Iranian Journal of Mathematical Sciences and Informatics
Vol.2, No.2 (2007), pp 29-33
DOI: 10.7508/ijmsi.2007.02.004

# THE AUTOMORPHISM GROUP OF FINITE GRAPHS

G. H. FATH-TABAR
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, UNIVERSITY OF KASHAN, KASHAN 87317-51167, IRAN
EMAIL: FATHTABAR@KASHANU.AC.IR

Abstract. Let $G = (V, E)$ be a simple graph with exactly $n$ vertices and $m$ edges. The aim of this paper is a new method for investigating non-triviality of the automorphism group of graphs. To do this, we prove that if $|E| \geq \lfloor (n-1)^2/2 \rfloor$ then $|Aut(G)| > 1$ and $|Aut(G)|$ is even number.

**Keywords and phrases**: Finite graph, graph automorphism, bipartite graph.

**2000 Mathematics subject classification:** Primary 20D45, Secondary 05C25.

## 1. Introduction

Throughout this paper all graphs mentioned are assumed to be finite simple graph. Let $G = (V, E)$ be a graph of order n with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$, $E \subseteq P_2(V)$ and $|E| = m$. The automorphism group of a graph $G$ is denoted by $Aut(G)$.

In [2, 3], the authors proved that the proportion of graphs which have a non-trivial automorphism group tends to zero as $n \to \infty$. This is true whether we take labeled or unlabeled graphs. Let $G_1, G_2$ be two graphs. Then $G_1 + G_2$ is join of $G_1$ and $G_2$ namely every vertex of $G_1$ is join to every vertex of $G_2$.

Let $G = (V, E)$ be a graph with $n$ vertices and $x \in V(G)$. We define $V_x = \{t \in V(G) | \ xt \in E(G)\}$. If $V_x - \{y\} = V_y - \{x\}$ then we call $x, y \in V(G)$ to be co-adjacent.

**Theorem 1.1.** *If $G = (V, E)$ is a finite simple graph with two vertices that are co-adjacent then $2 || Aut(G)|$ and $|Aut(G)| > 1$.*

Received 10 July 2007; Accepted 5 February 2008

**Proof.** Let x,y be co-adjacent. Our main proof consider two separate cases:

Case 1. If $x, y$ are not adjacent then $V_x = V_y$. We now define $f : V(G) \to V(G)$ by $f(x) = y, f(y) = x, f(t) = t$, for $t \notin \{x, y\}$. Since $V_x = V_y$, $f$ is an automorphism. One can see that $f \neq$ identity and $O(f) = 2$. Thus $2||Aut(G)|$ and $|Aut(G)| > 1$.

Case 2. Suppose $x, y$ are adjacent. Then $V_x - \{y\} = V_y - \{x\}$ and a similar argument as Case 1 shows that $f : V(G) \to V(G)$ is an isomorphism, where $f(x) = y, f(y) = x$ and $f(t) = t$, for $t \notin \{x, y\}$. Therefore $2||Aut(G)|$, proving the theorem. □

**Theorem 1.2.** *Suppose* $x_i, y_i$, $1 \leq i \leq k$, *are co-adjacent and* $\{x_i, y_i\} \bigcap \{x_j, y_j\} = \phi, i \neq j$, *then* $2^k||Aut(G)|$.

**Proof.** By the proof of Theorem 1, $(x_i, y_i) \in Aut(G)$ and $(x_i, y_i)(x_j, y_j) = (x_j, y_j)(x_i, y_i)$, because $\{x_i, y_i\} \bigcap \{x_j, y_j\} = \phi$ and $(x_i, y_i), (x_j, y_j)$ are disjoint permutation of order 2. Thus $< (x_1, y_1), (x_2, y_2), ..., (x_k, y_k) > = < (x_1, y_1) > \times < (x_2, y_2) > \times ... \times < (x_k, y_k) >$ is a subgroup of Aut(G) and by Lagrange's theorem $O(< (x_1, y_1), (x_2, y_2), ..., (x_k, y_k) >)||Aut(G)|$. Therefore $O(< (x_i, y_i), (x_j, y_j) >) = O((x_i, y_i))O((x_j, y_j))$ and hence $2^k||Aut(G)|$. □

**Example 1.3.** Suppose $G = (V, E)$ in which

$$V = \{1, 2, 3, 4\}, \quad E = \{13, 24, 32, 41, 34\}.$$

Then $\{1, 2\} \bigcap \{3, 4\} = \phi$ and so $4 \mid |\text{Aut(G)}|$. Thus $|\text{Aut}(G)| = 4$ and $\text{Aut}(G) \cong Z_2 \times Z_2$.

**Theorem 1.4.** *Let* $G$ *be a graph with* $n$ *vertices. If* $|E| \geq \lfloor (n-1)^2/2 \rfloor$ *then there exists a co-adjacent pair* $(x, y) \in V(G)$.

**Proof.** Since two vertices with the same degree $n - 1$ are co-adjacent, so it is enough to assume that $G$ have at most one vertex of degree $n - 1$. We consider the following two cases.

Case 1. $n$ is even. Then $\lfloor (n-1)^2/2 \rfloor = \frac{n(n-2)}{2}$. Since the number of edges in a $n - 2$-regular graph is $\frac{n(n-2)}{2}$, there are at least two co-adjacent vertices of degree $n - 1$, whenever $|E| > \frac{n(n-2)}{2}$. If $|E| = \frac{n(n-2)}{2}$ and G is $(n - 2)-$regular then every two non-adjacent vertices of degree $n - 2$ are co-adjacent. If $|E| = \frac{n(n-2)}{2}$ and $G$ is not $(n - 2)-$regular then there exist $x, y \in V(G)$ such that $deg(x) = deg(y) = n - 2$ and $x, y$ are not adjacent. Thus these are co-adjacent. Otherwise $2|E| \leq (n-2)(n-3) + (n-2) + (n-1) < n(n-2)$, which is a contradiction.

Case 2. Suppose $n$ is odd. Then $|E| = \lfloor (n-1)^2/2 \rfloor = \frac{(n-1)^2}{2}$. If there are two vertices of degree $n - 1$ then they are co-adjacent, otherwise if $G$ dose not have one vertex of degree $n - 1$, then a similar argument as above completes the proof. Suppose there exist one vertex of degree $n - 1$. Then by omitting this vertex $G-v$ has order $n-1$ and $n-1$ is even. Since $|E(G-v)| \geq (n-1)(n-3)/2$, a simple argument as Case 1 completes the proof. □

**Example 1.5.** Suppose $G = (V, E)$, where

$$V = \{1, 2, 3, 4\}, \quad E = \{12, 14, 15, 23, 24, 34, 45\}.$$

We can see that $G$ dose not satisfy the conditions of Theorem 3 with one edge less than $\lfloor (n-1)^2/2 \rfloor$ and there are not co-adjacent vertices. This shows that the bound given in Theorem 3 is sharp.

## 2. The Main Results

This section is concerned with the main theorem of the paper. Some new results are also presented.

**Theorem 2.1.** *Let $G$ be a graph with $|E| = m \geq \lfloor (n-1)^2/2 \rfloor$. Then $|Aut(G)| > 1$ and $|Aut(G)|$ is even number.*

**Proof.** Suppose $|E| \geq \lfloor (n-1)^2/2 \rfloor$. Then by Theorem 3, there are two vertices $x, y$ such that $x, y$ are co-adjacent and by Theorem 1, we can conclude that $2||Aut(G)|$, proving the theorem. □

**Theorem 2.2.** *Let $G = (V, E)$ be a graph and $A, B \subseteq V(G)$ such that every two member of $A$ or $B$ are co-adjacent. Then $Aut(G)$ contains a subgroup of order $|A|!|B|!$.*

**Proof.** Suppose $G_A = \{f \in Aut(G) | f(x) = x, \forall x \notin A\}$ and $G_B = \{f \in Aut(G) | f(x) = x, \forall x \notin B\}$. We can see that $G_A$ and $G_B$ are subgroups of $Aut(G)$ such that $G_A \cong S_{|A|}$ and $G_B \cong S_{|B|}$. Notice that if $f \in G_A$ and $g \in G_B$ then $f, g$ are disjoint permutation and $fg = gf$. Thus $G_A G_B = G_B G_A$ and so $G_A G_B$ is a subgroup of $Aut(G)$. Since $|G_A| = |A|!, |G_B| = |B|!$ and $G_A \bigcap G_B = \{e\}$, $|G_A G_B| = |G_A||G_B| = |A|!|B|!$. □

**Theorem 2.3.** *Let $G = (V, E)$ be a graph, $A, B \subseteq V$, $|V| = A \cup B$ and $deg(a) \neq deg(b)$, for all $a \in A, b \in B$. Then $Aut(G) \cong S_{|A|} \times S_{|B|}$.*

**Proof.** By Theorem 5, $G_A G_B \leq Aut(G)$. Since $deg(a) \neq deg(b)$, $a \in A$ is not commute with $b \in B$. This means that $Aut(G) = G_A G_B$. By Theorem 5, $|Aut(G)| = |A|!|B|!$ and $G_A \bigcap G_B = \{e\}$. Hence $G_A, G_B \trianglelefteq Aut(G)$ and $Aut(G) \cong G_A \times G_B$. Obviously, $G_A \cong S_{|A|}$, $G_B \cong S_{|B|}$ and so $Aut(G) \cong S_{|A|} \times S_{|B|}$. □

**Corollary 2.4.** *Suppose $n_i \neq n_j$, where $i, j$ are distinct. Then*

$$Aut(K_{n_1,n_2,n_3}) \cong S_{n_1} \times S_{n_2} \times S_{n_3}.$$

**Proof.** Suppose $A, B$ and $C$ are the part of $K_{n_1,n_2,n_3}$ containing $n_1, n_2$ and $n_3$ vertices, respectively. Apply Theorem 6. One can see that elements of $A, B$ and $C$ have degree $n_1 + n_2, n_1 + n_3$ and $n_2 + n_3$, as desired. □

**Theorem 2.5.** *Suppose $G_i, i = 1, 2$ are $(n, m_i)$-graph with $m_1 = C(n, 2) - 1$ and $m_2 = C(n, 2) - 2$. Then*

*a) $Aut(G_1) \cong Z_2 \times S_{n-2}$.*

*b)$Aut(G_2) \cong Z_2 \times S_{n-3}$ or $Aut(G_2) \cong D_4 \times S_{n-4}$.*

**Proof.** a) Suppose $A$ and $B$ are subsets with two and $n-2$ elements of $V(G)$, where elements of $A$ have degree $n-2$ and elements of $B$ have degree $n-1$. Thus elements of $A$ are co-adjacent and the same are true for elements of $B$. We now apply Theorem 6 to prove $Aut(G_1) \cong Z_2 \times S_{n-2}$.

b) By omitting two edges from the complete graph $K_n$, one can prove there are four vertices of degree $n-2$ or two vertices with degree $n-2$ and one vertex of degree $n-3$. Thus by Theorem 6, in the first case $A$ contains two element of degree two and $B$ contains $n-3$ elements of degree $n-1$. Thus $Aut(G_2) \cong Z_2 \times S_{n-3}$. In the second part one can see that there are four vertices of degree $n-2$ and $n-4$ vertices of degree $n-1$. By omitting this $n-4$ vertices, we obtain the cycle graph $C_4$, where $Aut(C_4) \cong D_4$. A similar argument shows that $Aut(G_2) \cong D_4 \times S_{n-4}$. □

**Theorem 2.6.** *Suppose $G_1$ and $G_2$ are two graphs. If $H_1 \leq Aut(G_1)$ and $H_2 \leq Aut(G_2)$ then $H_1 \times H_2 \leq Aut(G_1 + G_2)$. Also, if $|d(x_i) - d(y_j)| \neq |n_1 - n_2|, i = 1, 2, ..., n_1$ and $j = 1, 2, ..., n_2$ then*

$$Aut(G_1 + G_2) \cong Aut(G_1) \times Aut(G_2)$$

**Proof.** Let $H_1 \leq Aut(G_1)$ and $H_2 \leq Aut(G_2)$. Then it is obvious that $H_1 \times H_2 \leq Aut(G_1 + G_2)$. For proving the second part of the theorem, we assume that $H_1 = Aut(G_1)$ and $H_2 = Aut(G_2)$. Then $Aut(G_1) \times Aut(G_2) \leq Aut(G_1 + G_2)$. Suppose $f(x_i) = y_j$. Then $d(x_i) + n_2 = d(y_j) + n_1$ and so $d(x_i) - d(y_j) = n_1 - n_2$. This implies that $|d(x_i) - d(y_j)| = |n_1 - n_2|$, a contradiction. Thus vertices of $G_1$ and $G_2$ cannot interchange to each other and so $|Aut(G_1 + G_2)| = |Aut(G_1)||Aut(G_2)|$. Hence $Aut(G_1 + G_2) \cong Aut(G_1) \times Aut(G_2)$. □

In the end of this paper, we compute the automorphism groups of the complete bipartite graph $K_{m,n}$ and a summation of complete bipartite graphs. To do this, we notice that $K_{m,n} = \bar{K}_m + \bar{K}_n$.

**Corollary 2.7.** *Suppose $m = m_1 + m_2$, $m' = m'_1 + m'_2$ and $|m_i - m_j| \neq |m - m'|$. Then $Aut(K_{m_1 m_2} + K_{m'_1 m'_2}) \cong Aut(K_{m_1 m_2}) \times Aut(K_{m'_1 m'_2})$. In particular if $m_1 \neq m_2, m'_1 \neq m'_2$ then $Aut(K_{m_1 m_2} + K_{m'_1 m'_2}) \cong S_{m_1} \times S_{m_2} \times S_{m'_1} \times S_{m'_2}$.*

**Proof.** Apply Theorems 6 and 7. □

**Acknowledgement.** The work of the author was supported in part by a grant from the Center of Excellence of Algebraic Methods and Applications of the Isfahan University of Technology.

## References

[1] N. Biggs, *Algebraic Graph theory*, Cambridge Univ. Press, Cambridge, 1993.
[2] P. J. Cameron, *Atomorphism of graphs*, University of london, Queen Marry, Draft, April 2001.
[3] N. C. Wormald, *Models of random regular graphs, in: Surveys in combinatorics,* (ed. J.D. Lamb. and D.A. Preece), London Math. Soc, Lecture Notes Series 267, Cambridge University Press, Cambridge, (1999), 239-298.