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Generalization of Grüss Inequality on Time Scales

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Abstract. In the present paper we derive a generalized Grüss inequality on time scales and recapture several published results of different authors of various papers and thus unify corresponding discrete and continuous versions. Moreover, we use our obtained consequence to the case of quantum calculus.

Keywords: Grüss inequality, Jensen's inequality, Time scales, Quantum calculus.

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1. Introduction

In 1935, Gerhard Grüss proved an inequality to estimate the deviation of the integral mean of the product of two functions from the product of the integral

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means. The below inequality is called Grüss inequality which is extracted from [11] and for other details of Grüss type inequalities (see [7]).

$$\left|\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g(\tau)h(\tau)d\tau - \frac{1}{(\ell-\kappa)^2}\int_{\kappa}^{\ell} g(\tau)d\tau \int_{\kappa}^{\ell} h(\tau)d\tau\right| \leq \frac{(M_1-m_1)(M_2-m_2)}{4} \tag{1.1}$$

holds.

This paper was motivated by the discovery of a very elementary proof [11] of the Grüss inequality and the desire to understand the meaning of the quantity on the right-hand side of the inequality. As a result of Theorem 3.1, we find several inequalities that either imply or generalize the Grüss inequality in the discrete version, continuous version and quantum calculus version.

Many generalizations of this inequality have been proved by several authors and its applications of the Grüss type inequalities have been found in statistics, coding theory, and numerical analysis. If you are interested in the historical development and further refinements of the Grüss Inequality, you may want to explore research papers and publications in mathematical journals that discuss inequalities, inner product spaces, and related topics.

The pattern of current paper is consist of four sections. In the section one and two, we would present some preliminaries about Grüss inequality and time scales that are needed in the remainder of the paper. In the section three, we would get time scales version of generalized Grüss inequality (1.1) and use our established result for the especial cases of discrete, continuous, and quantum calculus. In the section four, we would give conclusion of the paper.

2. Time scales essentials

Time scale calculus is a mathematical framework that unifies and extends the concepts of difference equations and differential equations. It provides a way to study dynamic systems that involve both continuous and discrete aspects. It also provides a comprehensive approach to calculus for functions defined on time scales, which can include both continuous and discrete intervals. Time scales are a generalization of the real numbers that incorporates both continuous and discrete points. The idea of theory of time scales calculus was initiated and introduced by German mathematician Hilger (1988) in his PhD thesis [12] (supervized by Aulbach) in order to unify discrete and continuous analysis and to expend the discrete and continuous theories to cases "in between". Since then, mathematical research in this field has exceeded more than 1000 publications and a lot of applications in the fields of science i.e., operations research, economics, physics, engineering, statistics, finance and biology [6]. Even the theory of time scale calculus may be used in most of the branches of science in which dynamic processes are explained by discrete-time / continuous-time models. We prefer the researcher to the book [4] written by Bohner and Peterson about the introduction to the singled variable time scale calculus and its implementations.

In 2004, Bohner introduced the calculus of variations on time scale, he used the delta derivative and delta integral [5], and it has since then been further developed by several different authors in several different publications (see [1, 10, 13, 14, 15, 16, 17]). Many classical results of calculus of variations as necessary or sufficient conditions of optimality have been generalized to arbitrary time scale.

Definition 2.1. A time scale is an arbitrary nonempty closed subset of the real numbers.

The most important examples of time scales are $\mathbb{R}, \mathbb{Z}$ and $q^{\mathbb{N}_0} := \{q^\ell | \ell \in \mathbb{N}_0\}$.

Definition 2.2. If $\mathbb{T}$ is a time scale, then we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(\theta) := \inf\{\tau \in \mathbb{T} | \tau > \theta\}$ for all $\theta \in \mathbb{T}$, the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(\theta) := \sup\{\tau \in \mathbb{T} | \tau < \theta\}$ for all $\theta \in \mathbb{T}$, and the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(\theta) := \sigma(\theta) - \theta$ for all $\theta \in \mathbb{T}$. Furthermore for a function $g : \mathbb{T} \to \mathbb{R}$, we define $g^\sigma(\theta) = g(\sigma(\theta))$ for all $\theta \in \mathbb{T}$ and $g^\rho(\theta) = g(\rho(\theta))$ for all $\theta \in \mathbb{T}$. In this definition we use $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\rho(\theta) = \theta$ if θ is the maximum of $\mathbb{T}$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(\theta) = \theta$ if θ is the minimum of $\mathbb{T}$).

These definitions allow us to characterize every point in a time scale as following classification of points:

(i) θ right-scattered $\Longrightarrow \theta < \sigma(\theta)$,
(ii) θ right-dense $\Longrightarrow \theta = \sigma(\theta)$,
(iii) θ left-scattered $\Longrightarrow \rho(\theta) < \theta$,
(iv) θ left-dense $\Longrightarrow \rho(\theta) = \theta$,
(v) θ isolated $\Longrightarrow \rho(\theta) < \theta < \sigma(\theta)$,
(vi) θ dense $\Longrightarrow \rho(\theta) = \theta = \sigma(\theta)$.

Definition 2.3. A function $g : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* (denoted by C_{rd}) if it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points of $\mathbb{T}$.

Theorem 2.4. *(Existence of Antiderivatives). Let g be rd-continuous. Then g has an antiderivative G satisfying $G^\Delta = g$.*

Proof. See Theorem 1.74 of paper [4]. □

Definition 2.5. If g is *rd-continuous* and $\theta_0 \in \mathbb{T}$, then we define the integral

$$G(\theta) = \int_{\theta_0}^{\theta} g(s)\Delta s \qquad for \qquad \theta \in \mathbb{T}. \tag{2.1}$$

Therefore for $g \in C_{rd}$ we have $\int_\kappa^\ell g(s)\Delta s = G(\ell) - G(\kappa)$, where $G^\Delta = g$.

Theorem 2.6. *Let g, h be rd-continuous, $\kappa, \ell, l \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then*

(i) $\int_\kappa^\ell [\alpha g(\theta) + \beta h(\theta)]\Delta\theta = \alpha \int_\kappa^\ell g(\theta)\Delta\theta + \beta \int_\kappa^\ell h(\theta)\Delta\theta$,

(ii) $\int_\kappa^\ell g(\theta)\Delta\theta = -\int_\ell^\kappa g(\theta)\Delta\theta$,

(iii) $\int_\kappa^\ell g(\theta)\Delta\theta = \int_\kappa^l g(\theta)\Delta\theta + \int_l^\ell g(\theta)\Delta\theta$,

(iv) $\int_\kappa^\kappa g(\theta)\Delta\theta = 0$.

Proof. See Theorem 1.77 of paper [4]. □

Theorem 2.7. *(Jensen's Inequality). Let $\kappa, \ell \in \mathbb{T}$ and $\kappa_1, \ell_1 \in \mathbb{R}$. If $G : (\kappa_1, \ell_1) \to \mathbb{R}$ is continuous and convex and $h : [\kappa, \ell] \to (\kappa_1, \ell_1)$ is rd-continuous. Then*

$$G\left(\frac{\int_\kappa^\ell h(\theta)\Delta\theta}{\ell - \kappa}\right) \leq \frac{\int_\kappa^\ell G(h(\theta))\Delta\theta}{\ell - \kappa}. \tag{2.2}$$

Proof. See Theorem 6.17 of [4]. □

In 2007, Bohner et. al. gave a Grüss inequality on time scales and they obtained unified results to the literature and they also gave results to the case of quantum calculus. In this paper, we would obtain generalization of Grüss inequality on time scales and recapture the results of [3] for discrete and continuous versions. Moreover, we will use our weighted results to the case of quantum calculus.

3. Generalization of Grüss inequality on time scales

With the same manner as in [11] and [3], the weighted Grüss inequality may be derived for general time scales.

Theorem 3.1. *(Generalized Grüss Inequality). Let $\kappa, \ell, \tau, \theta \in \mathbb{T}$, $g, h \in C_{rd}$ and $w \geq 0$ with $\int_\kappa^\ell w(\tau)\Delta\tau = 1$ and $g, h : [\kappa, \ell] \to \mathbb{R}$. Then for*

$$m_1 \leq g(\tau) \leq M_1, \qquad m_2 \leq h(\tau) \leq M_2, \tag{3.1}$$

we have

$$\Bigg| \int_\kappa^\ell w^\sigma(\tau)\Delta\tau \int_\kappa^\ell w^\sigma(\tau)g^\sigma(\tau)h^\sigma(\tau)\Delta\tau - \int_\kappa^\ell w^\sigma(\tau)g^\sigma(\tau)\Delta\tau$$
$$\int_\kappa^\ell w^\sigma(\tau)h^\sigma(\tau)\Delta\tau\Bigg| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2)\left(\int_\kappa^\ell w^\sigma(\tau)\Delta\tau\right)^2. \tag{3.2}$$

Proof. For proving the theorem we 1st consider $g(\tau) = h(\tau)$ and

$$\int_\kappa^\ell w^\sigma(\tau)g^\sigma(\tau)\Delta\tau = 0. \tag{3.3}$$

If we define $\nu(\tau) = w(\tau)\frac{g(\tau)-m_1}{M_1-m_1}$, then we get

$$g(\tau) = \frac{(M_1 - m_1)\nu(\tau) + m_1 w(\tau)}{w(\tau)} \tag{3.4}$$

with $\nu(\tau) \in [0,1]$. Since

$$\begin{aligned}\int_\kappa^\ell (\nu^\sigma(\tau))^2 \Delta\tau &\leq \int_\kappa^\ell \nu^\sigma(\tau)\Delta\tau \\ &= \int_\kappa^\ell w^\sigma(\tau)\frac{g^\sigma(\tau)-m_1}{M_1-m_1}\Delta\tau \\ &= -\int_\kappa^\ell \frac{m_1 w^\sigma(\tau)}{M_1-m_1}\Delta\tau.\end{aligned}$$

We have

$$\begin{aligned}D(g,g) &:= \int_\kappa^\ell w^\sigma(\tau)\Delta\tau \int_\kappa^\ell w^\sigma(\tau)(g^\sigma(\tau))^2\Delta\tau - \left(\int_\kappa^\ell w^\sigma(\tau)g^\sigma(\tau)\Delta\tau\right)^2 \\ &= \int_\kappa^\ell w^\sigma(\tau)\Delta\tau \int_\kappa^\ell w^\sigma(\tau)\left(\frac{(M_1-m_1)\nu(\tau)+m_1 w^\sigma(\tau)}{w^\sigma(\tau)}\right)^2\Delta\tau.\end{aligned}$$

Now consider the case

$$\int_\kappa^\ell w^\sigma(\tau)g^\sigma(\tau)\Delta\tau = G \neq 0,$$

where $G \in \mathbb{R}$ and let $g_1(\tau) = g(\tau) - G$. Therefore

$$\begin{aligned}\int_\kappa^\ell w^\sigma(\tau)g_1^\sigma(\tau)\Delta\tau &= \int_\kappa^\ell w^\sigma(\tau)(g^\sigma(\tau)-G)\Delta\tau \\ &= \int_\kappa^\ell w^\sigma(\tau)g^\sigma(\tau)\Delta\tau - G\int_\kappa^\ell w^\sigma(\tau)\Delta\tau \\ &= G\Big(1-\int_\kappa^\ell w^\sigma(\tau)\Delta\tau\Big) \\ &= 0\end{aligned}$$

with

$$g_1(\tau) \in [m_1 - G, M_1 - G].$$

Hence g_1 satisfies the assumption from the earlier part of this proof so that

$$\begin{aligned}D(g_1,g_1) &\leq \frac{1}{4}[M_1 - G - (m_1 - G)]^2\Big(\int_\kappa^\ell w^\sigma(\tau)\Delta\tau\Big)^2 \\ &= \frac{1}{4}(M_1-m_1)^2\Big(\int_\kappa^\ell w^\sigma(\tau)\Delta\tau\Big)^2.\end{aligned}$$

Moreover we have

$$\begin{aligned}
D(g_1,g_1) &= \int_\kappa^\ell w^\sigma(\tau)\Delta\tau \int_\kappa^\ell w^\sigma(\tau)(g^\sigma(\tau)-G)^2\Delta\tau \\
&= \int_\kappa^\ell w^\sigma(\tau)\Delta\tau \int_\kappa^\ell w^\sigma(\tau)\Big((g^\sigma(\tau))^2-2g^\sigma(\tau)G+G^2\Big)\Delta\tau \\
&= \int_\kappa^\ell w^\sigma(\tau)\Delta\tau\Bigg(\int_\kappa^\ell w^\sigma(\tau)(g^\sigma(\tau))^2\Delta\tau-2\int_\kappa^\ell w^\sigma(\tau)g^\sigma(\tau)G\Delta\tau \\
&+ \int_\kappa^\ell w^\sigma(\tau)G^2\Delta\tau\Bigg) \\
&= \int_\kappa^\ell w^\sigma(\tau)\Delta\tau \int_\kappa^\ell w^\sigma(\tau)(g^\sigma(\tau))^2\Delta\tau - G^2\int_\kappa^\ell w^\sigma(\tau)\Delta\tau \\
&\times \left(2-\int_\kappa^\ell w^\sigma(\tau)\Delta\tau\right) \\
&= \int_\kappa^\ell w^\sigma(\tau)\Delta\tau \int_\kappa^\ell w^\sigma(\tau)(g^\sigma(\tau))^2\Delta\tau - G\int_\kappa^\ell w^\sigma(\tau)\Delta\tau \\
&\times G\left(1-\int_\kappa^\ell w^\sigma(\tau)\Delta\tau\right) - G^2\int_\kappa^\ell w^\sigma(\tau)\Delta\tau \\
&= D(g,g)
\end{aligned}$$

and thus

$$D(g,g)=D(g_1,g_1)\le\frac{1}{4}(M_1-m_1)^2\left(\int_\kappa^\ell w^\sigma(\tau)\Delta\tau\right)^2.$$

Now we consider the case of general functions g and h and assume (3.1). Then

$$\begin{aligned}
D(g,h) &:= \int_\kappa^\ell w^\sigma(\tau)\Delta\tau\int_\kappa^\ell w^\sigma(\tau)g^\sigma(\tau)h^\sigma(\tau)\Delta\tau - \int_\kappa^\ell w^\sigma(\tau)g^\sigma(\tau)\Delta\tau \\
&\qquad\qquad \times\int_\kappa^\ell w^\sigma(\tau)h^\sigma(\tau)\Delta\tau \\
&= \frac{1}{4}\Bigg[\int_\kappa^\ell w^\sigma(\tau)\Delta\tau\int_\kappa^\ell w^\sigma(\tau)\Big((g^\sigma(\tau)+h^\sigma(\tau))^2-(g^\sigma(\tau)-h^\sigma(\tau))^2\Big)\Delta\tau \\
&\qquad -4\left(\int_\kappa^\ell w^\sigma(\tau)g^\sigma(\tau)\Delta\tau\int_\kappa^\ell w^\sigma(\tau)h^\sigma(\tau)\Delta\tau\right)\Bigg].
\end{aligned}$$

Note that

$$D(g+h,g+h)\le\frac{1}{4}(M_1+M_2-m_1-m_2)^2\left(\int_\kappa^\ell w^\sigma(\tau)\Delta\tau\right)^2$$

by what we have shown earlier. By using the below notations

$$\int_{\kappa}^{\ell} w^{\sigma}(\tau)g^{\sigma}(\tau)\Delta\tau = \bar{g}, \qquad \int_{\kappa}^{\ell} w^{\sigma}(\tau)h^{\sigma}(\tau)\Delta\tau = \bar{h}$$

which results in

$$\left(\int_{\kappa}^{\ell} w^{\sigma}(\tau)(g^{\sigma}(\tau)-h^{\sigma}(\tau))\Delta\tau\right)^2 = (\bar{g}-\bar{h})^2$$

and

$$\left(\int_{\kappa}^{\ell} w^{\sigma}(\tau)(g^{\sigma}(\tau)+h^{\sigma}(\tau))\Delta\tau\right)^2 = (\bar{g}+\bar{h})^2$$

and Jensen's inequality (given in Theorem 2.7) for the convex quadratic function

$$\int_{\kappa}^{\ell} w^{\sigma}(\tau)(g^{\sigma}(\tau)-h^{\sigma}(\tau))^2\Delta\tau \geq \left(\int_{\kappa}^{\ell} w^{\sigma}(\tau)(g^{\sigma}(\tau)-h^{\sigma}(\tau))\Delta\tau\right)^2 = (\bar{g}-\bar{h})^2,$$

and

$$\int_{\kappa}^{\ell} w^{\sigma}(\tau)(g^{\sigma}(\tau)+h^{\sigma}(\tau))^2\Delta\tau \leq \left(\int_{\kappa}^{\ell} w^{\sigma}(\tau)(g^{\sigma}(\tau)+h^{\sigma}(\tau))\Delta\tau\right)^2 = (\bar{g}+\bar{h})^2,$$

we get

$$\begin{aligned}
D(g,h) &\leq \frac{1}{4}\Bigg[\frac{1}{4}(M_1+M_2-m_1-m_2)^2\left(\int_{\kappa}^{\ell} w^{\sigma}(\tau)\Delta\tau\right)^2 \\
&+ \int_{\kappa}^{\ell} w^{\sigma}(\tau)\Delta\tau\left(\int_{\kappa}^{\ell} w^{\sigma}(\tau)(g^{\sigma}(\tau)+h^{\sigma}(\tau))\Delta\tau\right)^2 \\
&- \int_{\kappa}^{\ell} w^{\sigma}(\tau)\Delta\tau\int_{\kappa}^{\ell} w^{\sigma}(\tau)(g^{\sigma}(\tau)-h^{\sigma}(\tau))^2\Delta\tau - 4\bar{g}\bar{h}\Bigg] \\
&= \frac{1}{16}(M_1+M_2-m_1-m_2)^2\left(\int_{\kappa}^{\ell} w^{\sigma}(\tau)\Delta\tau\right)^2 \\
&+ \frac{1}{4}\Bigg[(\bar{g}-\bar{h})^2\int_{\kappa}^{\ell} w^{\sigma}(\tau)\Delta\tau - \int_{\kappa}^{\ell} w^{\sigma}(\tau)\Delta\tau \\
&\times \int_{\kappa}^{\ell} w^{\sigma}(\tau)(g^{\sigma}(\tau)-h^{\sigma}(\tau))^2\Delta\tau\Bigg] \\
&\leq \frac{1}{16}(M_1+M_2-m_1-m_2)^2\left(\int_{\kappa}^{\ell} w^{\sigma}(\tau)\Delta\tau\right)^2 \\
&= \frac{1}{4}\left[(M_1-m_1)(M_2-m_2)+\frac{1}{4}(M_1-m_1-M_2+m_2)^2\right]
\end{aligned}$$

$$\times\left(\int_{\kappa}^{\ell} w^{\sigma}(\tau)\Delta\tau\right)^2.$$

If $M_1 - m_1 = M_2 - m_2$, then clearly

$$D(g,h) \leq \frac{1}{4}(M_1 - m_1)^2\left(\int_{\kappa}^{\ell} w^{\sigma}(\tau)\Delta\tau\right)^2,$$

but if $M_1 - m_1 \neq M_2 - m_2$, then we state

$$r = \sqrt{\frac{M_2 - m_2}{M_1 - m_1}}, \qquad s = \sqrt{\frac{M_1 - m_1}{M_2 - m_2}}$$

and suppose $g_1(\tau) = rg(\tau)$, $h_1(\tau) = sh(\tau)$ and obtain

$$\bar{m_1} = rm_1 \leq g_1(\tau) \leq rM_1 = \bar{M_1}, \qquad \bar{m_2} = sm_2 \leq h_1(\tau) \leq sM_2 = \bar{M_2}.$$

Now we get

$$\begin{aligned}\bar{M_1} - \bar{m_1} &= \sqrt{\frac{M_2 - m_2}{M_1 - m_1}}(M_1 - m_1) = \sqrt{(M_1 - m_1)(M_2 - m_2)} \\ &= \sqrt{\frac{M_1 - m_1}{M_2 - m_2}}(M_2 - m_2) = \bar{M_2} - \bar{m_2}\end{aligned}$$

and

$$\begin{aligned}D(g,h) = rsD(g,h) = D(g_1,h_1) &\leq \frac{1}{4}(\bar{M_1} - \bar{m_1})(\bar{M_2} - \bar{m_2}) \\ &\quad \times\left(\int_{\kappa}^{\ell} w^{\sigma}(\tau)\Delta\tau\right)^2 \\ &= \frac{1}{4}rs(M_1 - m_1)(M_2 - m_2)\left(\int_{\kappa}^{\ell} w^{\sigma}(\tau)\Delta\tau\right)^2 \\ &= \frac{1}{4}(M_1 - m_1)(M_2 - m_2)\left(\int_{\kappa}^{\ell} w^{\sigma}(\tau)\Delta\tau\right)^2.\end{aligned}$$

If we consider the case of $-g$, then we summarize

$$D(-g,h) = -D(g,h) \leq \frac{1}{4}(-m_1 + M_1)(M_2 - m_2)\left(\int_{\kappa}^{\ell} w^{\sigma}(\tau)\Delta\tau\right)^2,$$

and finally we get

$$|D(g,h)| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2)\left(\int_{\kappa}^{\ell} w^{\sigma}(\tau)\Delta\tau\right)^2.$$

Thus (3.2) holds. □

Remark 3.2. If put $w = \frac{1}{\ell - \kappa}$ in Theorem 3.1, then we recapture the result of Theorem 3.1 of [3].

If we implement the Grüss inequality to several time scales, we obtain some well-known and new results.

Discrete Case:

Corollary 3.3. *We suppose $\mathbb{T}=\mathbb{Z}$. Let $\kappa=0$, $\ell=n$, $\tau=b$ and $g(c)=y_c$. Then*

$$\left|\sum_{b=1}^{n} w_b \sum_{b=1}^{n} w_b y_b z_b - \sum_{b=1}^{n} w_b y_b \sum_{b=1}^{n} w_b z_b\right| \leq \frac{1}{4}(M_1-m_1)(M_2-m_2)\left(\sum_{b=1}^{n} w_b\right)^2,$$

where

$$m_1 \leq y_b \leq M_1, \qquad m_2 \leq z_b \leq M_2.$$

In the discrete case the sharpness of this inequality holds only for even n.

The following inequality holds for even and odd n, is sharp in our sense.

$$\left|\sum_{b=1}^{n} w_b \sum_{b=1}^{n} w_b y_b z_b - \sum_{b=1}^{n} w_b y_b \sum_{b=1}^{n} w_b z_b\right| \leq \frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(M_1-m_1) \times (M_2-m_2)\left(\sum_{b=1}^{n} w_b\right)^2. \tag{3.5}$$

Remark 3.4. If put $w=\frac{1}{\ell-\kappa}$ in Corollary 3.3, then we recapture Corollary 3.3 of [3] and Corollary 1.3 of [8].

Remark 3.5. If put $w=\frac{1}{\ell-\kappa}$ in (3.5), then we recapture discrete inequality (which are cited in [3] and [2]) for even and odd n .

Continuous Case:

Corollary 3.6. *Let $\mathbb{T}=\mathbb{R}$. We have*

$$\left|\int_\kappa^\ell w(\tau)d\tau \int_\kappa^\ell w(\tau)g(\tau)h(\tau)d\tau - \int_\kappa^\ell w(\tau)g(\tau)d\tau \int_\kappa^\ell w(\tau)h(\tau)d\tau\right| \leq \frac{1}{4}(M_1-m_1)(M_2-m_2)\left(\int_\kappa^\ell w(\tau)d\tau\right)^2, \tag{3.6}$$

where

$$m_1 \leq g(\tau) \leq M_1, \qquad m_2 \leq h(\tau) \leq M_2,$$

which is exactly the result of Theorem 1.1 of [9]*. The constant $\frac{1}{4}$ in the right-hand side of* (3.6) *is the best possible.*

Remark 3.7. If put $w=\frac{1}{\ell-\kappa}$ in Corollary 3.6, then we recapture the Grüss inequality shown in [11] and Corollary 3.2 of paper [3].

Quantum Calculus Case:

Corollary 3.8. *We suppose* $\mathbb{T} = q^{\mathbb{N}_0}$, $1 < q$, $\kappa = q^m$, $\ell = q^n$ *and* $w = w^c$. *Then*

$$\left| \frac{\sum_{c=m}^{n-1} w^c g(q^{c+1}) h(q^{c+1})}{\sum_{c=m}^{n-1} w^c} - \frac{1}{\left(\sum_{c=m}^{n-1} w^c\right)^2} \left(\sum_{c=m}^{n-1} w^c g(q^{c+1}) \right) \right.$$
$$\left. \times \left(\sum_{c=m}^{n-1} w^c h(q^{c+1}) \right) \right| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2),$$

where

$$m_1 \leq g(q^c) \leq M_1, \qquad m_2 \leq h(q^c) \leq M_2.$$

This corresponds to the result obtained in Corollary 3.4 of paper [3].

4. Conclusion

In this section, we have given generalization of Grüss inequality on time scales and obtained unified results to the previous research papers [2, 3, 8, 9, 11] and recaptured all results of [3] for discrete and continuous versions. Moreover, we have used our weighted results to the quantum calculus case.

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