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Duality of $\mathbb{F}_q\mathbb{F}_q[u]$-additive Skew Cyclic Codes

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Abstract. Li et al. (2021) obtained the generator polynomials and the minimal generating sets of $\mathbb{F}_q\mathbb{F}_q[u]$-linear skew cyclic codes, where q is a power of a prime integer and $u^2 = 0$. In this paper, we determine the structure of dual of these codes in terms of their generating polynomials and we illustrate the dual of some special $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes.

Keywords: Chain ring, Skew cyclic code, Additive skew cyclic code, Duality.

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1. Introduction

The class of constacyclic codes plays a very significant role in the theory of error-correcting codes. The most important class of these codes is the class of all cyclic codes. The $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are subgroups of the group $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. These codes are observed as a generalization of quaternary and binary codes for $\alpha = 0$ and $\beta = 0$, respectively. Abualrub et al. studied $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes in [1]. Inspired by this paper, Aydogdu et al. presented the structure

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of cyclic and constacyclic codes and their duals in [2]. Borges et al. carried the findings on $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes over $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic codes for any prime p, in [4]. In order to generalize the concepts appeared in [4] and [2], Mahmoudi and Samei (in [12]) introduced and investigated the SR-additive codes as R-submodule of $S^\alpha \times R^\beta$, where R is an arbitrary finite commutative ring and S is a finite R-algebra.

The detailed structures of all constacyclic codes of length p^s over $R_2 = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ (with $u^2 = 0$) has been classified in [7]. Dinh et al. [8] studied the stucture of self-dual cyclic codes of length p^s over R_2.

One of the most applicable generalizations of cyclic codes is the class of skew cyclic codes which were introduced by Boucher in [5]. The algebraic structure and some properties of these codes over finite chain rings and their Euclidean and Hermitian dual codes have been established in [10]. Hesari et al. in [9] determined the structure of (Euclidean) dual of some special skew cyclic codes of length p^s over R_2 and identified all self-dual codes in this category. Li et al. have studied the $\mathbb{F}_q\mathbb{F}_q[u]$-linear skew cyclic codes in [11].

The purpose of this paper is to continue the investigations done in [11] and to determine the structure of the (Euclidean) dual of $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes obtained there.

This paper has been organized as follows. Section 2 contains some basic definitions, notations and specifics of the $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes, where $u^2 = 0$. These codes have been classified into eight different types in terms of their explicit generator polynomials. Section 3 is divided into two parts. In Subection 3.1, we calculate the dual of $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes. Finally, in Subsection 3.2, as an application, we provide some examples of $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes and compute their duals in terms of their generating polynomials.

2. Preliminaries

In this section, we present some basic definitions, notations and previous results related to our work.

A ring R is a *principal left ideal ring* if it has unity and every left ideal is principally generated. R is called a *local ring* if R has a unique maximal right (left) ideal. Furthermore, a ring R is called a *chain ring* if the set of all ideals of R is linearly ordered under set-theoretic inclusion.

Definition 2.1. Let R be a finite commutative ring and σ be an automorphism of R. Consider the ring $R[x;\sigma] = \{a_0 + a_1x + a_2x^2 + ... + a_nx^n : \quad a_i \in R \text{ and } n \in \mathbb{N}_0\}$, where the addition is defined to be the usual addition of polynomials and the multiplication is defined by the basic rule $xa = \sigma(a)x$ ($a \in R$). The multiplication is extended to all elements in $R[x;\sigma]$ by associativity and distributivity. The ring $R[x;\sigma]$ is called a *skew polynomial ring* over R and every element in $R[x;\sigma]$ is called a *skew polynomial.* It is easily seen that the ring $R[x;\sigma]$ is non-commutative unless σ is the identity automorphism on R.

Proposition 2.2. [10, Proposition 2.2] *Let R be a finite commutative ring, n be a positive integer and λ be a unit in R. Then the following statements are equivalent:*
i) $x^n - \lambda$ is central in $R[x;\sigma]$.
ii) $\langle x^n - \lambda \rangle$ is two-sided.
iii) n is a multiple of the order of σ and λ is fixed by σ.

Definition 2.3. A code C of length n over R is a non-empty subset of R^n and the ring R is referred to as the alphabet of C. If C is also an R-submodule of R^n, then C is called a *linear code.*

For a given automorphism σ of R, a code C over R is called skew σ-cyclic, if C is closed under σ-cyclic shift $\rho_{\sigma} : R^n \longrightarrow R^n$ which is defined by

$$\rho_{\sigma}\big((a_0, a_1, ..., a_{n-1})\big) = \big(\sigma(a_{n-1}), \sigma(a_0), ..., \sigma(a_{n-2})\big).$$

Each codeword $c = (c_0, c_1, ..., c_{n-1})$ is customarily identified with its polynomial representation $c(x) = c_0 + c_1x + ... + c_{n-1}x^{n-1}$. In particular, if $o(\sigma) \mid n$, then $\frac{R[x;\sigma]}{\langle x^n-1\rangle}$ is a ring and the polynomial $xc(x)$ corresponds to the σ-cyclic shift of $c = (c_0, c_1, ..., c_{n-1})$. In this way, C is a skew cyclic code of length n over R if and only if C is a left ideal of $\frac{R[x;\sigma]}{\langle x^n-1\rangle}$. When there is no ambiguity, we say "skew cyclic" instead of "skew σ-cyclic".

Proposition 2.4. [10, Proposition 2.3] *Let $h(x), g(x) \in R[x;\sigma]$. If $h(x)g(x)$ is a monic central skew polynomial, then $h(x)g(x) = g(x)h(x)$.*

Let $\mathbb{F}_q$ denote the finite field with q elements and δ be a primitive $(q-1)$-th root of unity in $\mathbb{F}_q$, i.e.,

$$\mathbb{F}_q = \{0, \delta, \cdots, \delta^{q-2}, \delta^{q-1} = 1\}.$$

Suppose that $R_2 = \mathbb{F}_q + u\mathbb{F}_q$, with $u^2 = 0$. It is known that R_2 is a chain ring with the unique maximal ideal $u\mathbb{F}_q$.

Lemma 2.5. [10, Corollary 2.1] *For $\theta \in \mathrm{Aut}(\mathbb{F}_q)$ and $\eta \in \mathbb{F}_q^*$, let*

$$\Theta_{\theta,\eta} : \mathbb{F}_q + u\mathbb{F}_q \longrightarrow \mathbb{F}_q + u\mathbb{F}_q$$

be defined by

$$\Theta_{\theta,\eta}(a + bu) = \theta(a) + \eta\theta(b)u.$$

Then $\mathrm{Aut}(\mathbb{F}_q + u\mathbb{F}_q) = \{\Theta_{\theta,\eta} : \quad \theta \in \mathrm{Aut}(\mathbb{F}_q) \text{ and } \eta \in \mathbb{F}_q^*\}$.

As we saw in the above lemma, every automorphism of $\mathbb{F}_q + u\mathbb{F}_q$ is of the form $\Theta_{\theta,\eta}$. From now on, let $\eta = 1$ and $\Theta_{\theta,1}$ will be denoted by Θ.

We say that $f(x)$ is a *right divisor* (*left divisor*) of $g(x)$ in $\mathbb{F}_q[x,\theta]$ and we write $f(x) \mid_r g(x)$ ($f(x) \mid_l g(x)$) if there exists a skew polynomial $h(x)$ such that $g(x) = h(x)f(x)$ ($g(x) = f(x)h(x)$).

Definition 2.6. Suppose $f(x), g(x)$ are skew polynomials in $\mathbb{F}_q[x;\theta]$. The *greatest common right divisor* of $f(x)$ and $g(x)$ is the monic polynomial $d_r(x) \in \mathbb{F}_q[x;\theta]$, where $d_r(x) \mid_r f(x), d_r(x) \mid_r g(x)$ and for any $d'_r(x) \in \mathbb{F}_q[x;\theta]$ such that $d'_r(x) \mid_r f(x)$ and $d'_r(x) \mid_r g(x)$, we have $d'_r(x) \mid_r d_r(x)$. We denote $d_r(x)$ by $\gcd_r(f(x), g(x))$.

The *greatest common left divisor* $d_l(x)$ of $f(x)$ and $g(x)$, written

$$d_l(x) = \gcd_l(f(x), g(x))$$

is defined in the same manner using left division.

Definition 2.7. The *least common left multiple* (lcm_l) of $f(x)$ and $g(x)$ is the unique monic polynomial $m_l(x) = \mathrm{lcm}_l(f(x), g(x))$ such that $f(x) \mid_r m_l(x), g(x) \mid_r m_l(x)$ and for any $m'(x) \in \mathbb{F}_q[x;\theta]$ such that $f(x) \mid_r m'(x)$ and $g(x) \mid_r m'(x)$, then we have $m_l(x) \mid_r m'(x)$.

Lemma 2.8. [13, Page 486] *Let $f(x)$ and $g(x)$ be skew polynomials in $\mathbb{F}_q[x;\theta]$. Then,*

$$\deg(\mathrm{lcm}_l(f(x), g(x))) = \deg(f(x)) + \deg(g(x)) - \deg(\gcd_r(f(x), g(x))).$$

Lemma 2.9. *Let $f(x)$, $g(x)$, $h(x)$ and $c(x)$ be skew polynomials in the ring $\mathbb{F}_q[x;\theta]$ such that $c(x)$ is a central element. If*

$$f(x)g(x) \equiv 0 \pmod{c(x)}$$

and

$$f(x)h(x) \equiv 0 \pmod{c(x)}.$$

Then,

$$f(x)\gcd_l(g(x), h(x)) \equiv 0 \pmod{c(x)}.$$

Proof. By the hypothesis, $f(x)g(x) = k_1(x)c(x)$ and $f(x)h(x) = k_2(x)c(x)$, for some $k_1(x), k_2(x) \in \mathbb{F}_q[x;\theta]$. In other hand, there exist $a(x)$ and $b(x)$ in $\mathbb{F}_q[x;\theta]$ such that

$$g(x)a(x) + h(x)b(x) = \gcd_l(g(x), h(x)).$$

Hence, $f(x)g(x)a(x) + f(x)h(x)b(x) = f(x)\gcd_l(g(x), h(x))$. We have

$$k_1(x)c(x)a(x) + k_2(x)c(x)b(x) = f(x)\gcd_l(g(x), h(x)).$$

Since $c(x)$ is central element, then $c(x) | f(x)\gcd_l(g(x), h(x))$.

□

Throughout this paper, we use the following symbols for simplicity:

- $R_2 = \mathbb{F}_q[u] = \mathbb{F}_q + u\mathbb{F}_q$.
- $m = \mathrm{lcm}(\alpha, \beta)$.
- $\mathcal{R}_{1,k} = \frac{\mathbb{F}_q[x;\theta]}{\langle x^k - 1\rangle}$, for $k = \alpha, \beta, m$.
- $\mathcal{R}_\tau = \frac{R_2[x;\Theta]}{\langle x^\tau - 1\rangle}$, for $\tau = \beta, n, m$.
- $\mathcal{R} = \mathcal{R}_{1,\alpha} \times \mathcal{R}_\beta$.

In this paper, we assume that $o(\Theta) = o(\theta) \mid \gcd(\alpha, \beta)$, where α and β are positive integers and $o(\theta)$ is the order of θ. By Proposition 2.2, $\langle x^\alpha - 1\rangle$ and $\langle x^\beta - 1\rangle$ are two-sided ideals of $R_2[x;\Theta]$.

Since $x^\alpha - 1$ and $x^\beta - 1$ are monic central skew polynomials, by Proposition 2.4, right divisors of $x^\alpha - 1$ and $x^\beta - 1$ are two-sided divisors.

Let $\mu : R_2 \longrightarrow \mathbb{F}_q$, be the natural ring morphism, defined by $\mu(a_0 + ua_1) = a_0$. We consider the set

$$\mathbb{F}_q \times R_2 = \{(a|b) : \quad a \in \mathbb{F}_q, b \in R_2\}.$$

By the following scalar multiplication, $\mathbb{F}_q \times R_2$ is a left R_2-module,

$$\begin{aligned} * : R_2 \times (\mathbb{F}_q \times R_2) &\longrightarrow \mathbb{F}_q \times R_2, \\ \nu * (a|b) &= (\mu(\nu)a|\nu b). \end{aligned}$$

This multiplication can be extended to the set $\mathbb{F}_q^\alpha \times R_2^\beta$ in the following way. For any $\nu \in R_2$ and $(a_0, a_1, ..., a_{\alpha-1}|b_0, ..., b_{\beta-1}) \in \mathbb{F}_q^\alpha \times R_2^\beta$ define

$$\nu * (a_0, a_1, ..., a_{\alpha-1}|b_0, ..., b_{\beta-1}) = (\mu(\nu)a_0, \mu(\nu)a_1, ..., \mu(\nu)a_{\alpha-1}|\nu b_0, ..., \nu b_{\beta-1}).$$

Definition 2.10. A non-empty subset C of $\mathbb{F}_q^\alpha \times R_2^\beta$ is called an $\mathbb{F}_q\mathbb{F}_q[u]$-*additive skew cyclic code* of length $n = \alpha + \beta$, if C is a left R_2-submodule of $\mathbb{F}_q^\alpha \times R_2^\beta$.

There is a bijective map between $\mathbb{F}_q^\alpha \times R_2^\beta$ and $\mathcal{R} = \mathcal{R}_{1,\alpha} \times \mathcal{R}_\beta$ given by

$$(a_0, ..., a_{\alpha-1}|b_0, ..., b_{\beta-1}) \longmapsto (a_0 + \cdots + a_{\alpha-1}x^{\alpha-1}|b_0 + \cdots + b_{\beta-1}x^{\beta-1}) = (a(x)|b(x)).$$

Suppose $(f(x)|g(x)) \in \mathcal{R}$ and $\nu(x) \in R_2[x;\Theta]$, we have

$$\begin{aligned} &* : R_2[x;\Theta] \times \mathcal{R} \longrightarrow \mathcal{R}, \\ &\nu(x) * (f(x)|g(x)) = (\mu(\nu(x))f(x)|\nu(x)g(x)), \end{aligned}$$

where

$$\mu(\nu(x)) = \mu(\sum_{j=0}^{\alpha-1} \nu_j x^j) = \sum_{j=0}^{\alpha-1} \mu(\nu_j) x^j$$

and $\nu_j \in R_2$.

Note that the skew polynomial ring $\mathbb{F}_q[x;\theta]$ is not a unique factorization domain. In fact, many different factorizations may be possible in this domain. The ring $R_2[x;\Theta]$ is neither left nor right Euclidean. However, left and right divisions can be defined for some suitable elements. Let $f(x), g(x)$ be skew polynomials in $R_2[x;\Theta]$, where leading coefficient of $f(x)$ is a unit in R_2. Then there exist $q(x), r(x) \in R_2[x,\Theta]$ such that $g(x) = q(x)f(x)+r(x)$, where $r(x) = 0$ or $\deg(r(x)) < \deg(f(x))$. Note that $q(x)$ and $r(x)$ are unique.

Set $\mathcal{F}_k := \{a(x) \in \mathbb{F}_q[x;\theta] : \quad a(x) \text{ is a monic factor of } x^k - 1\}$, for $k = \alpha, \beta, m$.

Proposition 2.11. [5, Theorem 1] *The ring $\mathcal{R}_{1,k}$ is a principal left ideal ring, in which left ideals are generated by $a(x) + \langle x^k - 1\rangle$, where $a(x)$ is a monic divisor of $x^k - 1$ in $\mathbb{F}_q[x;\theta]$.*

It is well known that $R_2 = \mathbb{F}_q + u\mathbb{F}_q$ is a finite chain ring of nilpotency index 2 and the unique maximal ideal $u\mathbb{F}_q$. In [11], Li et al. determined the $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes of length $\alpha+\beta$, where α and β are multiples of the order of Θ.

Proposition 2.12. [10, Theorem 2.2] *A linear code $\mathcal{C}$ of length β is a skew cyclic code over R_2 if and only if $\mathcal{C}$ is a left ideal of $\mathcal{R}_\beta$.*

For each left ideal $\mathcal{I}$ in $\mathcal{R}_\beta$, the image $\mu(\mathcal{I} :_{\mathcal{R}_\beta} u) = \mu(\{v \in \mathcal{R}_\beta : \ vu \in \mathcal{I}\})$ is a left ideal in $\mathcal{R}_{1,\beta}$. Inasmuch as every skew cyclic code $\mathcal{C}$ over the ring R_2 is a left ideal of $\mathcal{R}_\beta$, the image $\mu(\mathcal{C} :_{\mathcal{R}_\beta} u)$ is in fact a skew cyclic code over $\mathbb{F}_q$ which is called the *torsion code* associated to $\mathcal{C}$ and it is denoted by $\mathrm{Tor}(\mathcal{C})$.

Lemma 2.13. [11, Lemma 3] *A code C is an $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic code of length $\alpha+\beta$ if and only if C is a left $R_2[x;\Theta]$-submodule of $\mathcal{R}$.*

Theorem 2.14. [11, Theorem 1] *Every left $R_2[x;\Theta]$-submodule of $\mathcal{R}$ is of the form*

$$\mathcal{R}_n\big((a(x)|0)\big) + \mathcal{R}_n\big((k_1(x)|a_1(x) + ug_1(x))\big) + \mathcal{R}_n\big((k_2(x)|ua_2(x))\big),$$

where $n := \alpha+\beta$, $a(x) \in \mathcal{F}_\alpha$, $a_i(x) \in \mathcal{F}_\beta$, $a_2(x) \mid_r a_1(x)$, $k_i(x) \in \mathcal{R}_{1,\alpha}$, $\deg(k_i(x)) < \deg(a(x))$ and $\deg(g_1(x)) < \deg(a_2(x))$.

We can list all $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes of length $\alpha+\beta$ as follows:

Theorem 2.15. [11, Theorem 2] *Every $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic code of length $n = \alpha+\beta$ can be presented in one of the following forms:*

- *Type* 1 : 0*,* $\mathcal{R}$.
- *Type* 2 : $\mathcal{R}_n\big((a(x)|0)\big)$*, where* $a(x) \in \mathcal{F}_\alpha$ *and* $0 \le \deg(a(x)) \le \alpha - 1$.
- *Type* 3 : $\mathcal{R}_n\big((k_1(x)|a_1(x) + ug_1(x))\big)$*, where* $k_1(x) \in \mathcal{R}_{1,\alpha}$*,* $a_1(x) \in \mathcal{F}_\beta$*,*

$0 \leq \deg(a_1(x)) \leq \beta - 1$, $g_1(x) \in \mathcal{R}_{1,\beta}$ *and* $\deg(g_1(x)) < \deg(a_1(x))$.

- *Type* 4 : $\mathcal{R}_n\big((k_2(x)|ua_2(x))\big)$, *where* $k_2(x) \in \mathcal{R}_{1,\alpha}$, $a_2(x) \in \mathcal{F}_\beta$ *and* $0 \leq \deg(a_2(x)) \leq \beta - 1$.
- *Type* 5 : $\mathcal{R}_n\big((a(x)|0)\big) + \mathcal{R}_n\big((k_1(x)|a_1(x) + ug_1(x))\big)$, *where* $a(x) \in \mathcal{F}_\alpha$, $0 \leq \deg(a(x)) \leq \alpha - 1$, $k_1(x) \in \mathcal{R}_{1,\alpha}$, $a_1(x) \in \mathcal{F}_\beta$, $0 \leq \deg(a_1(x)) \leq \beta - 1$, $g_1(x) \in \mathcal{R}_{1,\beta}$, $\deg(k_1(x)) < \deg(a(x))$ *and* $\deg(g_1(x)) < \deg(a_1(x))$.
- *Type* 6 : $\mathcal{R}_n\big((a(x)|0)\big)+\mathcal{R}_n\big((k_2(x)|ua_2(x))\big)$, *where* $a(x) \in \mathcal{F}_\alpha$, $0 \leq \deg(a(x)) \leq \alpha - 1$, $k_2(x) \in \mathcal{R}_{1,\alpha}$, $a_2(x) \in \mathcal{F}_\beta$, $0 \leq \deg(a_2(x)) \leq \beta - 1$ *and* $\deg(k_2(x)) < \deg(a(x))$.
- *Type* 7 : $\mathcal{R}_n\big((k_1(x)|a_1(x) + ug_1(x))\big) + \mathcal{R}_n\big((k_2(x)|ua_2(x))\big)$, *where* $k_i(x) \in \mathcal{R}_{1,\alpha}$, $a_i(x) \in \mathcal{F}_\beta$, $a_2(x) \mid_r a_1(x)$, $0 \leq \deg(a_1(x)) \leq \beta - 1$, $g_1(x) \in \mathcal{R}_{1,\beta}$ *and* $\deg(g_1(x)) < \deg(a_2(x))$.
- *Type* 8 : $\mathcal{R}_n\big((a(x)|0)\big) + \mathcal{R}_n\big((k_1(x)|a_1(x) + ug_1(x))\big) + \mathcal{R}_n\big((k_2(x)|ua_2(x))\big)$, *where* $a(x) \in \mathcal{F}_\alpha$, $0 \leq \deg(a(x)) \leq \alpha - 1$, $a_i(x) \in \mathcal{F}_\beta$, $0 \leq \deg(a_i(x)) \leq \beta - 1$, $a_2(x) \mid_r a_1(x)$, $k_i(x) \in \mathcal{R}_{1,\alpha}$, $\deg(k_i(x)) < \deg(a(x))$ *and* $\deg(g_1(x)) < \deg(a_2(x))$.

Lemma 2.16. *Let* $C = \mathcal{R}_n\big((a(x)|0)\big)+\mathcal{R}_n\big((k_1(x)|a_1(x)+ug_1(x))\big)+\mathcal{R}_n\big((k_2(x)|ua_2(x))\big)$, *be an* $\mathbb{F}_q\mathbb{F}_q[u]$*-additive skew cyclic codes of length* $\alpha + \beta$ *such that* $\deg(g_1(x)) < \deg(a_2(x))$. *Then,* $g_1(x)$ *with the above condition is unique.*

Proof. Let

$$\pi : C \longrightarrow \mathcal{R}_\beta,$$
$$(\lambda(x)|\lambda_1(x)) \longmapsto \lambda_1(x)$$

be a homomorphism between two left $R_2[x; \Theta]$-modules and assume that $g'(x)$ is a polynomial satisfying

$$C = \mathcal{R}_n\big((a(x)|0)\big) + \mathcal{R}_n\big((k_1(x)|a_1(x) + ug'(x))\big) + \mathcal{R}_n\big((k_2(x)|ua_2(x))\big).$$

Hence $(a_1(x)+ug_1(x))-(a_1(x)+ug'(x)) \in \mathrm{Im}(\pi)$. So $u(g_1(x)-g'(x)) \in \mathrm{Im}(\pi)$, implies that $g_1(x) - g'(x) \in \mathrm{Tor}(\mathrm{Im}(\pi)) = \mathcal{R}_{1,\beta}\big(a_2(x)\big)$. If $g_1(x) \neq g'(x)$, then $\deg(a_2(x)) \leq \deg(g_1(x) - g'(x))$. But by the hypothesis, $\deg(g_1(x) - g'(x)) < \deg(a_2(x))$, a contradiction. Thus $g_1(x) = g'(x)$. □

3. Duality of $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes

This section is devoted to discussing the structural properties of the dual codes of $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes.

3.1. **Dual of $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes of length $\alpha+\beta$.**

In this subsection, we determine the dual of $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes of length $\alpha+\beta$.
Let

$$x = (x_0, x_1, ..., x_{\alpha-2}, x_{\alpha-1}|x'_0, x'_1, ..., x'_{\beta-2}, x'_{\beta-1})$$

and

$$y = (y_0, y_1, ..., y_{\alpha-2}, y_{\alpha-1}|y'_0, y'_1, ..., y'_{\beta-2}, y'_{\beta-1})$$

be elements of $\mathbb{F}_q^\alpha \times R_2^\beta$. The inner product is defined as

$$x \cdot y = u\sum_{i=0}^{\alpha-1} x_i y_i + \sum_{j=0}^{\beta-1} x'_j y'_j.$$

The dual of an $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic code of length $\alpha+\beta$ is an $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic code of length $\alpha+\beta$, defined by

$$C^\perp = \{x \in \mathbb{F}_q^\alpha \times R_2^\beta : \quad x \cdot y = 0, \quad \text{for all } y \in C\}.$$

A code C is called *self-orthogonal* if $C \subseteq C^\perp$, and it is called *self-dual* if $C = C^\perp$.

Definition 3.1. [6, Definition 3] Let $f(x) = a_0 + a_1x + ... + a_tx^t \in \mathbb{F}_q[x;\theta]$, where $a_t \neq 0$. Then the polynomial $f^*(x) = a_t + \theta(a_{t-1})x + ... + \theta^t(a_0)x^t$ is called the *reciprocal polynomial* of $f(x)$. Equivalently, $f^*(x)$ can be expressed by $f^*(x) = \sum_{i=0}^{t} \theta^i(a_{t-i})x^i$.

Lemma 3.2. [6, Section 3] *Let ψ be the map*

$$\psi : R_2[x;\Theta] \longrightarrow R_2[x;\Theta],$$
$$\sum_{i=0}^{n} a_ix^i \longmapsto \sum_{i=0}^{n} \Theta(a_i)x^i,$$

where $a_i \in R_2$. Then ψ is a ring homomorphism.

Lemma 3.3. *Let $f(x), g(x)$ be skew polynomials in $R_2[x;\Theta]$.*
(1) *If* $\deg(f) \geq \deg(g)$, *then* $(f(x)+g(x))^* = f^*(x) + x^{\deg f - \deg g}g^*(x)$.
(2) $(fg)^* = \psi^{\deg f}(g^*)f^*$.
(3) $(f^*)^* = \psi^n(f)$, *where* $\deg(f) = n$.

Proof. It is similar to the proof of [9, Lemma 2.8]. □

Set $\Gamma_\nu(x) = \sum_{j=0}^{\nu-1} x^j$. As the unit element 1 remains fixed under automorphism θ, we have the following lemma whose proof is evident.

Lemma 3.4. *Let $n, n' \in \mathbb{N}$. Then* $x^{nn'} - 1 = (x^n - 1)\Gamma_{n'}(x^n) = \Gamma_{n'}(x^n)(x^n - 1)$.

Definition 3.5. Let $\mathbf{v}(x) = (v(x)|v'(x))$ and $\mathbf{w}(x) = (w(x)|w'(x))$ be any two elements in $\mathcal{R}$ and $m = \text{lcm}(\alpha, \beta)$. Define the map

$$\circ : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}_m$$

such that

$$\begin{aligned}\circ(\mathbf{v}(x), \mathbf{w}(x)) =& uv(x)\psi^{m-\deg(w(x))}(w^*(x))x^{m-1-\deg(w(x))}\Gamma_{\frac{m}{\alpha}}(x^\alpha) \\ &+ v'(x)\psi^{m-\deg(w'(x))}(w'^*(x))x^{m-1-\deg(w'(x))}\Gamma_{\frac{m}{\beta}}(x^\beta).\end{aligned}$$

The map $\circ$ is bilinear between left $R_2[x;\Theta]$-modules. We denote $\circ(\mathbf{v}(x), \mathbf{w}(x))$ by $\mathbf{v}(x) \circ \mathbf{w}(x)$.

Proposition 3.6. *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be elements in* $\mathbb{F}_q^\alpha \times R_2^\beta$ *with associated polynomials* $\mathbf{v}(x) = (v(x)|v'(x))$ *and* $\mathbf{w}(x) = (w(x)|w'(x))$*, respectively. Then,* $\mathbf{w}$ *is orthogonal to* $\mathbf{v}$ *and all its* Θ*-shifts if and only if*

$$\mathbf{v}(x) \circ \mathbf{w}(x) = 0.$$

Proof. Let

$$\mathbf{v} = (v_0, v_1, ..., v_{\alpha-1}|v'_0, v'_1, ..., v'_{\beta-1})$$

and

$$\mathbf{w} = (w_0, w_1, ..., w_{\alpha-1}|w'_0, w'_1, ..., w'_{\beta-1}).$$

Let

$$\mathbf{v}^{(i)} = (\theta^i(v_{0-i}), \theta^i(v_{1-i}), ..., \theta^i(v_{\alpha-1-i})|\Theta^i(v'_{0-i}), \Theta^i(v'_{1-i}), ..., \Theta^i(v'_{\beta-1-i}))$$

be the i-th cyclic Θ-shift of $\mathbf{v}$ such that $0 \le i \le m-1$. For $0 \le j_1 \le \alpha - 1$ and $0 \le j_2 \le \beta - 1$, indices $j_1 - i$ and $j_2 - i$ are computed modulo α and β, respectively. Hence

$$\mathbf{v}^{(i)}.\mathbf{w} = 0 \quad \text{if and only if} \quad u\sum_{j=0}^{\alpha-1}\theta^i(v_{j-i})w_j + \sum_{\nu=0}^{\beta-1}\Theta^i(v'_{\nu-i})w'_\nu = 0.$$

Let $S_i = u\sum_{j=0}^{\alpha-1}\theta^i(v_{j-i})w_j + \sum_{\nu=0}^{\beta-1}\Theta^i(v'_{\nu-i})w'_\nu$. We can get

$$\begin{aligned}\mathbf{v}(x)\circ\mathbf{w}(x) = u\Big[&\sum_{\mu=0}^{\alpha-1}\sum_{j=\mu}^{\alpha-1} v_{j-\mu}\theta^{m-\mu}(w_j)x^{m-1-\mu}\\ &+\sum_{\mu=1}^{\alpha-1}\sum_{j=\mu}^{\alpha-1} v_j\theta^{\mu}(w_{j-\mu})x^{m-1+\mu}\Big]\Gamma_{\frac{m}{\alpha}}(x^\alpha)\\ &+\Big[\sum_{\eta=0}^{\beta-1}\sum_{\nu=\eta}^{\beta-1} v'_{\nu-\eta}\Theta^{m-\eta}(w'_\nu)x^{m-1-\eta}\\ &+\sum_{\eta=1}^{\beta-1}\sum_{\nu=\eta}^{\beta-1} v'_\eta\Theta^{\eta}(w'_{\nu-\eta})x^{m-1+\eta}\Big]\Gamma_{\frac{m}{\beta}}(x^\beta)\\ =&\sum_{i=0}^{m-1}\Theta^{m-i}(S_i)x^{m-1-i}.\end{aligned}$$

So, $\mathbf{v}(x)\circ\mathbf{w}(x)=0$ if and only if $S_i=0$ for $0\le i\le m-1$.

□

Lemma 3.7. *Let $\mathbf{v}(x)=(v(x)|v'(x))$ and $\mathbf{w}(x)=(w(x)|w'(x))$ be elements in $\mathcal{R}$ such that $\mathbf{v}(x)\circ\mathbf{w}(x)=0$. If $v'(x)=0$ or $w'(x)=0$, then*

$$v(x)\psi^{m-\deg(w(x))}(w^*(x))\equiv 0 \pmod{(x^\alpha-1)}.$$

Respectively, if $v(x)=0$ or $w(x)=0$, then

$$v'(x)\psi^{m-\deg(w'(x))}(w'^*(x))\equiv 0 \pmod{(x^\beta-1)}.$$

Proof. Suppose that $v'(x)=0$ or $w'(x)=0$. Therefore

$$\mathbf{v}(x)\circ\mathbf{w}(x) = uv(x)\psi^{m-\deg(w(x))}(w^*(x))x^{m-1-\deg(w(x))}\Gamma_{\frac{m}{\alpha}}(x^\alpha)+0 \pmod{(x^m-1)}.$$

This imply that there exists $f(x)\in R_2[x,\Theta]$ such that

$$uv(x)\psi^{m-\deg(w(x))}(w^*(x))x^{m-1-\deg(w(x))}\Gamma_{\frac{m}{\alpha}}(x^\alpha) = uf(x)(x^m-1).$$

Since $\Gamma_{\frac{m}{\alpha}}(x^\alpha)=\frac{x^m-1}{x^\alpha-1}$ and $(x^m-1)(x^\alpha-1)=(x^\alpha-1)(x^m-1)$, we have

$$v(x)\psi^{m-\deg(w(x))}(w^*(x))x^m = f(x)x^{\deg(w(x))+1}(x^\alpha-1).$$

Hence

$$v(x)\psi^{m-\deg(w(x))}(w^*(x))\equiv 0 \pmod{(x^\alpha-1)}.$$

The same argument can be used to prove the other case. □

Notation 3.8. *Let C be an $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic code and α (resp. β) be the set of $\mathbb{F}_q$ (resp. R_2) coordinate positions. Denote C_α (resp. C_β) the punctured code of C by deleting the coordinates β (resp. α).*

Proposition 3.9. [11, Lemma 5] *Let C be an $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic code of length $\alpha+\beta$, then $C^{\perp}$ is also an additive skew cyclic code.*

If $C = \mathcal{R}_n\big((a(x)|0)\big) + \mathcal{R}_n\big((k_1(x)|a_1(x)+ug_1(x))\big) + \mathcal{R}_n\big((k_2(x)|ua_2(x))\big)$ is an $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic code of length $\alpha+\beta$. For simplicity, we denote the polynomial $a_1(x)+ug_1(x)$ by $A(x)$. The dual code $C^{\perp}$ is also an $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic code of length $\alpha+\beta$. We denote

$$C^{\perp} = \mathcal{R}_n\big((\bar{a}(x)|0)\big) + \mathcal{R}_n\big((\bar{k}_1(x)|\bar{A}(x))\big) + \mathcal{R}_n\big((\bar{k}_2(x)|u\bar{a}_2(x))\big),$$

where $\bar{a}(x) \in \mathcal{F}_\alpha, \bar{a}_2(x) \in \mathcal{F}_\beta,\ \bar{A}(x) \in \mathcal{R}_\beta,\ 0 \le \deg(\bar{a}(x)) \le \alpha,\ \bar{k}_i(x) \in \mathcal{R}_{1,\alpha}$ and $\deg(\bar{k}_i(x)) < \deg(\bar{a}(x))$, for $i=1,2$.

Theorem 3.10. *Let*

$$C = \mathcal{R}_n\big((a(x)|0)\big) + \mathcal{R}_n\big((k_1(x)|A(x))\big) + \mathcal{R}_n\big((k_2(x)|ua_2(x))\big)$$

be an $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic code of length $\alpha+\beta$. Let

$$C^{\perp} = \mathcal{R}_n\big((\bar{a}(x)|0)\big) + \mathcal{R}_n\big((\bar{k}_1(x)|\bar{A}(x))\big) + \mathcal{R}_n\big((\bar{k}_2(x)|u\bar{a}_2(x))\big)$$

be its dual code. Then we have

(1) $\bar{a}(x) = \frac{x^\alpha - 1}{\gcd_l(\psi^{m-\deg(a(x))}(a^*(x)),\psi^{m-\deg(k_1(x))}(k_1^*(x)),\psi^{m-\deg(k_2(x))}(k_2^*(x)))}$.

(2) $\bar{k}_1(x)\psi^{m-\deg(a(x))}(a^*(x)) = f(x)(x^\alpha - 1)$, *for some* $f(x) \in \mathbb{F}_q[x,\theta]$.

(3) $\bar{A}(x)\psi^{m-\deg(\frac{\mathrm{lcm}_l(a(x),k_1(x))}{k_1(x)}A(x))}((\frac{\mathrm{lcm}_l(a(x),k_1(x))}{k_1(x)}A(x))^*) = \mu(x)(x^\beta - 1)$, *for some* $\mu(x) \in R_2[x,\Theta]$.

(4) $\bar{k}_2(x)\psi^{m-\deg(a(x))}(a^*(x)) = \nu(x)(x^\alpha - 1)$, *for some* $\nu(x) \in \mathbb{F}_q[x,\theta]$.

(5) $\bar{a}_2(x)\psi^{m-\deg(\frac{\mathrm{lcm}_l(k_1(x),k_2(x))}{k_1(x)}A(x))}((\frac{\mathrm{lcm}_l(k_1(x),k_2(x))}{k_1(x)}A(x))^*) = \lambda(x)(x^\beta - 1)$, *for some* $\lambda(x) \in R_2[x,\Theta]$.

Proof. (1) Since $(\bar{a}(x)|0)$ is an element in $C^{\perp}$, it follows that

$$(\bar{a}(x)|0)\circ(a(x)|0) \equiv 0 \pmod{(x^m-1)},$$
$$(\bar{a}(x)|0)\circ(k_1(x)|A(x)) \equiv 0 \pmod{(x^m-1)}$$

and

$$(\bar{a}(x)|0)\circ(k_2(x)|ua_2(x)) \equiv 0 \pmod{(x^m-1)}.$$

By Lemma 3.7,

$$\bar{a}(x)\psi^{m-\deg(a(x))}(a^*(x)) \equiv 0 \pmod{(x^\alpha-1)},$$
$$\bar{a}(x)\psi^{m-\deg(k_1(x))}(k_1^*(x)) \equiv 0 \pmod{(x^\alpha-1)}$$

and

$$\bar{a}(x)\psi^{m-\deg(k_2(x))}(k_2^*(x)) \equiv 0 \pmod{(x^\alpha-1)}.$$

Using Lemma 2.9,

$\bar{a}(x)\mathrm{gcd}_l(\psi^{m-\deg(a(x))}(a^*(x)), \psi^{m-\deg(k_1(x))}(k_1^*(x)), \psi^{m-\deg(k_2(x))}(k_2^*(x))) \equiv 0 \ (\mathrm{mod}\ (x^\alpha - 1))$.

Therefore, there exists $h(x) \in \mathbb{F}_q[x, \theta]$ such that

$\bar{a}(x)\mathrm{gcd}_l(\psi^{m-\deg(a(x))}(a^*(x)), \psi^{m-\deg(k_1(x))}(k_1^*(x)), \psi^{m-\deg(k_1(x))}(k_1^*(x))) = h(x)(x^\alpha - 1)$.

Since $C_\alpha = \mathcal{R}_{1,\alpha}\big(\mathrm{gcd}_r(a(x), k_1(x), k_2(x))\big)$, we have

$$\bar{a}(x) = \frac{x^\alpha - 1}{\mathrm{gcd}_l(\psi^{m-\deg(a(x))}(a^*(x)), \psi^{m-\deg(k_1(x))}(k_1^*(x)), \psi^{m-\deg(k_2(x))}(k_2^*(x)))}.$$

(2) Inasmuch as $(\bar{k}_1(x)|\bar{A}(x)) \in C^\perp$, hence

$$(\bar{k}_1(x)|\bar{A}(x)) \circ (a(x)|0) \equiv 0 \pmod{(x^m - 1)}.$$

By Lemma 3.7, $\bar{k}_1(x)\psi^{m-\deg(a(x))}(a^*(x)) \equiv 0 \pmod{(x^\alpha - 1)}$. Thus there exists $f(x) \in \mathbb{F}_q[x, \theta]$ such that $\bar{k}_1(x)\psi^{m-\deg(a(x))}(a^*(x)) = f(x)(x^\alpha - 1)$.

(3) Let

$$\begin{aligned} c(x) &= \frac{\mathrm{lcm}_l(a(x), k_1(x))}{k_1(x)}.(k_1(x)|A(x)) - \frac{\mathrm{lcm}_l(a(x), k_1(x))}{a(x)}.(a(x)|0) \\ &= (0|\frac{\mathrm{lcm}_l(a(x), k_1(x))}{k_1(x)}A(x)). \end{aligned}$$

Hence $c(x) \in C$, which implies that

$$(\bar{k_1}(x)|\bar{A}(x)) \circ (0|\frac{\mathrm{lcm}_l(a(x), k_1(x))}{k_1(x)}A(x)) = 0.$$

Using Lemma 3.7, there exists $\mu(x) \in R_2[x; \Theta]$ such that

$$\bar{A}(x)\psi^{m-\deg(\frac{\mathrm{lcm}_l(a(x),k_1(x))}{k_1(x)}A(x))}((\frac{\mathrm{lcm}_l(a(x), k_1(x))}{k_1(x)}A(x))^*) = \mu(x)(x^\beta - 1).$$

(4) Similarly to the proof of (2).

(5) Let

$$\begin{aligned} c(x) &= \frac{\mathrm{lcm}_l(k_1(x), k_2(x))}{k_1(x)}.(k_1(x)|A(x)) - \frac{\mathrm{lcm}_l(k_1(x), k_2(x))}{k_2(x)}.(k_2(x)|ua_2(x)) \\ &= (0|\frac{\mathrm{lcm}_l(k_1(x), k_2(x))}{k_1(x)}A(x) - u\frac{\mathrm{lcm}_l(k_1(x), k_2(x))}{k_2(x)}a_2(x)). \end{aligned}$$

Hence $c(x) \in C$, which implies that

$$(\bar{k_2}(x)|u\bar{a}_2(x)) \circ (0|\frac{\mathrm{lcm}_l(k_1(x), k_2(x))}{k_1(x)}A(x) - u\frac{\mathrm{lcm}_l(k_1(x), k_2(x))}{k_2(x)}a_2(x)) = 0.$$

Using Lemma 3.7, there exists $f(x) \in R_2[x; \theta]$ such that

$$u\bar{a}_2(x)\psi^{m-\deg(\frac{\mathrm{lcm}_l(k_1(x),k_2(x))}{k_1(x)}A(x))}((\frac{\mathrm{lcm}_l(k_1(x), k_2(x))}{k_1(x)}A(x))^*) = uf(x)(x^\beta - 1).$$

□

When θ is the identity automorphism, we have the following corollary.

Corollary 3.11. *Let* $C = \langle (a(x)|0), (k_1(x)|A(x)), (k_2(x)|ua_2(x))\rangle$ *be an* $\mathbb{F}_q\mathbb{F}_q[u]$-*additive cyclic code of length* $\alpha+\beta$. *Let*

$$C^{\perp} = \langle (\bar{a}(x)|0), (\bar{k}_1(x)|\bar{A}(x)), (\bar{k}_2(x)|u\bar{a}_2(x))\rangle$$

be its dual code. Then we have

(1) $\bar{a}(x) = \dfrac{x^{\alpha}-1}{\gcd(a^*(x), k_1^*(x), k_2^*(x))}$.

(2) $\bar{k}_1(x)a^*(x) = f(x)(x^{\alpha}-1)$, *for some* $f(x) \in \mathbb{F}_q[x]$.

(3) *In the ring* $\mathbb{F}_q[x]$, *we have*

$$\operatorname{lcm}(a(x), k_1(x)).\gcd(a(x), k_1(x)) = a(x)k_1(x).$$

Hence, $\bar{A}(x)a^*(x)A^*(x) = \mu(x)\gcd(a^*(x), k_1^*(x))(x^{\beta}-1)$, *for some* $\mu(x) \in R_2[x]$.

(4) $\bar{k}_2(x)a^*(x) = \nu(x)(x^{\alpha}-1)$, *for some* $\nu(x) \in \mathbb{F}_q[x]$.

(5) $\bar{a}_2(x)k_1^*(x)A^*(x) = \lambda(x)\gcd(k_1^*(x), k_2^*(x))(x^{\beta}-1)$, *for some* $\lambda(x) \in R_2[x]$.

3.2. **Examples.**

In this subsection, we provide some examples of $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes of length $\alpha+\beta$, which are generated by different factors of $x^{\alpha}-1$ and $x^{\beta}-1$ and we specify their dual.

Example 3.12. Consider $\mathcal{R}_{1,3} = \frac{\mathbb{F}_{27}[x;\theta]}{\langle x^3-1\rangle}$ and $\mathcal{R}_9 = \frac{(\mathbb{F}_{27}+u\mathbb{F}_{27})[x;\Theta]}{\langle x^9-1\rangle}$, with Θ be an automorphism of $\mathbb{F}_{27}+u\mathbb{F}_{27}$ which is defined by $\Theta(a+ub) = \theta(a)+u\theta(b)$, where θ is the Frobenius map, $\theta(\alpha) = \alpha^3$, for all $\alpha \in \mathbb{F}_{27}$. Clearly, $o(\Theta) = o(\theta) = 3$. Let δ be a primitive 26th root of unity in $\mathbb{F}_{27}$, i.e, $\mathbb{F}_{27} = \{0, \delta, ..., \delta^{25}, \delta^{26} = 1\}$. A factorization of x^3-1 in $\mathbb{F}_{27}[x;\theta]$ is

$$x^3 - 1 = (x-\delta^2)(x-\delta^{18})(x-\delta^6).$$

- Let $C = \mathcal{R}_{12}\big(((x-\delta^{18})(x-\delta^6)|0)\big)$. Then $C^{\perp} = \mathcal{R}_{12}\big((\delta^6x-1|0)\big)$.

Consider the following code:

- $C = \mathcal{R}_{12}\big((x-\delta^2|0)\big) + \mathcal{R}_{12}\big((\delta|x^3-1)\big) + \mathcal{R}_{12}\big((\delta^2+1|u(x-\delta^{18})(x-\delta^6))\big)$.

By Theorem 3.10, we have

(1) $\bar{a}(x) = \frac{x^3-1}{\gcd_l(\psi^8(1-\delta^6 x), \psi^9(\delta), \psi^9(\delta^2+1))} = \delta^{25}(x^3-1)$.

(2) $\bar{k}_1(x)(1-\delta^2 x) = f(x)(x^3-1)$, for some $f(x) \in \mathbb{F}_{27}[x,\theta]$.

(3) $\bar{A}(x)(\delta^{25} - \delta x - \delta^{25} x^3 + \delta x^4) = \mu(x)(x^9-1)$, for some $\mu(x) \in R_2[x,\Theta]$.

(4) $\bar{k}_2(x)(1-\delta^2 x) = \nu(x)(x^3-1)$, for some $\nu(x) \in \mathbb{F}_{27}[x,\theta]$.

(5) $\bar{a}_2(x)((\delta^{25}+\delta)x^3 - \delta^{25} - \delta) = -\lambda(x)(x^9-1)$, for some $\lambda(x) \in R_2[x,\Theta]$.

EXAMPLE 3.13. Consider $\mathcal{R}_{1,2} = \frac{\mathbb{F}_{16}[x;\theta]}{\langle x^2-1\rangle}$ and $\mathcal{R}_4 = \frac{(\mathbb{F}_{16}+u\mathbb{F}_{16})[x;\Theta]}{\langle x^4-1\rangle}$ with Θ be an automorphism of $\mathbb{F}_{16}+u\mathbb{F}_{16}$ which is defined by $\Theta(a+ub) = \theta(a)+u\theta(b)$, where θ is the Frobenius map, $\theta(\nu) = \nu^2$, for all $\nu \in \mathbb{F}_{16}$. Then $o(\Theta) = o(\theta) = 4$. Let δ be a primitive 15th root of unity in $\mathbb{F}_{16}$, i.e, $\mathbb{F}_{16} = \{0, \delta, ..., \delta^{14}, \delta^{15} = 1\}$. Consider a factorization of x^4-1 in $\mathbb{F}_{16}[x;\theta]$

$$x^4 - 1 = (x-\delta^{10})(x-\delta^5)(x-\delta^{10})(x-\delta^5).$$

- Suppose $C = \mathcal{R}_6\big((x-\delta^5|0)\big)$. We have $\bar{a}(x) = \frac{x^2-1}{\psi^3(1-\delta^{10}x)} = \delta^5(x-\delta^{10})$ and

$$C^\perp = \mathcal{R}_6\big((\delta^5(x-\delta^{10})|0)\big).$$

Consider the following code:

- $C = \mathcal{R}_6\big((x-\delta^{10}|0)\big) + \mathcal{R}_6\big((\delta^2|(x-1)(x-\delta^{10}))\big) + \mathcal{R}_6\big((\delta^2+1|u(x-\delta^{10}))\big)$.

Using Theorem 3.10, we have

(1) $\bar{a}(x) = \frac{x^2-1}{\gcd_l(\psi(1-\delta^5 x), \psi^4(\delta^2), \psi^4(\delta^2+1))} = \delta^{13}(x^2-1)$.

(2) $\bar{k}_1(x)(1-\delta^{10}x) = f(x)(x^2-1)$, for some $f(x) \in \mathbb{F}_{16}[x,\theta]$.

(3) $\bar{A}(x)(\delta^7 + \delta^{10}x + \delta^{12}x^2 + \delta^3 x^3) = \mu(x)(x^4-1)$, for some $\mu(x) \in R_2[x,\Theta]$.

(4) $\bar{k}_2(x)(1-\delta^{10}x) = \nu(x)(x^2-1)$, for some $\nu(x) \in \mathbb{F}_{16}[x,\theta]$.

(5) $\bar{a}_2(x)(\delta^9 + \delta^8 x + \delta x^2) = \lambda(x)(x^4-1)$, for some $\lambda(x) \in R_2[x,\Theta]$.

EXAMPLE 3.14. Consider $\mathcal{R}_{1,4} = \frac{\mathbb{F}_{16}[x;\theta]}{\langle x^4-1\rangle}$ and $\mathcal{R}_8 = \frac{(\mathbb{F}_{16}+u\mathbb{F}_{16})[x;\Theta]}{\langle x^8-1\rangle}$ with Θ be an automorphism of $\mathbb{F}_{16}+u\mathbb{F}_{16}$ which is defined by $\Theta(a+ub) = \theta(a)+u\theta(b)$, where θ is the Frobenius map, $\theta(\nu) = \nu^4$, for all $\nu \in \mathbb{F}_{16}$. Then $o(\Theta) = o(\theta) = 2$. A factorization of x^8-1 in $\mathbb{F}_{16}[x;\theta]$ is

$$x^8 - 1 = (x-\delta)(x-\delta^3)(x-\delta^5)(x-\delta^6)(x-\delta)(x-\delta^3)(x-\delta^5)(x-\delta^6).$$

Consider:

$C = \mathcal{R}_{12}\big(((x-\delta)(x-\delta^3)|0)\big)+\mathcal{R}_{12}\big((\delta^5|(x^4-1)(x-\delta)(x-\delta^3))\big)+\mathcal{R}_{12}\big((\delta+1|u(x-\delta^3))\big).$

By Theorem 3.10, we can get

(1) $\bar{a}(x) = \frac{x^4-1}{\gcd_l(\psi^6(1+\delta^7x+\delta^4x^2),\psi^8(\delta^5),\psi^8(\delta+1))} = \delta^{11}(x^4-1)$.

(2) $\bar{k}_1(x)(1+\delta^7x+\delta^4x^2) = f(x)(x^4-1)$, for some $f(x) \in \mathbb{F}_{16}[x,\theta]$.

(3) $\bar{A}(x)(1+\delta^2x+\delta^{12}x^2+x^4+\delta^2x^5+\delta^{12}x^6) = \mu(x)(x^8-1)$, for some $\mu(x) \in R_2[x,\Theta]$.

(4) $\bar{k}_2(x)(1+\delta^7x+\delta^4x^2) = \nu(x)(x^4-1)$, for some $\nu(x) \in \mathbb{F}_{16}[x,\theta]$.

(5) $\bar{a}_2(x)(1-\delta^7x+\delta^4x^2+x^4+\delta^7x^5+\delta^4x^6) = \lambda(x)(x^8-1)$, for some $\lambda(x) \in R_2[x,\Theta]$.

4. Conclusions

In this paper, we studied the structure of the (Euclidean) dual of $\mathbb{F}_q\mathbb{F}_q[u]$-additive skew cyclic codes in terms of their generating polynomials, where q is a power of prime integer and $u^2 = 0$.

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