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# Some Properties of a Special Subalgebra of Central Derivations

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Abstract. Let $L$ be a Lie algebra, and $\mathrm{Der}(L)$ and $\mathrm{IDer}(L)$ be the set of all derivations and inner derivations of $L$, respectively. Let $\mathcal{D}$ be a subalgebra of $\mathrm{Der}(L)$ such that it contains $\mathrm{IDer}(L)$ and $H = \cap_{\alpha\in\mathcal{D}}\mathrm{Ker}\alpha$. If $\mathrm{Der}^H(L)$ denotes the set of all derivations of $L$ whose images are in $H$, then we give necessary and sufficient conditions under which $\mathrm{Der}^H(L)$ is equal to some subalgebras of $\mathrm{Der}(L)$ for finite dimensional nilpotent Lie algebras.

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## 1. Introduction and preliminaries

Let $L$ be a Lie algebra over an arbitrary field $F$. We denote by $L^2$ and $Z(L)$ the derived algebra and the center of $L$, respectively. The linear map $\alpha : L \to L$ is said to be a derivation if

$$\alpha([x, y]) = [\alpha(x), y] + [x, \alpha(y)],$$

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for all $x, y \in L$. The vector space of all derivations of $L$ is denoted by $\mathrm{Der}(L)$. It is well-known that $\mathrm{Der}(L)$ is a Lie algebra through the following bracket

$$[\alpha_1, \alpha_2] := \alpha_1 \circ \alpha_2 - \alpha_2 \circ \alpha_1,$$

for all $\alpha_1, \alpha_2 \in \mathrm{Der}(L)$. This Lie algebra is called the *derivation algebra* of $L$. The map $\mathrm{ad}_x : L \to L$ given by $y \mapsto [x, y]$ is a derivation called *inner derivation* for all $x \in L$. The set $\mathrm{IDer}(L) = \{\mathrm{ad}_x \mid x \in L\}$ of inner derivations is an ideal of $\mathrm{Der}(L)$.

Study of the necessary and sufficient conditions under which subalgebras of $\mathrm{Der}(L)$ coincide gose back to sixties when Jacobson [2], initiated study the complete Lie algebra. A Lie $L$ is said to be *complete*, if its center is zero and $\mathrm{Der}(L) = \mathrm{IDer}(L)$. Later these Lie algebras are studied in [3, 4, 5, 6, 7, 8, 9, 10, 15]. Also, Tôgô [17] characterized Lie algebras $L$ over a field of characteristic zero with $Z(L) \neq 0$ for which $\mathrm{Der}_{Z(L)}(L) = \mathrm{IDer}(L)$ and $\mathrm{Der}_{Z(L)}(L) = \mathrm{Der}(L)$, where $\mathrm{Der}_{Z(L)}(L)$ is a subalgebra of $\mathrm{Der}(L)$ consisting of all central derivations of $L$, that is, the set of all derivations of $L$ whose images lie in the center of $L$.

Sheikh-Mohseni et al. [11, 14] introduced the new subalgebra $\mathrm{Der}_J^I(L)$, the set of all derivations of $L$ whose images are in $I$ and send $J$ to zero, and $\mathrm{Der}_c(L)$, the set of all derivations $\alpha$ for which $\alpha(x) \in [x, L] := \{[x, y] \mid y \in L\}$ and investigated the equalities $\mathrm{Der}_J^I(L) = \mathrm{IDer}(L), \mathrm{Der}_c(L) = \mathrm{IDer}(L)$ and $\mathrm{Der}_c(L) = \mathrm{Der}_{Z(L)}(L)$. Also, they characterized those Lie algebras for which $\mathrm{ID}(L) = \mathrm{IDer}(L)$, $\mathrm{ID}^*(L) = \mathrm{Der}_{Z(L)}(L)$, where $\mathrm{ID}(L) = \mathrm{Der}^{L^2}(L)$ and $\mathrm{ID}^*(L) = \mathrm{Der}_{Z(L)}^{L^2}(L)$ (See [12, 13]).

Let $\mathcal{D}$ be a subalgebra of $\mathrm{Der}(L)$ and it contains $\mathrm{IDer}(L)$. Put $H = \cap_{\alpha \in \mathcal{D}} \mathrm{Ker}\alpha$. Clearly, $H$ is a central subalgebra of $L$. In fact, if $\mathcal{D} = \mathrm{IDer}(L)$, then $H = Z(L)$. Also, if $\mathcal{D} = \mathrm{Der}(L)$, then $H$ is a absolut center of $L$ was introduced by stitzinger [16] in 1944. The aim of this paper is characterized the Lie algebras $L$ such that $\mathrm{Der}^H(L)$ is equal to $\mathrm{Der}_c(L)$. Also, we give necessary and sufficient conditions under which $\mathrm{Der}_{Z(L)}^H(L) = \mathrm{ID}^*(L)$. In final, as an application of our results, we find those nilpotent Lie algebras $L$ for which $\dim L \leq 6$ and $\mathrm{Der}^H(L) = \mathrm{Der}_c(L)$ or $\mathrm{Der}_{Z(L)}^H(L) = \mathrm{ID}^*(L)$.

Recall that for a Lie algebra $L$, the *lower central series* of $L$ is defined as follows:

$$L = L^1 \supseteq L^2 \supseteq \cdots \supseteq L^n \supseteq \cdots,$$

where $L^2$ is the derived algebra of $L$ and $L^n = [L^{n-1}, L]$.

Also, the *upper central series* of $L$ is defined as

$$\{0\} = Z_0(L) \subseteq Z_1(L) \subseteq \cdots Z_n(L) \subseteq \cdots,$$

where $Z_1(L) = Z(L)$ is the center of $L$ and $Z_{n+1}(L)/Z_n(L) = Z(L/Z_n(L))$.

A Lie algebra $L$ is *nilpotent* if there exists a non-negative integer $k$ such that $L^k = 0$ (or $Z_k(L) = L$). The smallest integer $k$ for which $L^{k+1} = 0$ (or $Z_k(L) = L$) is called the *nilpotency class* of $L$. A Lie algebra $L$ of dimension $n$

is *filiform* if it has maximal nilpotency class $n-1$, $dimZ(L)=1$ and $dimL^2 = n-2$. de Graaf [1] classified all nilpotent Lie algebras of dimension at most 6.

We recall that the Heisenberg Lie algebras are nilpotent of class 2. A Lie algebra $L$ is called Heisenberg Lie algebra if $L^2 = Z(L)$ and $\dim L^2 = 1$. Heisenberg Lie algebras of finite dimension are odd dimensional with basis $x_1, \ldots, x_{2k}, x_{2k+1}$ and the only non-zero multiplication between basis elements are $[x_{2i-1}, x_{2i}] = x_{2i+1}$ for $i = 1, \ldots, k$. The symbol $H(k)$ denotes the Heisenberg Lie algebra of dimension $2k+1$.

The following results are of our interest and we use them to characterize all finite dimensional nilpotent Lie algebras of dimension $n \leqslant 6$, for which $\mathrm{Der}^H(L) = \mathrm{Der}_{Z(L)}(L)$, $\mathrm{Der}^H_{Z(L)}(L) = \mathrm{Der}_{Z(L)}(L)$ and $\mathrm{Der}^H_{Z(L)}(L) = \mathrm{IDer}(L)$.

**Theorem 1.1.** [11, Corollary 3] *Let $L$ be a finite dimensional Lie algebra and $I$, $J$ be two ideals of $L$ such that $I \subseteq Z(L)$. Then $\mathrm{Der}^I_J(L) = \mathrm{Der}_{Z(L)}(L)$ if and only if $I = Z(L)$ and $J \subseteq L^2$.*

If $I = H$ and $J = Z(L)$, then we have $\mathrm{Der}^H_{Z(L)}(L) = \mathrm{Der}_{Z(L)}(L)$ if and only if $H = Z(L)$ and $Z(L) \subseteq L^2$. Also if $I = Z(L)$, since every element of $L^2$ is sent to zero by every element of $\mathrm{Der}^H(L)$, by Theorem 1.1, $\mathrm{Der}^H(L) = \mathrm{Der}_{Z(L)}(L)$ if and only if $H = Z(L)$.

**Theorem 1.2.** [11, Corollary 4] *Let $I$ and $J$ be two ideals of a finitely generated non-abelian Lie algebra $L$ with $Z(L) \neq 0$ such that $I \subseteq Z(L) \subseteq J$. Then $\mathrm{Der}^I_J(L) = \mathrm{IDer}(L)$ if and only if $L$ is a finite dimensional nilpotent Lie algebra of class 2, $J = Z(L), L^2 \subseteq I$ and $\dim I = 1$*

If $I = H$ and $J = Z(L)$, then $\mathrm{Der}^H_{Z(L)}(L) = \mathrm{IDer}(L)$ if and only if $L^2 = H$ and $\dim H = 1$.

## 2. Main results

In this section, we give necessary and sufficient conditions under which the Lie algebra $\mathrm{Der}^H(L) = \mathrm{Der}_c(L)$ and $\mathrm{Der}^H_{Z(L)}(L) = \mathrm{ID}^*(L)$ for finite dimensional nilpotent Lie algebras.

Let $A, B$ be two Lie algebras and $\mathrm{Hom}(A, B)$ be the set of all homomorphism from $A$ to $B$. Clearly, if $B$ is an abelian Lie algebra, then $\mathrm{Hom}(A, B)$ equipped with Lie bracket $[f, g](x) = [f(x), g(x)]$ for all $x \in A$ and $f, g \in \mathrm{Hom}(A, B)$ is an abelian Lie algebra.

The following lemma is useful for the proof of our main results.

**Lemma 2.1.** *If $U, V$ and $W$ are finite dimensional Lie algebras and $W$ be Lie algebra abelian, then*

$$\mathrm{Hom}(U \oplus V, W) \cong \mathrm{Hom}(U, W) \oplus \mathrm{Hom}(V, W).$$

**Lemma 2.2.** *Let $I, J$ be two ideals of Lie algebra $L$ such that $I \subseteq J$ and $I \subseteq Z(L)$. Then*

$$\mathrm{Der}_J^I(L) \cong \mathrm{Hom}(L/J, I).$$

*Proof.* For any $\alpha \in \mathrm{Der}_J^I(L)$, the map $\psi_\alpha : L/J \to I$ defined by $\psi_\alpha(x+J) = \alpha(x)$ for all $x \in L$ is a Lie homomorphism. Since $\mathrm{Der}_J^I(L)$ and $\mathrm{Hom}(L/J, I)$ are abelian, it is easy to see that the map $\psi : \mathrm{Der}_J^I(L) \to \mathrm{Hom}(L/J), I)$ defined by $\psi(\alpha) = \psi_\alpha$ is a Lie isomorphism. □

**Lemma 2.3.** *Let $L$ be a non-abelian nilpotent Lie algebra of finite dimension and $\mathrm{Der}^H(L) = \mathrm{Der}_c(L)$. Then*

(i) $\mathrm{Der}_c(L) \cong \mathrm{Hom}(L/Z(L), L^2) \cong \mathrm{Hom}(L/Z(L), H) \cong \mathrm{Hom}(L/L^2, H)$.
(ii) $\dim \mathrm{Der}_c(L) = \sum_{i=1}^d \dim[x, L]$, *where $\{x_1, \cdots, x_d\}$ is a minimal generating set for $L$.*

*Proof.* (i) Since $\mathrm{IDer}(L) \subseteq \mathrm{Der}_c(L) = \mathrm{Der}^H(L)$, $L^2 \subseteq H$. Therefore

$$\mathrm{Der}_c(L) \subseteq \mathrm{ID}^*(L) \subseteq \mathrm{Der}_{Z(L)}^H(L) \subseteq \mathrm{Der}^H(L) = \mathrm{Der}_c(L).$$

Now, by Lemma 2.2, $\mathrm{Der}_c(L) = \mathrm{ID}^*(L) \cong \mathrm{Hom}(L/Z(L), L^2)$ and $\mathrm{Der}_c(L) = \mathrm{Der}_{Z(L)}^H(L) \cong \mathrm{Hom}(L/Z(L), H)$. Therefore $L^2 = H$. Since every element of $L^2$ is sent to zero by every element of $\mathrm{Der}^H(L)$, by Lemma 2.2,

$$\mathrm{Der}_c(L) = \mathrm{Der}_{Z(L)}^H(L) = \mathrm{Der}_{L^2}^H(L) \cong \mathrm{Hom}(L/L^2, H).$$

(ii) By using Proposition 2.3 in [14], we have $\dim \mathrm{Der}_c(L) \leq \sum_{i=1}^d \dim[x, L]$. On the other hand,

$$\begin{aligned}\dim \mathrm{Der}_c(L) &= \dim \mathrm{Hom}(L/Z(L), L^2)\\ &= \dim \mathrm{Hom}(\langle \overline{x_1} \rangle \oplus \ldots \oplus \langle \overline{x_d} \rangle, L^2)\\ &= \sum_{i=1}^d \dim \mathrm{Hom}(\langle \overline{x_i} \rangle, L^2)\\ &= \sum_{i=1}^d \dim L^2 \geq \sum_{i=1}^d \dim[x_i, L].\end{aligned}$$

This completes the proof. □

**Definition.** Let $L$ be a finite dimensional Lie algebra and $I \neq 0$ be an ideal of $L$. Then $(L, I)$ is called a Camina pair if $I \subseteq [x, L]$ for all $x \in L \setminus I$. $L$ is called a Camina Lie algebra if $(L, L^2)$ is a Camina pair.

**Theorem 2.4.** *Let $L$ be a finite dimensional non-abelian nilpotent Lie algebra. Then $\mathrm{Der}^H(L) = \mathrm{Der}_c(L)$ if and only if $L^2 = H = Z(L)$ and $L$ is a Camina Lie algebra.*

*Proof.* First suppose that $L$ is a Camina Lie algebra and $L^2 = H = Z(L)$. Then $\mathrm{Der}^H(L) = \mathrm{Der}_{Z(L)}(L)$ and $\mathrm{Der}_c(L) \subseteq \mathrm{Der}_{Z(L)}(L)$. Now, let $\alpha \in \mathrm{Der}_{Z(L)}(L)$ and $x \in L^2$. Therefore $\alpha(x) = 0 \in [x, L]$. If $x \in L \setminus L^2$, then $\alpha(x) \in [x, L]$, because $L$ is a Camina Lie algebra. Therefore $\alpha \in \mathrm{Der}_c(L)$ and the result follows.

Conversely suppose that $\mathrm{Der}^H(L) = \mathrm{Der}_c(L)$. Then by Lemma 2.1, $L^2 = H \subseteq Z(L)$. On the other hand, by Lemma 2.3(i),

$$\dim \mathrm{Der}_c(L) = \dim \mathrm{Hom}(L/Z(L), H) = \dim \mathrm{Hom}(L/L^2, H).$$

Therefore $L^2 = Z(L)$ and $L^2 = H = Z(L)$. If there exists $x \in L \setminus L^2$ such that $[x, L] \subsetneq L^2$, then we can assume without loss of generality that $\{x = x_1, x_2, \ldots, x_d\}$ is a minimal generating set for $L$. Now, by Lemma 2.3(ii),

$$\dim \mathrm{Der}_c(L) = \sum_{i=1}^{d} \dim[x_i, L] \lneqq d \dim L^2.$$

On the other hand by Lemma 2.3(i), we have

$$\dim \mathrm{Der}_c(L) = \dim \mathrm{Hom}(L/Z(L), L^2) = d \dim L^2,$$

which is a contradiction. Therefore, $L$ is a Camina Lie algebra. □

As an immediate consequence of Theorem 2.4, we obtain the following result.

**Corollary 2.5.** [14, Corollary 4.4] *Let $L$ be a finite dimensional non-abelian nilpotent Lie algebra. Then $\mathrm{Der}_c(L) = \mathrm{Der}_{Z(L)}(L)$ if and only if $L$ is a Camina Lie algebra of class 2.*

*Proof.* By putting $\mathcal{D} = \mathrm{IDer}(L)$, we have $H = Z(L)$ and the result is clear. □

**Theorem 2.6.** *Let $L$ be a finite dimensional non-abelian nilpotent Lie algebra. Then $\mathrm{Der}^H_{Z(L)}(L) = \mathrm{ID}^*(L)$ if and only if $L^2 = H$.*

*Proof.* First, suppose that $\mathrm{Der}^H_{Z(L)}(L) = \mathrm{ID}^*(L)$. Then, since $\mathrm{IDer}(L) \subseteq \mathrm{ID}^*(L) = \mathrm{Der}^H_{Z(L)}(L)$, we have $L^2 \subseteq H$. Also, by Lemma 2.2,

$$\mathrm{Der}^H_{Z(L)}(L) \cong \mathrm{Hom}(L/Z(L), H), \mathrm{ID}^*(L) \cong \mathrm{Hom}(L/Z(L), L^2).$$

Therefore

$$\dim \mathrm{Der}^H_{Z(L)}(L) = \dim(L/Z(L) \times \dim H, \dim \mathrm{ID}^*(L) = \dim L/Z(L) \times \dim L^2.$$

Since $L^2 \subseteq H$ and $\dim \mathrm{ID}^*(L) = \dim \mathrm{Der}^H_{Z(L)}(L)$, we have $L^2 = H$.

The converse is clear. □

**Corollary 2.7.** *If $L$ is a non-abelian filiform Lie algebra of finite dimension, then $\mathrm{Der}^H_{Z(L)}(L) = \mathrm{ID}^*(L)$ if and only if $L^2 = H = Z(L)$.*

Let $L$ be an $n$-dimensional Lie algebra with basis $x_1, \ldots, x_n$ and $\alpha \in \mathrm{Der}(L)$. Using this basis, we may define scalars $\alpha_{i,j}$ by $\alpha(x_i) = \sum_{j=1}^{n} \alpha_{i,j} x_j$. We say that the following matrix is the matrix of $\alpha$ with respect to our chosen basis and $\alpha$ have the following matrix forms.

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n,1} & \alpha_{n,2} & \cdots & \alpha_{n,n} \end{bmatrix}.$$

The following example shows that the conditions given in Theorems 2.4 and 2.6 are necessary.

**Example.** Let $L$ be a Lie algebra with basis $x_1, \cdots, x_4$ and the only non-zero multiplication between basis elements is $[x_1, x_2] = x_3$. Then $L^2 = \langle x_3 \rangle, Z(L) = \langle x_3, x_4 \rangle$ and $L^3 = 0$. It is obvious that elements of $\mathrm{Der}_c(L)$ has the following matrix form

$$\begin{bmatrix} 0 & 0 & \alpha_{1,3} & 0 \\ 0 & 0 & \alpha_{2,3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore $\dim \mathrm{Der}_c(L) = 2$. If $\mathcal{D} = \mathrm{Der}_c(L)$, then $H = Z(L) = \langle x_3, x_4 \rangle$ and $\mathrm{Der}^H(L)$ has the following matrix form

$$\begin{bmatrix} 0 & 0 & \alpha_{1,3} & \alpha_{1,4} \\ 0 & 0 & \alpha_{2,3} & \alpha_{2,4} \\ 0 & 0 & 0 & \alpha_{3,4} \\ 0 & 0 & \alpha_{4,3} & \alpha_{4,4} \end{bmatrix}$$

Thus $\dim \mathrm{Der}^H(L) = 7$ and $\mathrm{Der}^H(L) \neq \mathrm{Der}_c(L)$. Also, iFt is easy to see that $\mathrm{ID}^*(L)$ and $\mathrm{Der}^H_{Z(L)}(L)$ have the following matrix forms

$$\begin{bmatrix} 0 & 0 & \alpha_{1,3} & 0 \\ 0 & 0 & \alpha_{2,3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathrm{ID}^*(L),$$

$$\begin{bmatrix} 0 & 0 & \alpha_{1,3} & \alpha_{1,4} \\ 0 & 0 & \alpha_{2,3} & \alpha_{2,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathrm{Der}^H_{Z(L)}(L).$$

Therefore $\mathrm{Der}^H_{Z(L)}(L) \neq \mathrm{ID}^*(L)$.

## 3. Applications

In this section, we use the classification of all finite dimensional nilpotent Lie algebras of dimension $n$, where $n \leq 6$ given by de Graaf [1]. As an application of our results, we find those nilpotent Lie algebras $L$ of dimension $n \leq 6$ for which $\mathrm{Der}^H(L) = \mathrm{Der}_c(L)$ or $\mathrm{Der}^H_{Z(L)}(L) = \mathrm{ID}^*(L)$. Also, we use theorem 1.1 and theorem 1.2 to characterize all nilpotent Lie algebras of dimension $n \leq 6$, for which $\mathrm{Der}^H(L) = \mathrm{Der}_{Z(L)}(L)$, $\mathrm{Der}^H_{Z(L)}(L) = \mathrm{Der}_{Z(L)}(L)$ and $\mathrm{Der}^H_{Z(L)}(L) = \mathrm{IDer}(L)$.
Clearly, if $L$ is an abelian Lie algebra, then

$$\mathrm{IDer}(L) = \mathrm{Der}_c(L) = \mathrm{ID}^*(L) = \mathrm{ID}(L) = 0.$$

In this case, if $\mathcal{D} = \mathrm{Der}(L)$, then $H = 0$ and $\mathrm{Der}^H(L) = 0$, if $\mathcal{D} = \mathrm{IDer}(L)$, then $H = Z(L) = L$ thus $\mathrm{Der}^H(L) \neq \mathrm{Der}_c(L)$ and $\mathrm{Der}^H_{Z(L)} \neq \mathrm{ID}^*(L)$. Therefore in the sequal, we consider non-abelian Lie algebras.

**Corollary 3.1.** *Let $L$ be a non-abelian nilpotent Lie algebras with* $\dim L \leq 6$. *Then*

(i) *If $\mathcal{D} = \mathrm{IDer}(L)$, then $\mathrm{Der}^H(L) = \mathrm{Der}_c(L)$ if and only if $L$ is one of the following Lie algebra.*

$$H(1), H(2) \quad \textit{or} \quad L_{6,22}(\varepsilon), \varepsilon \neq 0.$$

(ii) *If $\mathcal{D} = \mathrm{Der}_c(L)$, then $\mathrm{Der}^H(L) = \mathrm{Der}_c(L)$ if and only if $L$ is one of the following Lie algebra.*

$$H(1), H(2) \quad \textit{or} \quad L_{6,22}(\varepsilon), \varepsilon \neq 0.$$

(iii) *If $\mathcal{D} = \mathrm{ID}^*(L)$, then $\mathrm{Der}^H(L) = \mathrm{Der}_c(L)$ if and only if $L$ is one of the following Lie algebra.*

$$H(1), H(2) \quad \textit{or} \quad L_{6,22}(\varepsilon), \varepsilon \neq 0.$$

(iv) *If $\mathcal{D} = \mathrm{ID}(L)$, then $\mathrm{Der}^H(L) = \mathrm{Der}_c(L)$ if and only if $L$ is one of the following Lie algebra.*

$$H(1), H(2) \quad \textit{or} \quad L_{6,22}(\varepsilon), \varepsilon \neq 0.$$

**Corollary 3.2.** *Let $L$ be a non-abelian nilpotent Lie algebras with* $\dim L \leq 6$. *Then*

(i) *If $\mathcal{D} = \mathrm{IDer}(L)$, then $\mathrm{Der}^H_{Z(L)}(L) = \mathrm{ID}^*(L)$ if and only if $L$ is one of the following Lie algebras.*

$$H(1), H(2), L_{5,8}, L_{6,22}(\varepsilon) \quad \textit{or} \quad L_{6,26}, \varepsilon \neq 0.$$

(ii) *If $\mathcal{D} = \mathrm{Der}_c(L)$, then $\mathrm{Der}^H_{Z(L)}(L) = \mathrm{ID}^*(L)$ if and only if $L$ is one of the following Lie algebras.*

$$H(1), H(2), L_{5,8}, L_{6,22}(\varepsilon) \quad \textit{or} \quad L_{6,26}, \varepsilon \neq 0.$$

(iii) *If* $\mathcal{D} = \mathrm{ID}^*(L)$*, then* $Der^H_{Z(L)}(L) = \mathrm{ID}^*(L)$ *if and only if* $L$ *is one of the following Lie algebras.*

$$H(1), H(2), L_{5,8}, L_{6,22}(\varepsilon) \quad or \quad L_{6,26}, \varepsilon \neq 0.$$

(iv) *If* $\mathcal{D} = \mathrm{ID}(L)$*, then* $Der^H_{Z(L)}(L) = \mathrm{ID}^*(L)$ *if and only if* $L$ *is one of the following Lie algebras.*

$$H(1), L_{4,2}, L_{5,2}, H(2), L_{5,8}, L_{6,2}, L_{6,4}, L_{6,8}, L_{6,22}(\varepsilon) \quad or \quad L_{6,26}, \varepsilon \neq 0.$$

**Corollary 3.3.** *Let* $L$ *be a non-abelian nilpotent Lie algebras with* $\dim L \leq 6$*. Then*

(i) *If* $\mathcal{D} = \mathrm{IDer}(L)$*, then* $\mathrm{Der}^H_{Z(L)}(L) = \mathrm{IDer}(L)$ *if and only if* $L$ *is one of the following Lie algebras.*

$$H(1), H(2).$$

(ii) *If* $\mathcal{D} = \mathrm{Der}_c(L)$*, then* $\mathrm{Der}^H_{Z(L)}(L) = \mathrm{IDer}(L)$ *if and only if* $L$ *is one of the following Lie algebras.*

$$H(1), H(2).$$

(iii) *If* $\mathcal{D} = \mathrm{ID}^*(L)$*, then* $\mathrm{Der}^H_{Z(L)}(L) = \mathrm{IDer}(L)$ *if and only if* $L$ *is one of the following Lie algebras.*

$$H(1), H(2).$$

(iv) *If* $\mathcal{D} = \mathrm{ID}(L)$*, then* $\mathrm{Der}^H_{Z(L)}(L) = \mathrm{IDer}(L)$ *if and only if* $L$ *is one of the following Lie algebras.*

$$H(1), L_{4,2}, L_{5,2}, H(2), L_{6,2}, L_{6,4}.$$

**Corollary 3.4.** *Let* $L$ *be a non-abelian nilpotent Lie algebras with* $\dim L \leq 6$*. Then*

(i) *If* $\mathcal{D} = \mathrm{IDer}(L)$*, then* $\mathrm{Der}^H(L) = \mathrm{Der}_{Z(L)}(L)$ *if and only if* $L$ *is one of the following Lie algebras.*

$$H(1), L_{4,2}, L_{4,3}, L_{5,i}(2 < i < 9), L_{6,i}(2 \leq i \leq 20), L_{6,21}(\varepsilon), L_{6,22}(\varepsilon) \quad or \quad L_{6,24}(\varepsilon), \varepsilon \neq 0.$$

(ii) *If* $\mathcal{D} = \mathrm{Der}_c(L)$*, then* $\mathrm{Der}^H(L) = \mathrm{Der}_{Z(L)}(L)$ *if and only if* $L$ *is one of the following Lie algebras.*

$$H(1), L_{4,2}, L_{4,3}, L_{5,i}(2 \leq i \leq 9), L_{6,i}(2 \leq i \leq 20), L_{6,21}(\varepsilon), L_{6,22}(\varepsilon) \quad or \quad L_{6,24}(\varepsilon), \varepsilon \neq 0.$$

(iii) *If* $\mathcal{D} = \mathrm{ID}^*(L)$*, then* $\mathrm{Der}^H(L) = \mathrm{Der}_{Z(L)}(L)$ *if and only if* $L$ *is one of the following Lie algebras.*

$$H(1), L_{4,2}, L_{4,3}, L_{5,i}(2 \leq i \leq 9), L_{6,i}(10 \leq i \leq 20), L_{6,21}(\varepsilon), L_{6,22}(\varepsilon) \quad or \quad L_{6,24}(\varepsilon), \varepsilon \neq 0.$$

(iv) *If* $\mathcal{D} = \mathrm{ID}(L)$*, then* $\mathrm{Der}^H(L) = \mathrm{Der}_{Z(L)}(L)$ *if and only if* $L$ *is one of the following Lie algebras.*

$$H(1), L_{4,3}, L_{5,i}(4 \leq i \leq 9), L_{6,i}(10 \leq i \leq 20), L_{6,3}, L_{6,21}(\varepsilon) \quad or \quad L_{6,24}(\varepsilon), \varepsilon \neq 0.$$

**Corollary 3.5.** *Let* $L$ *be a non-abelian nilpotent Lie algebras with* $\dim L \leq 6$*. Then*

(i) *If* $\mathcal{D} = \mathrm{IDer}(L)$*, then* $\mathrm{Der}^{H}_{Z(L)}(L) = \mathrm{Der}_z(L)$ *if and only if* $L$ *is one of the following Lie algebras.*

$$H(1), L_{4,3}, L_{5,i}(4 \le i \le 9), L_{6,i}(10 \le i \le 26), \varepsilon \ne 0.$$

(ii) *If* $\mathcal{D} = \mathrm{Der}_c(L)$*, then* $\mathrm{Der}^{H}_{Z(L)}(L) = \mathrm{Der}_z(L)$ *if and only if* $L$ *is one of the following Lie algebras.*

$$H(1), L_{4,3}, L_{5,i}(4 \le i \le 9), L_{6,i}(10 \le i \le 26), \varepsilon \ne 0.$$

(iii) *If* $\mathcal{D} = \mathrm{ID}^*(L)$*, then* $\mathrm{Der}^{H}_{Z(L)}(L) = \mathrm{Der}_z(L)$ *if and only if* $L$ *is one of the following Lie algebras.*

$$H(1), L_{4,3}, L_{5,i}(4 \le i \le 9), L_{6,i}(10 \le i \le 26), \varepsilon \ne 0.$$

(iv) *If* $\mathcal{D} = \mathrm{ID}(L)$*, then* $\mathrm{Der}^{H}_{Z(L)}(L) = \mathrm{Der}_z(L)$ *if and only if* $L$ *is one of the following Lie algebras.*

$$H(1), L_{4,3}, L_{5,i}(4 \le i \le 9), L_{6,i}(10 \le i \le 20), \varepsilon \ne 0.$$

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