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Certain Reduction Formulas for Kampé De Fériet Functions and Their Application in Laser Physics

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Abstract. Many works are elaborated to derive interesting identities for hypergeometric-type series containing as a factor a digamma function. In the present paper, new reduction formulae for Kampé de Fériet series of types $F^{1:2;1}_{2:1;0}$ and $F^{2:2;1}_{3:1;0}$ are performed. By specializing certain parameters, series identities and related reduction identities are deduced. An interesting application is also studied concerning the evaluation of the average intensity of a multi-Gaussian beam propagating through a turbulent atmosphere.

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1. Introduction

The pioneer of the present investigation is the work of Chu and Liu [3]. Based on quadratic approximation and δ expansion the first order, these authors have investigated the on-and off-axis average intensity errors due to quadratic approximation for the multi-Gaussian beam spreading in turbulent atmosphere. Following up, the average intensity in this paper is evaluated as a summation formula for hypergeometric-type series containing a digamma function as a factor. This idea, in view of previous works (see [5, 9, 10, 11, 12, 14]), helps us to obtain many interesting results. The following definitions are necessary to recall for the present study:

The Gamma function $\Gamma(z)$ is defined by the definite integral (see [2])

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1}\, dt \quad (\Re(z) > 0). \tag{1.1}$$

At a point $x \geq 0$, this last equation yields two incomplete gamma functions (see [6],[7])

$$\gamma(z, x) := \int_0^x e^{-t} t^{z-1}\, dt \tag{1.2}$$

and

$$\Gamma(z, x) := \int_x^\infty e^{-t} t^{z-1}\, dt, \tag{1.3}$$

where $\gamma(z, x)$ is the lower incomplete gamma function and $\Gamma(z, x)$ is the upper incomplete gamma function with z is a complex parameter with a positive real part.

The generalization of Gauss's hypergeometric function ${}_2F_1$ is called the generalized hypergeometric series ${}_pF_q$ and defined by (see [13], [15, pp. 71-75])

$$ {}_pF_q\left(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!} \tag{1.4}$$

$$(\beta_j \neq 0, -1, -2, \ldots, j = 1, ..., q),$$

where $(\alpha)_n$ is the Pochhammer symbol defined by (see [15, p. 2 and p. 5])

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}; \quad \alpha \neq 0, -1, -2, \ldots$$

In 1921, the four Appell functions were unified and generalized by Kampé de Fériet, who defined a general hypergeometric function of two variables (see [16]). We recall here the definition of a more general double hypergeometric function in a slightly modified notation (see [8, 16])

$$F_{q:m;n}^{p:l;k}\left[\begin{matrix}(a)_p: & (b)_l; & (c)_k;\\ (\alpha)_q: & (\beta)_m; & (\gamma)_n;\end{matrix} x,y\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{l}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{q}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}, \tag{1.5}$$

where, for convergence

(i) $p+l<q+m+1$, $p+k<q+n+1$, $|x|<\infty$, $|y|<\infty$, or

(ii) $p+l=q+m+1$, $p+k=q+n+1$, and

$$\begin{cases}|x|^{1/(p-q)}+|y|^{1/(p-q)}<1, & if\ p>q,\\ \max\{|x|,\ |y|\}<1, & if\ p\le q.\end{cases}$$

The main objective of the present note is to give new reduction formulas of Kampé de Fériet functions.

2. Reduction formulae for Kampé de fériet function $F_{2:1;0}^{1:2;1}$

In this Section, we prove two theorems, which yield the reduction formulae for the Kampé de fériet function $F_{2:1;0}^{1:2;1}[-z,-z]$, given by (2.2).

Theorem 2.1. *The following formula holds true:*

$$\begin{aligned}S &=\sum_{s=0}^{\infty}\frac{(s+1)}{s!}(-z)^s[\ln a+\psi(s+2)]\\ &=(1-z)e^{-z}\ln a+(1-z)(1-\gamma)e^{-z}-zF_{2:1;0}^{1:2;1}[-z,-z],\end{aligned} \tag{2.1}$$

where

$$F_{2:1;0}^{1:2;1}[-z,-z]=F_{2:1;0}^{1:2;1}\left[\begin{matrix}3: & 1,2; & 1;\\ 2,2: & 3; & -;\end{matrix} -z,-z\right]. \tag{2.2}$$

Proof. Applying the identity

$$(s+\lambda)=\frac{\lambda(\lambda+1)_s}{(\lambda)_s}, \tag{2.3}$$

we can write the summation S as

$$S=I_1+I_2,$$

where

$$I_1=\ln(a).\ {}_1F_1\,(2;1;-z)$$

and

$$I_2=\psi(2)+\sum_{s=1}^{\infty}\frac{(2)_s}{[(1)_s]^2}(-z)^s\psi(s+2).$$

Firstly, we now make use of the useful identity called as the Kummer first formula for the confluent hypergeometric function (see [2, 13])

$$ {}_1F_1\left(a;b;z\right) = e^z\,{}_1F_1\left(b-a;b;-z\right). \tag{2.4}$$

It helps us in rearranging I_1 to write

$$I_1 = (1-z)\ e^{-z}.\ \ln a. \tag{2.5}$$

To evaluate the expression of I_2, we use the identity (see [4])

$$\sum_{n=1}^{\infty}[\psi(\lambda+n)-\psi(\lambda)]\tfrac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}(-z)^n$$
$$= \tfrac{(-z)}{\lambda}\tfrac{\alpha_1\cdots\alpha_p}{\beta_1\cdots\beta_q}F^{p:2;1}_{q:1;0}\left[\begin{array}{llll}\alpha_1+1,\ldots,\alpha_p+1:1, & \lambda; & 1; & \\ \beta_1+1,\ldots,\beta_q+1: & \lambda+1; & -; & \end{array} -z;-z\right], \tag{2.6}$$

and the identity

$$\sum_{s=1}^{\infty}\frac{(2)_s}{(1)_s}\frac{(-z)^s}{s!} = (1-z)\ e^{-z}-1. \tag{2.7}$$

Now in view of (2.6) and (2.7), and for $\psi(2) = 1-\gamma$, where γ is the Euler number, I_2 becomes

$$I_2 = (1-z)(1-\gamma)e^{-z} - zF^{1:2;1}_{2:1;0}[-z,-z]. \tag{2.8}$$

Consequently, we get the summation (2.1). This completes the proof of Theorem 2.1. □

Theorem 2.2. *The following transformation holds true:*

$$e^z\sum_{s=0}^{\infty}\frac{(s+1)}{s!}(-z)^s\psi(s+2) = 2-e^{-z}+(z-1)[Shi(z)+Chi(z)-\ln(z)], \tag{2.9}$$

where $Chi(z)$ and $Shi(z)$ are the hyperbolic cosine and sine integral functions (see [1]*).*

Proof. With the help of the following identities (see [6])

$$\psi(s+2) = \psi(s+1)+\frac{1}{s+1}, \tag{2.10}$$

and

$$\frac{(s+1)}{s!} = \frac{1}{(s-1)!}+\frac{1}{s!}, \tag{2.11}$$

the summation S can be expressed as

$$S = \sum_{s=0}^{\infty}\frac{(s+1)}{s!}(-z)^s[\ln a+\psi(s+2)] = T_1+T_2+T_3+T_4+T_5, \tag{2.12}$$

where

$$T_1 = \ln a\left[\sum_{s=0}^{\infty}\frac{(-z)^s}{(s-1)!}+\sum_{s=0}^{\infty}\frac{(-z)^s}{s!}\right], \tag{2.13}$$

$$T_2=\sum_{s=0}^{\infty}\frac{(-z)^s}{(s-1)!}\psi(s+1), \tag{2.14}$$

$$T_3=\sum_{s=0}^{\infty}\frac{(-z)^s}{(s+1)(s-1)!}, \tag{2.15}$$

$$T_4=\sum_{s=0}^{\infty}\frac{(-z)^s}{s!}\psi(s+1), \tag{2.16}$$

and

$$T_5=\sum_{s=0}^{\infty}\frac{(-z)^s}{s!(s+1)!}. \tag{2.17}$$

To evaluate these above summations, we make use of the following identities (see [6])

$$\lim_{k\to 0}\frac{\psi(-k)}{\Gamma(-k)}=-1, \tag{2.18}$$

and

$$\gamma(\alpha,z)=z^{\alpha}\sum_{s=0}^{\infty}\frac{(-z)^s}{s!(s+\alpha)}. \tag{2.19}$$

For T_1, one finds

$$\frac{T_1}{\ln a}=\sum_{s=0}^{\infty}\frac{(-z)^s}{(s-1)!}+e^{-z}. \tag{2.20}$$

To evaluate T_2, we introduce the variable change $s-1=l$ and use the identity (2.10), such that T_2 becomes

$$T_2=\sum_{l=-1}^{\infty}\frac{(-z)^{l+1}}{l!}\psi(l+1)+e^{-z}. \tag{2.21}$$

With the help of (2.18), the first right term of (2.21) can be rearranged as

$$\sum_{l=-1}^{\infty}\frac{(-z)^{l+1}}{l!}\psi(l+1)=-z\left[\sum_{l=0}^{\infty}\frac{(-z)^l}{l!}\psi(l+1)-\frac{1}{z}\right], \tag{2.22}$$

and by using the following identity (see [12])

$$\sum_{l=0}^{\infty}\psi(l+1)\frac{(-z)^l}{l!}=\mathrm{e}^{-z}[\ln z-\mathrm{Ei}(z)], \tag{2.23}$$

where Ei is the exponential integral function (see [6]), then (2.21) becomes

$$T_2=-z\ \mathrm{e}^{-z}[\ln z-\mathrm{Ei}(z)]-1+\mathrm{e}^{-z}. \tag{2.24}$$

In what follows, we shall utilize the formula (2.19)

$$\gamma(\alpha,z)=\Gamma(\alpha)-\Gamma(\alpha,z). \tag{2.25}$$

Now, (2.15) can be derived as

$$T_3=\frac{-1}{z}\left[1-\Gamma(2,z)\right]. \tag{2.26}$$

It's easy to derive (2.16) by using (2.25) and we have

$$T_4 = \mathrm{e}^{-z}[\ln z - \mathrm{Ei}(z)]. \tag{2.27}$$

By using the identity

$$\sum_{s=1}^{\infty} \frac{(-z)^s}{s!} = \mathrm{e}^{-z} - 1, \tag{2.28}$$

T_5 can be written as

$$T_5 = \frac{1}{z}(1 - \mathrm{e}^{-z}). \tag{2.29}$$

Finally, with the help of the following identity (see [7])

$$\Gamma(2, z) = \mathrm{e}^{-z}(1 + z), \tag{2.30}$$

and the relation

$$\mathrm{Ei}(z) = Shi(z) + Chi(z), \tag{2.31}$$

(2.12) can be written as

$$S = \mathrm{e}^{-z}\left\{(z-1)[Shi(z) + Chi(z) - \ln(az)] + 2 - \mathrm{e}^{z}\right\}, \tag{2.32}$$

and eventually (2.9) is proved. This completes the proof of Theorem 2.2.

Note that (2.32) is equivalent to (16) of (see [3]), found by using another method based on the expansion of the hyperbolic cosine and sine integral functions, Chi and Shi. □

Corollary 2.3. *The following reduction formula for $F^{1:2;1}_{2:1;0}$ holds true:*

$$F^{1:2;1}_{2:1;0}[-z, -z] = \frac{\mathrm{e}^{-z}}{z}\{(1-z)[Shi(z)+Chi(z)-\ln z+(1-\gamma)]+(\mathrm{e}^{z}-2)\}, \tag{2.33}$$

with γ is the Euler number defined as $\gamma = 1 - \psi(2) = 0.577$.

Proof. By equating (2.1) and (2.32), one finds easily, a new reduction for the Kampé de Fériet function given by (2.33). □

Theorem 2.4. *The following closed-from summation holds true:*

$$\begin{aligned}&\sum_{s=1}^{\infty} \frac{(s+\lambda+\alpha)}{s!}(-z)^s[\psi(s+\lambda) + \psi(\lambda)] \\ &= \tfrac{z\mathrm{e}^{-z}}{\lambda}\{\tfrac{z\lambda}{(\lambda+1)}{}_2F_2\left(1,1;2,\lambda+2;z\right) - (\lambda+\alpha){}_2F_2\left(1,1;2,\lambda+1;z\right) + 1\}.\end{aligned} \tag{2.34}$$

Proof. We denote the left side of (2.34) by S' which can be written as

$$S' = (\lambda+\alpha)\sum_{s=1}^{\infty} \frac{(\lambda+\alpha+1)_s}{(\lambda+\alpha)_s}\frac{(-z)^s}{s!}[\psi(s+\lambda) - \psi(\lambda)]. \tag{2.35}$$

Again, with the help of the identity (2.6), (2.35) can be written as

$$S' = -z\tfrac{(\lambda+\alpha+1)}{\lambda}F^{1:2;1}_{2:1;0}\left[\begin{array}{lll} \lambda+\alpha+2:\lambda, & 1; & 1; \\ \lambda+\alpha+1, 2: & \lambda+1; & -; \end{array} -z; -z\right]. \tag{2.36}$$

On the other hand, (2.35) can be expanded as

$$S^{'} = I_1' + (\lambda+\alpha) I_2', \tag{2.37}$$

where

$$I_1' = \sum_{s=1}^{\infty} \frac{(-z)^s}{(s-1)!}[\psi(s+\lambda) - \psi(\lambda)], \tag{2.38}$$

and

$$I_2' = \sum_{s=1}^{\infty} \frac{(-z)^s}{s!}[\psi(s+\lambda) - \psi(\lambda)]. \tag{2.39}$$

By using the results of (see [12]), (2.38) and (2.39) can be rewritten as

$$I_1' = \frac{z^2 e^z}{(\lambda+1)} {}_2F_2\left(1,1;2,\lambda+2;z\right) - \frac{z e^{-z}}{\lambda} \tag{2.40}$$

and

$$I_2' = \frac{-z e^{-z}}{\lambda} {}_2F_2\left(1,1;2,\lambda+1;z\right), \tag{2.41}$$

from which, the result (2.34) easily follows. This completes the proof of Theorem 2.3. □

Corollary 2.5. *The undermentioned reduction formula holds true:*

$$F^{1:2;1}_{2:1;0}\left[\begin{array}{lll} \lambda+\alpha+2:\lambda, & 1; & 1; \\ \lambda+\alpha+1,2: & \lambda+1; & -; \end{array} -z;-z\right]$$
$$= \tfrac{e^{-z}}{(\lambda+\alpha+1)}\{1+(\lambda+\alpha){}_2F_2\left(1,1;2,\lambda+1;z\right) - \tfrac{\lambda z}{(\lambda+1)}{}_2F_2\left(1,1;2,\lambda+2;z\right)\}. \tag{2.42}$$

Proof. It is easy to derive (2.42) from (2.34) and (2.36). So, we skip the details of the proof.

From the corollary 2.5, we intend to deduce a new formula. By specializing the parameters $\lambda = 2$ and $\alpha = -1$, (2.42) becomes

$$F^{1:2;1}_{2:1;0}[-z,-z] = \tfrac{e^{-z}}{2}\left[1+{}_2F_2\left(1,1;2,3;z\right) - \tfrac{2z}{3}{}_2F_2\left(1,1;2,4;z\right)\right]. \tag{2.43}$$

□

Now, from (2.33) and (2.43), we derive the undermentioned corollary.

Corollary 2.6. *The following relation holds true:*

$$\begin{aligned} \mathrm{Ei}(z) = {} & \frac{z}{2(1-z)}\left[1+{}_2F_2\left(1,1;2,3;z\right) - \frac{2z}{3}{}_2F_2\left(1,1;2,4;z\right)\right] \\ & + \frac{(2-e^z)}{(1-z)} + \ln z - (1-\gamma). \end{aligned} \tag{2.44}$$

3. New formula for the Kampé de Fériet function $F^{2:2;1}_{3:1;0}$

Theorem 3.1. *The following transformation holds true:*

$$F^{2:2;1}_{3:1;0}\left[\begin{array}{lll}\lambda+\alpha+2,\chi+1:1, & \lambda; & 1;\\ \lambda+\alpha+1,\lambda+1,2: & \lambda+1; & -;\end{array} -z;-z\right] = F^{2:2;1}_{3:1;0}[-z,-z]$$
$$= \tfrac{1}{(\lambda+\alpha+1)}\{{}_1F_1\left(\chi+1;\lambda+1;-z\right)+(\lambda+\alpha)F^{1:2;1}_{2:1;0}\left[\begin{array}{lll}\chi+1:1, & \lambda; & 1;\\ \lambda+1,2: & \lambda+1; & -;\end{array} -z;-z\right]$$
$$-z\tfrac{(\chi+1)}{(\lambda+1)}F^{1:2;1}_{2:1;0}\left[\begin{array}{lll}\chi+2:1, & \lambda+1; & 1;\\ \lambda+2,2: & \lambda+2; & -;\end{array} -z;-z\right]\}. \tag{3.1}$$

Proof. To prove (3.1), we introduce the following summation (see [4])

$$S^{''}=\sum_{s=1}^{\infty}(s+\lambda+\alpha)\frac{(\chi)_s}{(\lambda)_s}\frac{(-z)^s}{s!}[\psi(s+\lambda)-\psi(\lambda)]. \tag{3.2}$$

By using the identity

$$(s+\lambda+\alpha)=(\lambda+\alpha)\frac{(\lambda+\alpha+1)_s}{(\lambda+\alpha)_s}, \tag{3.3}$$

and (3.2), we obtain

$$S^{''}=-z\chi\tfrac{(\lambda+\alpha+1)}{\lambda^2}F^{2:2;1}_{3:1;0}[-z,-z], \tag{3.4}$$

which can be written with the help of the following identity

$$\frac{(s+\lambda+\alpha)}{s!}=\frac{1}{(s-1)!}+\frac{(\lambda+\alpha)}{s!}, \tag{3.5}$$

as

$$S^{''}=I_1^{''}+(\lambda+\alpha)I_2^{''}, \tag{3.6}$$

where

$$I_1^{''}=\sum_{s=1}^{\infty}\frac{(\chi)_s}{(\lambda)_s(s-1)!}(-z)^s[\psi(s+\lambda)-\psi(\lambda)], \tag{3.7}$$

and

$$I_2^{''}=\sum_{s=1}^{\infty}\frac{(\chi)_s}{(\lambda)_s s!}(-z)^s[\psi(s+\lambda)-\psi(\lambda)]. \tag{3.8}$$

Starting with the functional relation for the digamma function given by (2.10) and using the expression (see [4])

$$(a)_{l+1}=a(a+1)_l,$$

and (3.2), (3.7) and (3.8) can be expressed respectively as

$$I_1^{''}=\tfrac{\chi z}{\lambda}\{\tfrac{z(\chi+1)}{\lambda(\lambda+1)}F^{1:2;1}_{2:1;0}\left[\begin{array}{lll}\chi+2:1, & \lambda+1; & 1;\\ \lambda+2,2: & \lambda+2; & -;\end{array} -z;-z\right], \tag{3.9}$$
$$-\tfrac{1}{\lambda}{}_1F_1\left(\chi+1;\lambda+1;-z\right)\}$$

and

$$I_2^{''}=\tfrac{-\chi z}{\lambda^2}F^{1:2;1}_{2:1;0}\left[\begin{array}{lll}\chi+1:1, & \lambda; & 1;\\ \lambda+1,2: & \lambda+1; & -;\end{array} -z;-z\right]. \tag{3.10}$$

And, we can deduce the following expression of (3.2)

$$S'' = \frac{\chi z}{\lambda^2}\{\frac{z(\chi+1)}{(\lambda+1)}F^{1:2;1}_{2:1;0}\left[\begin{array}{lll}\chi+2:1, & \lambda+1; & 1; \\ \lambda+2,2: & \lambda+2; & -;\end{array} -z;-z\right] - (\lambda+\alpha)$$
$$\times F^{1:2;1}_{2:1;0}\left[\begin{array}{lll}\chi+1:1, & \lambda; & 1; \\ \lambda+1,2: & \lambda+1; & -;\end{array} -z;-z\right] - {}_1F_1\left(\chi+1;\lambda+1;-z\right)\}. \tag{3.11}$$

Finally, according to (3.4) and (3.11), one obtains the new reduction formula for the Kampé de Fériet function given by (3.1). This completes the proof of Theorem 3.1. □

Now, we reduce an expression of the Kampé de Fériet function $F^{2:2;1}_{3:1;0}[-z,-z]$ as a function of the hypergeometric functions ${}_2F_2$ and ${}_1F_1$ in the next corollary.

Corollary 3.2. *The following formula holds true:*

$$\begin{aligned}&F^{2:2;1}_{3:1;0}[-z,-z] = \frac{e^{-z}}{(\lambda+1)(\lambda+\alpha+1)}\{e^z(\lambda+1){}_1F_1\left(\lambda+2;\lambda+1;-z\right)\\ &+(\lambda+\alpha-z)+[\lambda(\lambda+\alpha)-z(\lambda+1)]{}_2F_2\left(1,1;2,\lambda+1;z\right)\\ &-(\lambda+\alpha-z)\frac{\lambda z}{(\lambda+1)}{}_2F_2\left(1,1;2,\lambda+2;z\right)\}.\end{aligned} \tag{3.12}$$

Proof. It's easy to deduce from (3.1) for $\chi=\lambda+1$ the following expression

$$\begin{aligned}&F^{2:2;1}_{3:1;0}[-z,-z] = \frac{1}{(\lambda+\alpha+1)}\{{}_1F_1\left(\lambda+2;\lambda+1;-z\right)\\ &+(\lambda+\alpha)F^{1:2;1}_{2:1;0}\left[\begin{array}{lll}\lambda+2:1, & \lambda; & 1; \\ \lambda+1,2: & \lambda+1; & -;\end{array} -z;-z\right]\\ &-z\frac{(\lambda+2)}{(\lambda+1)}F^{1:2;1}_{2:1;0}\left[\begin{array}{lll}\lambda+3:1, & \lambda+1; & 1; \\ \lambda+2,2: & \lambda+2; & -;\end{array} -z;-z\right]\}.\end{aligned} \tag{3.13}$$

We can evaluate the Kampé de Fériet function in the left side of this last equation from (2.42). Consequently, one finds the following expressions, for $\alpha=0$ and $\alpha=1$, respectively

$$\begin{aligned}&F^{1:2;1}_{2:1;0}\left[\begin{array}{lll}\lambda+2:\lambda, & 1; & 1; \\ \lambda+1,2: & \lambda+1; & -;\end{array} -z;-z\right]\\ &= \frac{e^{-z}}{(\lambda+1)}\{1+\lambda{}_2F_2\left(1,1;2,\lambda+1;z\right) - \frac{\lambda z}{(\lambda+1)}{}_2F_2\left(1,1;2,\lambda+2;z\right),\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}&F^{1:2;1}_{2:1;0}\left[\begin{array}{lll}\lambda+3:\lambda, & 1; & 1; \\ \lambda+2,2: & \lambda+1; & -;\end{array} -z;-z\right]\\ &= \frac{e^{-z}}{(\lambda+2)}\{1+(\lambda+1){}_2F_2\left(1,1;2,\lambda+1;z\right) - \frac{\lambda z}{(\lambda+1)}{}_2F_2\left(1,1;2,\lambda+2;z\right).\end{aligned} \tag{3.15}$$

Consequently, by substituting (3.14) and (3.15) into (3.13), (3.12) is proved. □

4. Application

In recent years, many works are developed in laser physics, especially studies of the effects of the turbulent atmosphere on the characteristics of some fields as laser, radar and acoustic waves. Chu and Liu [3], have expressed the average intensity. From this work, one can deduce the spreading of some particular fields as Gaussian, flattened Gaussian, and annular beams.

We consider the propagation of a multi-Gaussian beam expressed as

$$u_0(r_0,0)=\sum_{n=1}^{N}A_n e^{-i\varphi_n}e^{-\left(\frac{1}{w_n^2}+\frac{ik}{2R}\right)r_0^2}, \tag{4.1}$$

where k, A_n, φ_n, w_n and R are the parameters of the Gaussian beams family. By using the Huygens-Fresnel principle, the average intensity at z plane of this beam propagating through turbulent atmosphere, is given by

$$\begin{aligned}<\Delta I(r,z)>=&\sum_{m,n=1}^{N}A_mA_ne^{i(\varphi_m-\varphi_n)}\frac{w_m^2w_n^2t_3^2}{3\left(w_m^2+w_n^2\right)t^4}\\ &\times\sum_{s=0}^{\infty}\frac{(s+1)}{s!}\left[ln\left(\frac{t_3^2}{t^2}\right)+\psi(s+2)\right]\left(-2\frac{r^2}{t^2}\right)^s,\end{aligned} \tag{4.2}$$

where $t_3=2\sqrt{2}z$ and $t=\sqrt{t_3^2+\frac{8z^2}{k^2a^2}}$. In this last expression, a depends on z and the parameters of the beam.

(4.2) can be evaluated with the help of our Theorem 2.1, which gives the expression of the average intensity in a weak turbulence in terms of the Kampé de Fériet function $F_{2:1;0}^{1:2;1}$

$$\begin{aligned}<\Delta I(r,z)>=&\sum_{m,n=1}^{N}A_mA_ne^{i(\varphi_m-\varphi_n)}\frac{w_m^2w_n^2t_3^2}{3\left(w_m^2+w_n^2\right)t^4}\\ &\times(1-\tfrac{2r^2}{t^2})e^{-\frac{2r^2}{t^2}}\left[ln\left(\tfrac{t_3^2}{t^2}\right)+1-\gamma\right]-\tfrac{2r^2}{t^2}F_{2:1;0}^{1:2;1}[-\tfrac{2r^2}{t^2},-\tfrac{2r^2}{t^2}].\end{aligned} \tag{4.3}$$

Also, by applying the second Theorem, we find (4.2) in terms of hyperbolic cosine and sine integral functions. Finally, by applying Theorem 2.3, one finds the average intensity in terms of the hypergeometric function ${}_2F_2$ given by

$$\begin{aligned}<\Delta I(r,z)>=&\sum_{m,n=1}^{N}A_mA_ne^{i(\varphi_m-\varphi_n)}\frac{w_m^2w_n^2t_3^2}{3\left(w_m^2+w_n^2\right)t^4}\\ &\times(1-\tfrac{2r^2}{t^2})e^{-\frac{2r^2}{t^2}}\left[ln\left(\tfrac{t_3^2}{t^2}\right)+1-\gamma\right]-\tfrac{r^2}{t^2}\mathrm{e}^{-\frac{2r^2}{t^2}}\\ &\left[1+{}_2F_2\left(1,1;2,3;\tfrac{2r^2}{t^2}\right)-\tfrac{4r^2}{3t^2}{}_2F_2\left(1,1;2,4;\tfrac{2r^2}{t^2}\right)\right].\end{aligned} \tag{4.4}$$

5. Conclusion

In this paper, we have investigated some new representations of the Kampé de Fériet functions $F^{1:2;1}_{2:1;0}$ and $F^{2:2;1}_{3:1;0}$. The new formulae are expressed in terms of hypergeometric functions ${}_1F_1$ and ${}_2F_2$. Some corollaries are derived from our main results as particular cases. We have also given a novel expression of the Kampé de Fériet function $F^{1:2;1}_{2:1;0}$ in terms of hyperbolic cosine and sine integral functions. An interesting application is studied in the end of this paper concerning the evaluation of the average intensity of a multi-Gaussian beam propagating through a turbulent atmosphere. Finally, the results derived in the present investigation are interesting and may potentially have influence and be useful in applied problems of the future as optics, radar, acoustic waves and laser physics.

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