On the Decomposition of Hilbert Spaces

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ABSTRACT. A basic relation between numerical range and Davis-Wielandt shell of an operator A acting on a Hilbert space with orthonormal basis $\xi = \{e_i | i \in I\}$ and its conjugate $\bar{A}$ which is introduced in this paper are obtained. The results are used to study the relation between point spectrum, approximate spectrum and residual spectrum of A and $\bar{A}$. A necessary and sufficient condition for A to be self-conjugate ($A = \bar{A}$) is given using a subgroup of H.

Keywords: Numerical range, Davis-Wielandt shell, Spectra, Conjugate of an operator.

1. INTRODUCTION

Let H be a Hilbert space and $B(H)$ be the algebra of bounded linear operators acting on H. The numerical range of $A \in B(H)$ is defined by

$$W(A) = \{\langle Ax, x\rangle | x \in H, \|x\| = 1\};$$

see [4, 5, 6, 7]. The numerical range is useful to study matrices, operators and to classify them. For example $W(A) = \{\mu\}$ if and only if $A = \mu I$ and $W(A) \subseteq \mathbb{R}$ if and only if $A = A^*$. Also, there are nice connections between $W(A)$ and spectrum $\sigma(A)$. For instance $\sigma(A) \subseteq \operatorname{cl} W(A)$ where $\operatorname{cl} W(A)$ is the closure of $W(A)$ and $\operatorname{cl} W(A) = \operatorname{conv} \sigma(A)$ if A is normal ($\operatorname{conv} \sigma(A)$ is the

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convex hull of $\sigma(A)$). The Davis-Wielandt shell of an operator $A \in B(H)$ is a generalization of the classical numerical range and it is defined by

$$DW(A) = \{(\langle Ax, x\rangle, \langle Ax, Ax\rangle) : x \in H, \|x\| = 1\};$$

see [1, 2, 9]. In fact $W(A)$ is the projection of $DW(A)$ on the first coordinate.

In this paper, we introduce the conjugate of $A \in B(H)$ and obtain the relation between $DW(A)$ and $DW(\bar{A})$. Also we obtain some relations between point spectrum, approximate spectrum and residual spectrum of A and $\bar{A}$. In section 3 we obtain a necessary and sufficient condition for A to be self-conjugate ($A = \bar{A}$) by a subgroup of H. Also we find the form of an operator $D \in B(H)$ such that $W(D) \subseteq \{(x, x) | x \in \mathbb{R}\}$.

2. The conjugate of an operator

In [8], for $A \in B(H)$, A^t is defined. In the following we introduce a new operator that has similar properties.

Definition 2.1. *Let H be a Hilbert space, $A \in B(H)$ and $\xi = \{e_i | i \in I\}$ be an orthonormal basis of H. The conjugate of A with respect to ξ is the operator $\bar{A}_\xi \in B(H)$ defined by*

$$\langle \bar{A}_\xi e_i, e_j \rangle = \overline{\langle Ae_i, e_j \rangle},$$

for every $i, j \in I$.

Let $x = \sum_{i \in I} x_i e_i \in H$. Define $\bar{x} = \sum_{i \in I} \bar{x}_i e_i$, where $\bar{x}_i$ is the complex conjugate of x_i. Let $y = \sum_{j \in I} y_j e_j \in H$. we have

$$\langle \bar{A}_\xi x, y \rangle = \sum_i \sum_j x_i \bar{y}_j \langle \bar{A}_\xi e_i, e_j \rangle = \overline{\sum_i \sum_j \bar{x}_i y_j \langle Ae_i, e_j \rangle} = \overline{\langle A\bar{x}, \bar{y} \rangle}.$$

By the above discussion we see that the definition of $\bar{A}_\xi$ is independent of ξ, so we denote the conjugate of A by $\bar{A}$.
Also we know that $\langle \bar{x}, \bar{y} \rangle = \overline{\langle x, y \rangle}$ and we have $\overline{\langle A\bar{x}, \bar{y} \rangle} = \langle \bar{A}x, y \rangle = \overline{\langle \bar{A}\bar{x}, y \rangle}$ for every $x, y \in H$. Hence $\bar{A}x = \overline{A\bar{x}}$. Furthermore, we recall that

$$\langle A^t x, y \rangle = \langle A\bar{y}, \bar{x} \rangle,$$

for every $x, y \in H$ [8].

Lemma 2.2. *Let $A, B \in B(H)$ and $\alpha \in \mathbb{C}$ be an arbitrary constant. Then:*
a) $\bar{A} = (A^t)^*$;
b) $\overline{(A + \alpha B)} = \bar{A} + \bar{\alpha}\bar{B}$ and $\overline{AB} = \bar{A}\bar{B}$.

Proof. a)We have

$$\langle (A^t)^* x, y \rangle = \langle x, A^t y \rangle = \langle \bar{y}, A\bar{x} \rangle = \overline{\langle A\bar{x}, \bar{y} \rangle} = \langle \bar{A}x, y \rangle,$$

for every $x, y \in H$. Thus the assertion follows.
b) Let $x, y \in H$ be arbitrary. then
$$\langle \overline{(A + \alpha B)}x, y \rangle = \overline{\langle (A + \alpha B)\bar{x}, \bar{y} \rangle} = \overline{\langle A\bar{x}, \bar{y} \rangle} + \overline{\langle \alpha B\bar{x}, \bar{y} \rangle} = \langle \bar{A}x, y \rangle + \bar{\alpha}\langle \bar{B}x, y \rangle =$$

$\langle(\bar{A} + \bar{\alpha}\bar{B})x, y\rangle$.

Since x, y are arbitrary, the result follows. On the other hand,

$\langle\overline{(AB)}x, y\rangle = \overline{\langle(AB)\bar{x}, \bar{y}\rangle} = \overline{\langle A(B\bar{x}), \bar{y}\rangle} = \overline{\langle B\bar{x}, A^*\bar{y}\rangle} = \langle \bar{B}x, \overline{A^*\bar{y}}\rangle = \langle\overline{\bar{B}x}, A^*\bar{y}\rangle = \langle A(\overline{\bar{B}x}), \bar{y}\rangle = \langle \bar{A}(\bar{B}x), y\rangle = \langle(\bar{A}\bar{B})x, y\rangle$

and hence $\overline{AB} = \bar{A}\bar{B}$. $\qquad\square$

Recall that the *point spectrum* of $A \in B(H)$ is the set $\sigma_p(A)$ of eigenvalues of A. The *residual spectrum* of A is the set $\sigma_r(A)$ of complex numbers λ such that the range of $\lambda I - A$ is not dense in H. The *approximate spectrum* of A is the set $\sigma_a(A)$ of complex numbers λ such that there exists a sequence of unit vectors $\{x_n\}_{n=1}^{\infty}$ in H such that $\lim_{n\to\infty} \|(\lambda I - A)x_n\| = 0$. It is well known that $\sigma_p(A) \subseteq \sigma_a(A)$ and $\sigma(A) = \sigma_a(A) \bigcup \sigma_r(A)$; see [3].

In the following theorem, we obtain some relations between the Davis-Wielandt shell and the spectrum of A and $\bar{A}$.

Theorem 2.3. . *Let $A \in B(H)$. Then:*
a) $DW(\bar{A}) = \{(\bar{\mu}, r)|(\mu, r) \in DW(A)\}$
b) $\sigma_p(\bar{A}) = \overline{\sigma_p(A)}$, $\sigma_a(\bar{A}) = \overline{\sigma_a(A)}$ and $\sigma_r(\bar{A}) = \overline{\sigma_r(A)}$.

Proof. a) $(\mu, r) \in DW(\bar{A})$ if and only if $\langle\bar{A}x, x\rangle = \mu, \langle\bar{A}x, \bar{A}x\rangle = r$ for some unit vector x. Since $\langle\bar{A}x, x\rangle = \overline{\langle A\bar{x}, \bar{x}\rangle}$ and $\|x\| = 1$, then $\bar{\mu} = \langle A\bar{x}, \bar{x}\rangle \in W(A)$. Also

$$\langle\bar{A}x, \bar{A}x\rangle = \overline{\langle A\bar{x}, \overline{\bar{A}x}\rangle} = \overline{\langle \bar{A}x, \overline{A\bar{x}}\rangle} = \langle A\bar{x}, A\bar{x}\rangle.$$

Therefore
$(\mu, r) = (\langle\bar{A}x, x\rangle, \langle\bar{A}x, \bar{A}x\rangle) \in DW(\bar{A}))$ if and only if $(\langle A\bar{x}, \bar{x}\rangle, \langle A\bar{x}, A\bar{x}\rangle) = (\bar{\mu}, r) \in DW(A)$, and the result follows.

b) $\mu \in \sigma_p(\bar{A})$ if and only if $\langle(\bar{A} - \mu I)x, y\rangle = 0$ for all $y \in H$ and some nonzero vector x. We have $\langle(\bar{A} - \mu I)x, y\rangle = 0$ if and only if $\langle(A - \bar{\mu}I)\bar{x}, \bar{y}\rangle = \overline{\langle(\bar{A} - \mu I)x, y\rangle} = 0$. Thus the first assertion holds.

We know that $\mu \in \sigma_a(\bar{A})$ if and only if there is a sequence $\{x_n\}_{n=1}^{\infty}$ of unit vectors such that $\lim_{n\to\infty}((\mu I - \bar{A})x_n) = 0$. We have

$$0 = \lim_{n\to\infty}(\mu x_n - \bar{A}x_n) = \lim_{n\to\infty}(\mu x_n - \overline{A\bar{x}_n}),$$

which implies that $0 = \lim_{n\to\infty}\overline{(\mu x_n - \overline{A\bar{x}_n})} = \lim_{n\to\infty}(\overline{\mu}\bar{x}_n - A\bar{x}_n)$, and the second assertion holds. If $\mu \in \sigma_r(\bar{A})$, then $\mathrm{cl}(\mathrm{Im}(\mu I - \bar{A})) \neq H$ and vice versa. This holds if and only if there exists a nonzero vector $z \in H$ which is orthogonal to $\mathrm{Im}(\mu I - \bar{A})$. Hence for any $x \in H$ we have
$0 = \langle(\mu I - \bar{A})x, z\rangle = \langle x, (\mu I - \bar{A})^*z\rangle = \langle x, (\bar{\mu}I - A^t)z\rangle = \langle x, \bar{\mu}z\rangle - \langle x, A^t z\rangle = \mu\langle x, z\rangle - \langle \bar{z}, A\bar{x}\rangle = \mu\langle \bar{z}, \bar{x}\rangle - \langle \bar{z}, A\bar{x}\rangle = \langle \bar{z}, (\bar{\mu}I - A)\bar{x}\rangle$.
Since x is arbitrary, thus $\bar{z} \in (\mathrm{Im}(\bar{\mu}I - A))^{\perp}$ and $\mathrm{cl}(\mathrm{Im}(\bar{\mu}I - A)) \neq H$. Therefore $\bar{\mu} \in \sigma_r(A)$ and vice versa. $\qquad\square$

3. Decomposition of a Hilbert space

Let H be a Hilbert space and $\xi = \{e_i | i \in I\}$ be an orthonormal basis of H. Define $H_{\mathbb{R}}^{\xi} = \{x \in H | x = \bar{x}\}$. Since $0 = \bar{0}$, $0 \in H_{\mathbb{R}}^{\xi}$. It is clear that $H_{\mathbb{R}}^{\xi}$ is a subgroup of H.

Theorem 3.1. *Let H be a Hilbert space with orthonormal basis $\xi = \{e_i | i \in I\}$. Then $H \equiv H_{\mathbb{R}}^{\xi} \times H_{\mathbb{R}}^{\xi}$ and each $x \in H$ can be uniquely written as $x = a + ib$, where $a, b \in H_{\mathbb{R}}^{\xi}$.*

Proof. Let $x \in H$ be arbitrary. We have $\overline{x + \bar{x}} = \bar{x} + x$. Thus $\bar{x} + x \in H_{\mathbb{R}}^{\xi}$. Also $\overline{i(\bar{x} - x)} = i(\bar{x} - x)$. Then $i(\bar{x} - x) \in H_{\mathbb{R}}^{\xi}$. Therefore $x = a + ib$, where $a = \frac{x + \bar{x}}{2}$, $b = \frac{i(\bar{x} - x)}{2}$ and $a, b \in H_{\mathbb{R}}^{\xi}$. Now we show that the above decomposition is unique. Let $x = a + ib = a' + ib'$ where $a', b' \in H_{\mathbb{R}}^{\xi}$ and

$$a = \sum_{i \in I} a_i e_i, b = \sum_{i \in I} b_i e_i, a' = \sum_{i \in I} a_i' e_i, b' = \sum_{i \in I} b_i' e_i.$$

Since $a, b, a', b' \in H_{\mathbb{R}}^{\xi}$, then for every $i \in I$, $a_i, b_i, a_i', b_i' \in \mathbb{R}$,

$$\sum_{i \in I} ((a_i - a_i') + i(b_i - b_i')) e_i = 0.$$

So we have $a_i = a_i'$ and $b_i = b_i'$ for every $i \in I$, which proves the uniqueness. $\square$

We remark that if $x = a + ib \in H$ and $a, b \in H_{\mathbb{R}}^{\xi}$, then $\bar{x} = a - ib$.

Theorem 3.2. *Let $A \in B(H)$ with orthonormal basis $\xi = \{e_i | i \in I\}$. The following conditions are equivalent:*
a) $A\bar{x} = \overline{Ax}$ for all $x \in H$;
b) $A = \bar{A}$;
c) $H_{\mathbb{R}}^{\xi}$ is A- invariant $(A(H_{\mathbb{R}}^{\xi}) \subseteq H_{\mathbb{R}}^{\xi})$.

Proof. $a \Rightarrow b$ and $b \Rightarrow c$ are trivial. we prove $c \Rightarrow a$.
Let $x = \alpha + i\beta \in H$ be arbitrary and $\alpha, \beta \in H_{\mathbb{R}}^{\xi}$. We have
$A\bar{x} = A(\alpha - i\beta) = A\alpha - iA\beta$. Since we assumed that $H_{\mathbb{R}}^{\xi}$ is A- invariant, then by previous remark we have

$$A\alpha - iA\beta = \overline{A\alpha + iA\beta} = \overline{A(\alpha + i\beta)} = \overline{Ax}.$$

$\square$

Corollary 3.3. *Let H be a Hilbert space with orthonormal basis $\xi = \{e_i | i \in I\}$. Then for all $x = a + ib \in H$ (where $a, b \in H_{\mathbb{R}}^{\xi}$), $\|x\|^2 = \|a\|^2 + \|b\|^2$.*

Proof. We have

$$\|x\|^2 = \langle x, x \rangle = \langle a + ib, a + ib \rangle = \|a\|^2 + \|b\|^2 + i\langle b, a \rangle - i\langle a, b \rangle.$$

We show that for all $a, b \in H_{\mathbb{R}}^{\xi}$, $\langle b, a \rangle = \langle a, b \rangle$.
Let $a = \sum_{i \in I} a_i e_i, b = \sum_{j \in I} b_j e_j$. (Since $a, b \in H_{\mathbb{R}}^{\xi}$, we have $a_i, b_j \in \mathbb{R}$, for any

$i, j \in I$). It follows that

$\langle a, b \rangle = \langle \sum_{i \in I} a_i e_i, \sum_{j \in I} b_j e_j \rangle = \sum_{i \in I} \sum_{j \in I} a_i \bar{b}_j \langle e_i, e_j \rangle = \sum_{j \in I} \sum_{i \in I} b_j a_i \langle e_j, e_i \rangle = \langle \sum_{j \in I} b_j e_j, \sum_{i \in I} a_i e_i \rangle = \langle b, a \rangle.$

Thus $\|x\|^2 = \|a\|^2 + \|b\|^2$ as asserted.

$\square$

Definition 3.4. *Let H be a Hilbert space. The operator $A \in B(H)$ is called self-conjugate, if $A = \bar{A}$.*

Let

$$B(H)_{\mathbb{R}} = \{A \in B(H) : A = \bar{A}\}.$$

It is clear that for every Hilbert space H, $B(H)_{\mathbb{R}}$ is a subring of H.
Note that for elements of a Hilbert space the concept of conjugate depends on the orthonormal basis. However, for the sake of simplicity, we put aside the notation of orthonormal basis when we refer to the conjugate of Hilbert space elements.

Corollary 3.5. *Assume that $A \in B(H)_{\mathbb{R}}$. Then $A^t, A^* \in B(H)_{\mathbb{R}}$ and $A^{-1} \in B(H)_{\mathbb{R}}$ if A is invertible.*

Proof. Let $A \in B(H)_{\mathbb{R}}$ and $x, y \in H$ be arbitrary. We have

$$\overline{\langle (A^t)x, y \rangle} = \overline{\langle A^t \bar{x}, \bar{y} \rangle} = \overline{\langle \overline{Ay}, x \rangle} = \overline{\langle Ay \rangle}, \bar{x} \rangle = \langle A\bar{y}, \bar{x} \rangle = \langle A^t x, y \rangle.$$

Since x, y are arbitrary, we have $A^t \in B(H)_{\mathbb{R}}$. Similarly, $A^* \in B(H)_{\mathbb{R}}$. Now let A be invertible and $x, y \in H$ be arbitrary. Consider $A^{-1}\bar{y} = y_0, A^{-1}\bar{x} = x_0$. By Theorem 3.2, $A\bar{x}_0 = \overline{Ax_0} = x$ and $\overline{Ay_0} = y$. Thus,
$\overline{\langle (A^{-1})x, y \rangle} = \overline{\langle (A^{-1})\bar{x}, \bar{y} \rangle} = \overline{\langle x_0, Ay_0 \rangle} = \langle \bar{x}_0, \overline{Ay_0} \rangle = \langle A^{-1}(A\bar{x}_0), \overline{Ay_0} \rangle = \langle A^{-1}(\overline{Ax_0}), \overline{Ay_0} \rangle = \langle A^{-1}x, y \rangle.$
Hence, our claim follows. $\square$

In the following theorem we deduce a necessary and sufficient condition for $W(D)$ to be a subset of $\{(r, r) : r \in \mathbb{R}\}$ when $D \in B(H)$.

Theorem 3.6. *Let $D = A + iB \in B(H)$ and $A, B \in B(H)_{\mathbb{R}}$. Then $W(D) \subseteq \{(r, r) : r \in \mathbb{R}\}$ if and only if $B = A^t$.*

Proof. Let H be a Hilbert space and $\xi = \{e_i | i \in I\}$ be an orthonormal basis. Let $x = a + ib \in H$ be arbitrary where $a, b \in H_{\mathbb{R}}^{\xi}$ and let $D = A + iA^t$. Then
$\langle Dx, x \rangle = \langle (A + iA^t)(a + ib), a + ib \rangle = (\langle Aa, a \rangle - \langle A^t b, a \rangle + \langle Ab, b \rangle + \langle A^t a, b \rangle) + i(\langle Ab, a \rangle + \langle A^t a, a \rangle - \langle Aa, b \rangle + \langle A^t b, b \rangle) = (\langle Aa, a \rangle - \langle Aa, b \rangle + \langle Ab, b \rangle + \langle Ab, a \rangle) + i(\langle Ab, a \rangle + \langle Aa, a \rangle - \langle Aa, b \rangle + \langle Ab, b \rangle).$
Since $\langle Aa, a \rangle - \langle Aa, b \rangle + \langle Ab, b \rangle + \langle Ab, a \rangle \in \mathbb{R}$, the result follows.
Conversely let $W(D) \subseteq \{(r, r) : r \in \mathbb{R}\}$. Then D is essentially self adjoint[4,5,6]. Therefore $D = \alpha H + \beta I$ for some $\alpha, \beta \in \mathbb{C}$ and H is Hermitian and there exists a Hermitian operator T such that $D = e^{\frac{i\pi}{4}}T$. Set $T = T_1 + iT_2$ where $T_1, T_2 \in B(H)_{\mathbb{R}}$. Since $T = T^*$, thus $T_1 + iT_2 = T_1^* - iT_2^*$. By Corollary 3.5,

$T_1^*, T_2^* \in B(H)_{\mathbb{R}}$ and $T_1 = T_1^*, T_2 = -T_2^*$. Then $T_1 = T_1^t, T_2 = -T_2^t$. Now we have

$$D = e^{\frac{i\pi}{4}}(T_1 + iT_2) = \frac{\sqrt{2}}{2}(T_1 - T_2) + i\frac{\sqrt{2}}{2}(T_1 - T_2)^t.$$

If $A = \frac{\sqrt{2}}{2}(T_1 - T_2)$, the proof is complete. $\qquad\qquad\qquad\qquad$ $\square$

Theorem 3.7. *Let $A \in B(H)$ and $Ax = \bar{A}x$ for some $0 \neq x \in H$. Then*

$$W(A)\bigcap W(\bar{A})\bigcap \mathbb{R} \neq \emptyset.$$

Proof. We may assume that x is a unit vector. Then $\mu = \langle Ax, x \rangle = \langle \bar{A}x, x \rangle \in W(\bar{A})$. Hence $\mu \in W(A)\bigcap W(\bar{A})$. Since $W(\bar{A}) = \overline{W(A)}$ by Theorem 2.3, we have $\bar{\mu} \in W(\bar{A})\bigcap W(A)$. By convexity of numerical range the line segment joins μ and $\bar{\mu}$ lies in $W(A)$ and $W(\bar{A})$. This line segment intersects the real line and the result follows. $\qquad\qquad$ $\square$

Theorem 3.8. *Let H be a Hilbert space and $A \in B(H)$. $A = \lambda I$ for some $\lambda \in \mathbb{C}$ if and only if for every $\alpha \in \mathbb{C}$, there exists a scalar $\beta \in \mathbb{C}$ such that $\alpha A + \beta I$ is self-conjugate.*

Proof. The implication ($\Rightarrow$) is clear. Suppose that for every $\alpha \in \mathbb{C}$, there exists $\beta \in \mathbb{C}$ such that $\alpha A + \beta I$ is self-conjugate. If H is one dimensional the result holds. Let $\xi = \{e_i\}_{i \in I}$ be an orthonormal basis for H and $e_j, e_k \in \xi$ be arbitrary and distinct. First we prove that A is diagonal. Suppose $Ae_j = \Sigma_{i \in I}\alpha_i e_i$ and $Ae_k = \Sigma_{i \in I}\beta_i e_i$. For every $\alpha \in \mathbb{C}$, there exists $\beta \in \mathbb{C}$ such that $(\alpha A + \beta I)e_j = \Sigma_{i \in I}r_i e_i, (\alpha A + \beta I)e_k = \Sigma_{i \in I}s_i e_i$, and $r_i, s_i \in \mathbb{R}$ for every $i \in I$. Hence, for every $i \neq j$, $\alpha\alpha_i = r_i \in \mathbb{R}$ and for every $i \neq k$, $\alpha\beta_i = s_i \in \mathbb{R}$. Since α is arbitrary, we must have $\alpha_i = 0$ for every $i \neq j$ and $\beta_i = 0$ for every $i \neq k$. Since $e_j, e_k \in \xi$, A is diagonal. Now, suppose that $Ae_j = \lambda_j e_j$ and $Ae_k = \lambda_k e_k$. It is enough to show that $\lambda_j = \lambda_k$. But for every $\alpha \in \mathbb{C}$, there exists $\beta \in \mathbb{C}$, such that $\alpha\lambda_j e_j + \beta e_j = r_j e_j$ and $r_j \in \mathbb{R}$, and we have $\alpha\lambda_k e_k + \beta e_k = s_k e_k$ where $s_k \in \mathbb{R}$. Thus for every $\alpha \in \mathbb{R}$, $\alpha(\lambda_j - \lambda_k) \in \mathbb{R}$. Hence $\lambda_j - \lambda_k = 0$ and the result follows. $\qquad\qquad$ $\square$

REFERENCES

[1] C. Davis, The shell of a Hilbert-space operator, *Acta Sci. Math.*, **29**, (1968), 69-86.
[2] C. Davis, The shell of a Hilbert-space operator. II, *Acta Sci. Math.*, **31**, (1970), 301-318.
[3] H. R. Dowson, *Spectral theory of linear operators*, Academic press, London, 1978.
[4] K. E. Gustafson and D. K. M. Rao, *Numerical ranges, Springer*, New York, 1997.
[5] P. Halmos, *A Hilbert Space Problem Book*, Second edition, Spring-Verlag, New York, 1982.

[6] M. T. Heydari, C*-Algebra Numerical Range of Quadratic Elements, *Iranian Journal of Mathematical Sciences and Informatics*, **5**(1), (2010), 49-53.

[7] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[8] C. K. Li, Y. T. Poon and N. S. Sze, Davis-Wielandt shells of operators, *Acta Sci. Math.*, **2**, (2008), 341-357.

[9] H. Wielandt, On eigenvalues of sums of normal matrices, *Pacific J. Math.*, **5** (1955), 633–638.