepsilon_t)$. It is easy to show that h is the unique A-homomorphism such that $A^{(X)}$ is a free object on K.

(iii) Let M be an A-module and X be a set of generators for M. Then by (ii), the inclusion map $f : X \longrightarrow M$ induces an A-homomorphism $h : A^{(X)} \longrightarrow M$ such that $X \subseteq Im\ h$. Since $M =< X >$, we have $Im\ h = M$. Therefore, M is a homomorphic image of a free A-module. □

Definition 3.15. A-module M is called a faithful A-module if $\{a \in A : aM = 0\} = \{0\}$.

EXAMPLE 3.16. In Example 3.3(ii), M is a faithful A-module.

Theorem 3.17. (i) *If one of the following conditions holds, then for every $A_{\mathbb{N}}$-module M,*

$$Rad(B) = \prec (E_M(B) \succ$$

where B is an A-ideal of M:
(a) *For every free A-module M, $Rad(B) = \prec (E_M(B) \succ$,*
(b) *For every faithful A-module M, $Rad(B) = \prec E_M(B) \succ$,*

(ii) *If one of the following conditions holds, then for every A-module M,*

$$Rad(B) = \prec E_M(B) \succ$$

where B is an A-ideal of M:
(c) *$Rad(0) = \prec (E_M(0) \succ$, for every A-module M.*
(d) *There is a PMV-algebra A' such that $\phi : A' \longrightarrow A$ is an PMV-epimorphism and $Rad(B') = \prec (E_{M'}(B') \succ$, where M' is any A'-module and B' is any A'-ideal of M'.*

Proof. (a) Let Ω be an $A_{\mathbb{N}}$-module and W be an A-ideal of Ω. By Lemma 3.14(iii), there is a free A-module F such that $\phi : F \longrightarrow \Omega$ is an A-epimorphism. By hypothesis, $Rad(\phi^{-1}(W)) = \prec (E_F(\phi^{-1}(W)) \succ$. Now, by Theorem 3.8$(i)$, we have

$$Rad(W) = Rad(\phi(\phi^{-1}(W))) = \prec (E_\Omega(\phi(\phi^{-1}(W))) \succ = \prec E_\Omega(W) \succ .$$

(b) It is clear that every free A-module is a faithful A-module. Then the proof is similar to (i).
(c) Let Ω be an A-module and W be an A-ideal of Ω. Consider the A-epimorphism $\phi : \Omega \longrightarrow \dfrac{\Omega}{W}$. By hypothesis, for the A-module $\frac{\Omega}{W}$, we have $Rad(\frac{0}{W}) = \prec E_M(\frac{0}{W}) \succ$. Hence

$$Rad_\Omega(\phi^{-1}(\frac{0}{W})) = \prec (E_\Omega(\phi^{-1}(\frac{0}{W})) \succ$$

and so $Rad_\Omega(W) = \prec (E_\Omega(W) \succ$.
(d) It is the same as Proposition 3.9(ii). □

4. Conclusion

The radical of A-ideals in MV-modules was studied. The definition of envelope of an A-ideal was presented, and by using of that definition, a method (a formula) for identifying the radical elements of an A-ideal was introduced. The study of radicals of A-ideals plays an important role in the study of A-ideals and their decomposition. In the future, we will continue on this path.

Acknowledgments

The authors would like to thank referee for some very helpful comments in improving several aspects of this paper.

References

1. R. A. Borzooei, S. Saidi Goraghani, Free MV-Modules, *Journal of Intelligent and Fuzzy System*, **31**(1), (2016), 151-161.
2. C. C. Chang, Algebric Analysis of Many-valued Logic, *Transactions of the American Mathematical Society*, **88**, (1958), 467-490.
3. C. C. Chang, A New Proof of the Completeness of the Łukasiewicz Axioms, *Transactions of the American Mathematical Society*, **93**, (1959), 74-80.
4. R. Cignoli, M. L. D'Ottaviano, D. Mundici, *Algebric Foundation of Many-valued Reasoning*, Kluwer Academic, Dordrecht, 2000.
5. A. Di Nola, A. Dvurečenskij, Product MV-Algebras, *Multiple-Valued Logics*, **6**, (2001), 193-215.
6. A. Di Nola, P. Flondor, I. Leustean, MV-Modules, *Journal of Algebra*, **267**, (2003), 21-40.
7. A. Dvurečenskij, On Partial Addition in Pseudo MV-Algebras, *Proceedings of the Fourth International Symposium on Economic Informatics*, (1999), 952-960.
8. A. Dvurečenskij, A Short Note on Categorical Equivalences of Proper Weak Pseudo EMV-Algebras, *Journal of Algebraic Hyperstructures and Logical Algebras*, **3**(1), (2022), 35-44.
9. A. Dvurečenskij, O. Zahiri, What Are Pseudo EMV-Algebras?, *Journal of Algebraic Hyperstructures and Logical Algebras*, **1**(1), (2020), 1-20.
10. F. Forouzesh, E. Eslami, A. Borumand Saeid, Radical of A-Ideals in MV-Modules, *Annals of the Alexandru Loan Cuza University-Math.*, **1**, (2016), 1-24.
11. T. Kroupa, Conditional Probability on MV-Algebras, *Fuzzy Sets and Systems*, **149**, (2005), 369-381.
12. J. Meng, Y. B. Jun, *BCK-Algebras*, Kyungmoon Sa Co, Korea, 1994.
13. D. Mundici, Interpretation of AF C^*-Algebras in Łukasiewicz Sentential Calculus, *Journal of Functional Analysis*, **65**, (1986), 15-63.
14. S. Saidi Goraghani, R. A. Borzooei, Most Results on A-Ideals in MV-Modules, *Journal of Algebraic Systems*, **5**(1), (2017), 1-13.
15. S. Saidi Goraghani, R. A. Borzooei, Results on Prime Ideals in PMV-Algebras and MV-Modules, *Italian Journal of Pure and Applied Mathematics*, **37**, (2017), 183-196.
16. S. Saidi Goraghani, R. A. Borzooei, Decomposition of A-Ideals in MV-Modules, *Annals of the University of Craiova-mathematics and Computer Science Series*, **45** (1), (2018), 66-77.
17. S. Saidi Goraghani, R. A. Borzooei, On Injective MV-Modules, *Bulletin of of the Section of Logic*, **47**(4), (2018), 283-298.
18. S. Saidi Goraghani, R. A. Borzooei, S. S. Ahn, Y. B. Jun, New Kind of MV-Modules, *International Journal of Computational intelligence Systems*, **13**(1), (2020), 794-801.
19. F. Xie, H. Liu,, Ideals in Pseudo-hoop Algebras, *Journal of Algebraic Hyperstructures and Logical Algebras*, **1**(4), (2020), 39-53.