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Almost Simple Groups of Lie Rank Two and Genus Two

Haval M. Mohammed Salih

Department of Mathematics, Faculty of Science, Soran University, Kawa St.-Soran, Erbil, Iraq

E-mail: havalmahmood07@gmail.com

Abstract. For a finite group G, the Hurwitz space $\mathcal{H}^{in}_{r,g}(G)$ is the space of genus g covers of the Riemann sphere $\mathbb{P}^1$ with r branch points and the monodromy group G. In this paper, we study the connectedness of the Hurwitz space $\mathcal{H}^{in}_{r,g}(G)$ where G is almost simple groups of Lie rank two, with at least four branch points and genus two. Our approach uses computational tools, relying on the computer algebra system GAP and the MAPCLASS package, to find the connected components of $\mathcal{H}^{in}_{r,g}(G)$. This work gives us the complete classification of G.

Keywords: Genus two groups, Almost simple groups, Braid orbits, Connected components.

2000 Mathematics subject classification: 20B05, 20B15, 20E32

1. Introduction

Let Ω be a finite set of order n and G be a transitive subgroup of S_n such that

$$G = \langle x_1, x_2, ..., x_r \rangle \tag{1.1}$$

$$\prod_{i=1}^{r} x_i = 1,\ x_i \in G^{\#} = G \setminus \{1\},\ i = 1, ..., r. \tag{1.2}$$

$$\sum_{i=1}^{r} ind\, x_i = 2(n + g - 1) \tag{1.3}$$

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where $ind\, x_i$ is the minimal number of 2-cycles needed to express x_i as a product. Then we call G a group of genus g and the triple $(G, \Omega, \langle x_1, x_2, ..., x_r \rangle)$ a genus g system. These conditions correspond to the existence of an n sheeted branched covering of Riemann surface X of genus g with r-branch points and monodromy group G. Our work relates to a conjecture made by Guralnick and Thompson in 1990, in [3]. In this paper they conjectured that the set $\mathcal{E}^*(g)$ of possible isomorphism classes of composition factors of G, which are neither cyclic nor alternating, is finite for all $g \geq 0$ [3]. In 2001 the conjecture was proved by Frohardt and Magaard [2]. The proof of the conjecture shows that we can determine $\mathcal{E}^*(g)$ explicitly for $g \leq 2$. As the conjecture is now a theorem, these sets are finite. It is known (see [7, 8]) that there are four types of classification of genus g system as follows:

(1) Up to signature
(2) Up to ramification type
(3) Up to the braid action and diagonal conjugation by $Aut(G)$
(4) Up to the braid action and diagonal conjugation by $Inn(G)$.

If G acts primitively on Ω, then the structure of G is reduced in the five cases and one of the case which is an almost simple group [1]. A group G is said to be almost simple, if it contains a non-abelian simple group S and $S \leq G \leq Aut(S)$. In [4], she worked on almost simple groups of type projective special linear group $PSL(3, q)$. Let G be a group such that $PSL(3, q) \leq G \leq P\Gamma L(3, q)$ where $P\Gamma L(3, q)$ is the projective semilinear group. G acts on points in the natural module, that is the set of projective points of 2-dimensional projective geometry $PG(2, q)$. In [4] Kong, gave a complete list for some almost simple groups of Lie rank 2 up to ramification type in her PhD thesis for a genus 0,1 and 2 system (That is 2.). Furthermore, she shows that the almost simple groups do not possess genus low tuples if $q \geq 16$. So we have $3 \leq q \leq 13$ where q is a prime power.

In this paper, we will classify primitive almost simple groups of Lie rank two for genus two up to the braid action and diagonal conjugation by $Inn(G)$ (That is 4.).

Before starting our computations, we give an interesting results which can be found in [9], which help us to show that the connectedness of Hurwitz spaces.

Lemma 1.1. [9] *We obtain a bijection* $\Psi_{\mathrm{A}}^{-1}(P_0) \longrightarrow \epsilon_r^{\mathrm{A}}(G)$ *by sending* $[P_0, \phi]_{\mathrm{A}}$ *to the generators* $(x_1, ..., x_r)$ *where* $x_i = \phi([\gamma_i])$ *for* $i = 1, ..., r$.

Proposition 1.2. [9] *Let* $\bar{C}$ *be a fixed ramifcation type in* G*, and the subset* $\mathcal{H}_r^A(G, \bar{C})$ *of* $\mathcal{H}_r^A(G)$ *consists of all* $[B, \phi]_A$ *with* $B = \{b_1, ..., b_r\}$*,* $\phi\colon \pi_1(\mathbb{P}^1 \setminus P, \infty) \to G$ *and* $\phi(\theta_{b_i})) \in C_i$ *for* $i = 1, \ldots, r$*. Then* $\mathcal{H}_r^A(G, \bar{C})$ *is a union of connected components in* $\mathcal{H}_r^A(G)$*. Under the bijection from Lemma 1.1, the fiber in* $\mathcal{H}_r^A(G, \bar{C})$ *over* B_0 *corresponds the set* $\mathcal{N}^A(\bar{C})$*. This yields a one to one correspondence between components of* $\mathcal{H}_r^A(G, \bar{C})$ *and the braid orbits on*

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$\mathcal{N}^A(\bar{C})$. In particular, $\mathcal{H}_r^{in}(G,\bar{C})$ is connected if and only if there is only one braid orbit.

Finally some words on the organization of this paper. In Section 2 we describe our method and illustrate it with an example to compute braid orbits for primitive genus 2 systems for all almost simple groups of Lie rank 2. Finally, Proposition 4.1 and Proposition 4.2 are given which describe the connectedness of Hurwitz spaces for given groups.

2. Methodology: Listing primitive genus two systems

In this section we briefly describe the methods which will be used to obtain Tables 3 and 4. To achieve these tables we needed to do the following steps:

- We extract all primitive permutation group G by using the GAP function
  AllPrimitiveGroups(DegreeOperation,n).
- For every almost simple group G, compute the conjugacy class representatives and permutation indices on n points.
- For given $n, g = 2$ and G we use the GAP function RestrictedPartions to compute all possible ramification types satisfying the Riemann-Hurwitz formula, that is Equation(1.3).
- Compute the character table of G if possible and remove those types which have zero structure constant, that is it satisfies Equation(1.2).
- Compute all generating tuple of G, that is it satisfies Equation(1.1). One can use the programs in [10] with few modification on it by removing the affine condition.
- We use the same rules for labeling and ordering conjugacy classes of G as in [6].
- For each of the remaining types of length greater than or equal to 4, we use MAPCLASS package to compute braid orbits.

The next example show that how to compute generating tuples and braid orbits for the group $G = PGammal(3,4)$.

Example 2.1. We use the MAPCLASS package to illustrate some of these computations more concretely. Suppose that $G = PGammal(3,4)$ and $|\Omega| = 21$.

```
gap> a:=AllPrimitiveGroups(DegreeOperation,21);
[ PGL(2, 7), A(7), S(7), PSL(3, 4)=M(21), PSigmaL(3, 4), PGL(3, 4),
PGammaL(3, 4), A(21), S(21) ]
gap> k:=a[7];
PGammaL(3, 4)
gap> Read("qu1.g");
gap> CheckingTheGroup(k);
gap> gt:=GeneratingType(k,21,2);
[ [ 5, 7, 19, 20 ], [ 5, 7, 7, 4 ], [ 5, 7, 6, 11 ], [ 5, 7, 6, 8 ],
[ 5, 7, 6, 3 ], [ 5, 5, 20, 20 ], [ 5, 5, 19, 16 ],[ 5, 5, 19, 15 ],
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[ 5, 5, 11, 14 ], [ 5, 5, 8, 14 ], [ 5, 5, 7, 13 ], [ 5, 5, 7, 12 ],
[ 5, 5, 7, 7, 7 ],[ 5, 5, 6, 4 ], [ 5, 5, 5, 7, 6 ],[ 5, 5, 5, 5, 14 ],
[ 5, 5, 5, 5, 2, 2 ], [ 5, 5, 5, 2, 20 ], [ 5, 5, 2, 7, 19 ],
[ 5, 5, 2, 2, 11 ],[ 5, 5, 2, 2, 8 ], [ 5, 4, 13 ], [ 5, 4, 12 ],
[ 5, 2, 11, 20 ], [ 5, 2, 8, 20 ],[ 5, 2, 7, 18 ], [ 5, 2, 7, 17 ],
[ 2, 7, 6, 6 ] ]
```

We can take one of the generating tuple t as follows:

```
gap> t:=List(gt[12],x->CC[x]);;
```

Here we compute braid orbits for given t:

```
gap> orbit:=GeneratingMCOrbits(k,0,t);;
Total Number of Tuples: 5080320
Orbit1:
Length=42
Generating Tuple  =[ ( 2, 6)( 4,16)( 7,13)( 8,14)( 9,15)(17,20)
(18,21), ( 1,18)( 2, 9)( 3,12)( 4,11)( 8,10)(14,16)(15,17),
( 2,19, 8)( 3,20,11)( 5,15, 6)( 7,10,18)(12,21,16),
 ( 1,21, 3,16,18,14, 4,11,17, 9, 6,20,12, 8,19,15, 5, 2,10,13, 7) ]
Centralizer size=1
```

Remark 2.2. Let us make a couple comments on the MAPCLASS package. For each of the remaining types of length greater than or equal to 4, we use MAPCLASS package. In particular we use the function **GeneratingMCOrbits** to compute braid orbits. However it does not work for some types. For example if $G = P\Gamma L(3,4)$, $g = 2$ and types $(2B, 2B, 5A, 6A)$ and $(2B, 2B, 3C, 15B)$. In this case, we use the function **AllMCOrbits**.

3. Main results

It is well known that deciding whether or not $\mathcal{H}_r(G, \bar{C})$ is connected is still an open problem, both computationally and theoretically. The MAPCLASS package of James, Magaard, Shpectorov and Volklein, is constructed to do braid orbit computations for a given finite group and given type. There were several known results about it for example see [5, 6, 8].

In this paper we find the connected components $\mathcal{H}_r(G, \bar{C})$ of G-curves X of genus 2 such that $g(X/G) = 0$. The computation shows that there is exactly 3465 braid orbits of primitive genus 2 systems for some almost simple groups of Lie rank two. The degree and the number of the branch points are given in Table 1.

Theorem 3.1. *Up to isomorphism, there exist exactly six primitive genus two groups with socle $PSL(3, q)$ for some q, $3 \leq q \leq 13$. The corresponding primitive genus two groups are enumerated in Table 3 and Table 4.*

The following results which tells us the connectedness of the Hurwitz space for some almost simple groups of Lie rank two for genus two.

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Degree	#Group Iso types	# RTs	# comp's $r = 3$	# comp's $r = 4$	# comp's $r = 5$	# comp's $r = 6$	# comp's $r = 7$	# comp's total
13	1	115	274	43	16	4	1	338
21	4	161	1988	117	18	3	- -	2126
31	1	50	463	7	-	-	-	470
57	2	17	310	-	-	-	-	310
73	2	3	128	-	-	-	-	128
91	2	8	81		-	-	-	81
133	1	8	8		-	-	-	8
183	1	4	4		-	-	-	4
Totals	14	365	3256	167	34	7	1	3465

TABLE 1. Primitive Genus Two Systems: Number of Components.

Proposition 3.2. *If G is isomorphic to one of the following groups $PGammal(3,4)$, $PSL(3,5)$ or $PSL(3,3)$ and $r \geq 4$, then $\mathcal{H}^{in}_{r,2}(G,\bar{C})$ is connected.*

Proof. Since we have just one braid orbit for all types $\bar{C}$ and the Nielsen classes $\mathcal{N}(\bar{C})$ are the disjoint union of braid orbits. From Proposition 1.2, we obtain that the Hurwitz space $\mathcal{H}^{in}_{r,2}(G,\bar{C})$ is connected. □

Proposition 3.3. *If G is isomorphic to one of the following groups ($PSigmal(3,4)$, $PSL(3,4)$ and $r \geq 4$) or ($PGL(3,4)$ and $r = 4$), then $\mathcal{H}^{in}_{r,2}(G,\bar{C})$ is disconnected.*

Proof. Since we have at least two braid orbits for some type $\bar{C}$ and the Nielsen classes $\mathcal{N}(\bar{C})$ are the disjoint union of braid orbits. From Proposition 1.2, we obtain that the Hurwitz space $\mathcal{H}^{in}_{r,2}(G,\bar{C})$ is disconnected. □

Proposition 3.4. *If G is isomorphic to $PGL(3,4)$ and $\bar{C} = (2A, 2A, 2A, 3A, 3B)$, then $\mathcal{H}^{in}_{r,2}(G,\bar{C})$ is connected.*

Proof. The proof is similar as Proposition 3.2. □

The detial of the symbols which will be used in the sequel tables lie in Table 2.

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group	ramification type
N.O	number of orbits
L.O	largest length of the orbit
GTS	genus 2 system
PGL	projective general linear group
PSL	projective special linear group
PGammaL	projective gamma linear group
PSigmaL	projective sigma linear group

TABLE 2. Notations.

group	ramification type	N.O	L.O	ramification type	N.O	L.O
	(2B,2B,4B,6A)	2	720	(2B,3A,3A,4B)	2	520
	(2A,2B,4B,5A)	2	260	(2A,2B,4C,6B)	1	640
	(2A,2B,3A,8A)	2	160	(2A,2B,4A,6A)	1	120
	(2A,2A,4B,8A)	2	64	(2A,2A,4C,5A)	1	280
	(2A,2A,6A,6A)	5	144	(2A,2A,3A,7A)	3	28
	(2A,2A,4A,5A)	3	35	(2A,2A,3A,7B)	3	28
$PSigmaL(3,4)$	(2A,2B,2B,2B,4B)	2	3504	(2A,2A,2B,2B,4C)	1	3328
	(2A,2A,2A,2B,6A)	2	1080	(2A,2A,2B,2B,4A)	2	624
	(2A,2A,2A,2A,5A)	3	300	(2A,2A,2A,2A,2B,2B)	2	5952
	(2B,3A,3A,6B)	1	804	(2B,3A,3A,8A)	1	80
	(2B,3A,4A,3B)	1	408	(2B,3A,4A,6A)	1	408
	(2B,3A,4A,4A)	1	540	(2B,2B,6B,6B)	1	4664
	(2B,2B,3C,15A)	1	160	(2B,2B,3C,15B)	1	160
	(2B,2B,3B,5A)	1	270	(2B,2B,6A,5A)	1	890
	(2B,2B,3A,21A)	1	42	(2B,2B,3A,21B)	1	42
	(2B,2B,4B,8A)	1	928	(2B,2A,3B,6B)	1	588
	(2B,2A,6A,6B)	1	2196	(2B,2A,3A,14A)	1	63
	(2A,3A,4B,4B)	1	468	(2B,2A,3A,14B)	1	63
$PGammaL(3,4)$	(2B,2B,3A,3A,3A)	1	840	(2B,2B,2B,3A,4B)	1	12816
	(2B,2B,2B,2B,5A)	1	10000	(2B,2B,2B,2A,6B)	1	23904
	(2B,2B,2A,3A,3C)	1	4008	(2B,2B,2A,2A,3B)	1	3096
	(2B,2B,2A,2A,6A)	1	11088	(2B,2B,2B,2B,2A,2A)	1	125528
	(3B,3B,3B,4A)	2	24	(3A,3B,3E,3E)	1	150
	(3A,3B,3B,6B)	1	18	(3A,3B,3B,3C)	1	18
	(3A,3A,3B,6A)	1	18	(3A,3A,3B,3D)	1	18
	(3A,3A,3A,4A)	2	24	(2A,3B,3E,6B)	1	468
	(2A,3A,3E,6A)	1	168	(2A,3A,3E,3D)	1	168
	(2A,3A,3B,5A)	2	20	(2A,3A,3B,5A)	2	20
	(2A,2A,6A,6B)	3	468	(2A,2A,3D,6B)	1	378
	(2A,2A,3C,3D)	1	78	(2A,2A,3C,6A)	1	378
$PGL(3,4)$	(2A,2A,3B,15A)	3	20	(2A,2A,3B,15B)	3	20
	(2A,2A,3B,15C)	3	20	(2A,2A,3B,15D)	3	20
	(2B,3B,3E,3C)	1	168	(2A,2A,2A,3A,3B)	1	576
$PSL(3,5)$	(2A,2A,4C,4C)	1	762	(2A,2A,3A,4C)	1	1098
	(2A,2A,3A,3A)	1	762	(2A,2A,2A,24A)	1	36
	(2A,2A,2A,3C)	1	36	(2A,2A,2A,24B)	1	36
				(2A,2A,2A,24D)	1	36
$PSL(3,4)$	(2A,2A,4C,4C)	5	72	(2A,2A,4B,4C)	3	120
	(2A,2A,4B,4B)	5	72	(2A,2A,4A,4C)	3	120
	(2A,2A,4A,4A)	5	72	(2A,2A,4A,4B)	3	120
	(2A,3A,3A,3A)	4	1560	(2A,2A,3A,5B)	4	360
	(2A,2A,3A,5B)	4	360	(2A,2A,2A,2A,3A)	4	8640

TABLE 3. GTSs for Almost Simple Groups of Lie Rank Two.

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group	ramification type	N.O	L.O	ramification type	N.O	L.O
$PSL(3,3)$	(3A,3A,3B,3B)	1	144	(3A,3A,4A,3B)	1	144
	(3A,3A,4A,4A)	1	120	(3A,3A,6A,3B)	1	132
	(3A,3A,4A,6A)	1	108	(3A,3A,4A,4A)	1	108
	(3A,3A,3A,8A)	1	16	(3A,3A,3A,8B)	1	16
	(2A,3B,3B,3B)	1	2052	(2A,3B,3B,4A)	1	1748
	(2A,3B,4A,4A)	1	1512	(2A,4A,4A,4A)	1	1202
	(2A,3B,3B,6A)	1	1620	(2A,3B,4A,6A)	1	1332
	(2A,4A,4A,6A)	1	1104	(2A,3B,6A,6A)	1	1224
	(2A,4A,6A,6A)	1	958	(2A,6A,6A,6A)	1	860
	(2A,3A,3B,8A)	1	144	(2A,3A,3B,8B)	1	144
	(2A,3A,4A,8A)	1	129	(2A,3A,4A,8B)	1	129
	(2A,3A,6A,8A)	1	117	(2A,3A,6A,8B)	1	117
	(2A,3A,3A,13A)	1	13	(2A,3A,3A,13B)	1	13
	(2A,3A,3A,13C)	1	13	(2A,3A,3A,13D)	1	13
	(2A,3A,3B,13A)	1	156	(2A,3A,3B,13B)	1	156
	(2A,3A,3B,13C)	1	156	(2A,3A,3B,13D)	1	156
	(2A,2A,8A,8A)	1	138	(2A,2A,8B,8A)	1	128
	(2A,2A,8B,8B)	1	138			
	(2A,2A,4A,13A)	1	130	(2A,2A,4A,13B)	1	130
	(2A,2A,4A,13C)	1	130	(2A,2A,4A,13D)	1	130
	(2A,2A,6A,13A)	1	117	(2A,2A,6A,13B)	1	117
	(2A,2A,6A,13C)	1	117	(2A,2A,6A,13D)	1	117
	(2A,3A,3A,3A,3A)	1	216	(2A,2A,3A,3A,3B)	1	2192
	(2A,3A,3A,3A,4A)	1	1976	(2A,2A,3A,3A,6A)	1	1836
	(2A,2A,2A,3B,3B)	1	30312	(2A,2A,2A,3B,4A)	1	26208
	(2A,2A,2A,3B,6A)	1	23976	(2A,2A,2A,4A,4A)	1	22164
	(2A,2A,2A,5A,6A)	1	19944	(2A,2A,2A,6A,6A)	1	17928
	(2A,2A,2A,3A,8A)	1	2040	(2A,2A,2A,3A,8B)	1	2040
	(2A,2A,2A,2A,13A)	1	2197	(2A,2A,2A,2A,13B)	1	2197
	(2A,2A,2A,2A,13C)	1	2197	(2A,2A,2A,2A,13D)	1	2197
	(2A,2A,2A,2A,3A,3A)	1	32436	(2A,2A,2A,2A,2A,3B)	1	455760
	(2A,2A,2A,2A,2A,5A)	1	392160	(2A,2A,2A,2A,2A,6A)	1	356400
	(2A,2A,2A,2A,2A,2A,2A)	1	6924960			

Table 4. GTSs for Almost Simple Groups of Lie Rank Two.

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